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Double Generating Functions for Jacobi and Laguerre Polynomials of Several Variables

Research Article

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Abstract: The aim of the present paper is to obtain some double generating functions for Jacobi and Laguerre polynomials of several variables. Some deductions from these results lead us to obtain a number of double generating functions for Sister Celine's polynomials of two variables. A number of interesting special cases of our main results are also considered.

MSC: 33C15, 33C20, 33C45, 33C65.

Keywords: Generating functions, Jacobi polynomials , Laguerre polynomials, Sister Celine's polynomials, Multivariable extension of the Kampé de Fériet function

1. Introduction

The Jacobi polynomials of several variables $P_n^{(\alpha_1,\beta_1;\cdots;\alpha_r,\beta_r)}(x_1,\cdots,x_r)$ of Shrivastava [4] are defined by

$$P_n^{(\alpha_1,\beta_1;\cdots;\alpha_r,\beta_r)}(x_1,\cdots,x_r)=\frac{(1+\alpha_1)_n(1+\alpha_2)_n\cdots(1+\alpha_r)_n}{(n!)^r}$$
$$\times F\begin{matrix}1 & : & 1 & ; & \cdots & ; & 1\\ 0 & : & 1 & ; & \cdots & ; & 1\end{matrix}\left[\begin{matrix}-n & : & 1+\alpha_1+\beta_1+n & ; & \cdots & ; & 1+\alpha_r+\beta_r+n & ;\\ - & : & \alpha_1+1 & ; & \cdots & ; & \alpha_r+1 & ;\end{matrix}\ \frac{1-x_1}{2},\cdots,\frac{1-x_r}{2}\right] \tag{1}$$

where $(a)_n$ is the Pochhammer symbol defined as :

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}=\begin{cases}1, & \text{if } n=0\\ a(a+1)(a+2)\ldots(a+n-1), & \text{if } n=1,2,\ldots.\end{cases} \tag{2}$$

and $F\begin{matrix}p: & q_1; & \ldots & ;q_n\\ l: & m_1; & \ldots & ;m_n\end{matrix}[x_1,\ldots,x_n]$ is the multivariable extension of the Kampé de Fériet function (see Srivastava and Manocha [7]):

$$F\begin{matrix}p: & q_1; & \ldots & ;q_n\\ l: & m_1; & \ldots & ;m_n\end{matrix}\left(\begin{matrix}(a_p): & (b^{(1)}_{q_1}) & ;\ldots; & (b^{(n)}_{q_n});\\ & & & & x_1, & \ldots & ,x_n\\ (c_l): & (d^{(1)}_{m_1}) & ;\ldots; & (d^{(n)}_{m_n});\end{matrix}\right)=\sum_{S_1,\ldots,S_n=0}^{\infty}\Lambda(S_1,\ldots,S_n)\frac{x_1^{S_1}}{S_1!}\cdots\frac{x_n^{S_n}}{S_n!}, \tag{3}$$

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where

$$\Lambda(S_1,\ldots,S_n)=\frac{\prod_{j=1}^{p}(a_j)_{S_1+\cdots+S_n}\prod_{j=1}^{q_1}(b_j^{(1)})_{S_1}\cdots\prod_{j=1}^{q_n}(b_j^{(n)})_{S_n}}{\prod_{j=1}^{l}(c_j)_{S_1+\cdots+S_n}\prod_{j=1}^{m_1}(d_j^{(1)})_{S_1}\cdots\prod_{j=1}^{m_n}(d_j^{(n)})_{S_n}}. \tag{4}$$

and, for convergence of the multiple hypergeometric series in (3)

$$1+l+m_k-p-q_k\geq 0,\ k=1,\cdots,n;$$

the equality holds when, in addition, either

$$p>l \text{ and } |x_1|^{\frac{1}{p-l}}+\cdots+|x_n|^{\frac{1}{p-l}}<1;$$

or

$$p\leq l \text{ and } \max\{|x_1|,\cdots,|x_n|\}<1.$$

The Laguerre polynomials of several variables $L_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$ of Khan and Shukla [3] are defined by

$$L_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)=\frac{\prod_{j=1}^{r}(\alpha_j+1)_n}{(n!)^r}\Psi_2^{(r)}\left[-n;\ \alpha_1+1,\ \cdots,\ \alpha_r+1;\ x_1,\cdots,x_r\right] \tag{5}$$

where $\Psi_2^{(r)}$ is the confluent hypergeometric function of r-variables [7]

$$\Psi_2^{(r)}\left[a;\ c_1,\cdots,c_r;\ x_1,\cdots,x_r\right]=\sum_{m_1,\cdots,m_r=0}^{\infty}\frac{(a)_{m_1+\cdots+m_r}}{(c_1)_{m_1}\cdots(c_r)_{m_r}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_r^{m_r}}{m_r!} \tag{6}$$

The Appell's double hypergeometric function F_2 [7] is defined by

$$F_2\left[a,b,b';c,c';x,y\right]=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n}\frac{x^m}{m!}\frac{y^n}{n!} \tag{7}$$

The Humbert's double hypergeometric function Ψ_1 [7] is defined by

$$\Psi_1\left[a,b;c,c';x,y\right]=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m}{(c)_m(c')_n}\frac{x^m}{m!}\frac{y^n}{n!} \tag{8}$$

The triple hypergeometric functions ${}_3\Phi_A^{(1)}$ and ${}_3\Phi_A^{(2)}$ of Jain [2] are defined by

$${}_3\Phi_A^{(1)}\left[a,b_1,b_2;c_1,c_2,c_3;x,y,z\right]=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n+p}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \tag{9}$$

$${}_3\Phi_A^{(2)}\left[a,b_1;c_1,c_2,c_3;x,y,z\right]=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n+p}(b_1)_m}{(c_1)_m(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \tag{10}$$

2. Double Generating Functions

In this section, we have proved the following double generating functions:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m!)^{r-1}(n!)^{s-1}(\lambda)_{m+n}}{\prod_{j=1}^{r}(1+\alpha_j)_m\prod_{j=1}^{s}(1+\gamma_j)_n}\times P_m^{(\alpha_1,\beta_1-m;\cdots;\alpha_r,\beta_r-m)}(x_1,\cdots,x_r)P_n^{(\gamma_1,\delta_1-n;\cdots;\gamma_s,\delta_s-n)}(y_1,\cdots,y_s)t^m(-t)^n$$

$$=F\begin{matrix}1:1;\cdots;1\\0:1;\cdots;1\end{matrix}\left[\begin{matrix}\lambda: 1+\alpha_1+\beta_1;\cdots;1+\alpha_r+\beta_r;1+\gamma_1+\delta_1;\cdots;1+\gamma_s+\delta_s;\\ -:\alpha_1+1;\cdots;\alpha_r+1;\gamma_1+1;\cdots;\gamma_s+1;\end{matrix}\right.$$
$$\left.\frac{1}{2}(x_1-1)t,\cdots,\frac{1}{2}(x_r-1)t,\frac{1}{2}(1-y_1)t,\cdots,\frac{1}{2}(1-y_s)t\right], \tag{11}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda)_{m+n}\,(m\,!)^{r-1}\,(n\,!)^{s-1}}{\prod\limits_{j=1}^{r}(1+\alpha_j)_m\,\prod\limits_{j=1}^{s}(1+\beta_j)_n}\,L_m^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)\,L_n^{(\beta_1,\cdots,\beta_s)}(y_1,\cdots,y_s)t^m\,(-t)^n$$
$$=\Psi_2^{(r+s)}\,[\lambda;1+\alpha_1,\cdots,1+\alpha_r,1+\beta_1,\cdots,1+\beta_s;-x_1t,\cdots,-x_rt,y_1t,\cdots,y_st] \quad (12)$$

and

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m\,!)^{r-1}\,(n\,!)^{s-1}(\lambda)_{m+n}}{\prod\limits_{j=1}^{r}(1+\alpha_j)_m\,\prod\limits_{j=1}^{s}(1+\gamma_j)_n}P_m^{(\alpha_1,\beta_1-m;\cdots;\alpha_r,\beta_r-m)}(x_1,\cdots,x_r)L_n^{(\gamma_1,\cdots,\gamma_s)}(y_1,\cdots,y_s)\,t^m(-t)^n$$
$$=F\begin{matrix}1&:&1&;&\cdots&;&1&;&0&;&\cdots&;&0\\0&:&1&;&\cdots&;&1&;&1&;&\cdots&;&1\end{matrix}\left[\begin{matrix}\lambda&:&1+\alpha_1+\beta_1&;&\cdots&;&1+\alpha_r+\beta_r&;&-&;&\cdots&;&-&;\\-&:&\alpha_1+1&;&\cdots&;&\alpha_r+1&;&\gamma_1+1&;&\cdots&;&\gamma_s+1&;\end{matrix}\right.$$
$$\left.\frac{1}{2}(x_1-1)t,\cdots,\frac{1}{2}(x_r-1)t,y_1t,\cdots,y_st\right]. \quad (13)$$

Proof of (11): Denoting the left - hand side of (11) by S, expressing the two Jacobi polynomials of several variables as in (1) and using certain well-known properties of Pochhammer symbol, we get

$$S=\sum_{m=0}^{\infty}\sum_{p_1=0}^{m}\sum_{p_2=0}^{m-p_1}\cdots\sum_{p_r=0}^{m-p_1-\cdots-p_r}\frac{(\lambda)_m\,(-1)^{p_1+\cdots+p_r}}{(m-p_1-\cdots-p_r)\,!}$$
$$\times\frac{(1+\alpha_1+\beta_1)_{p_1}\cdots(1+\alpha_r+\beta_r)_{p_r}}{(1+\alpha_1)_{p_1}\cdots(1+\alpha_r)_{p_r}\,p_1!\cdots p_r!}\,t^m\left(\frac{1-x_1}{2}\right)^{p_1}\cdots\left(\frac{1-x_r}{2}\right)^{p_r}$$
$$\times\sum_{n=0}^{\infty}\sum_{q_1=0}^{n}\sum_{q_2=0}^{n-q_1}\cdots\sum_{q_s=0}^{n-q_1-\cdots-q_s}\frac{(\lambda+m)_n(-1)^{q_1+\cdots+q_s}}{(n-q_1-\cdots-q_s)\,!}$$
$$\times\frac{(1+\gamma_1+\delta_1)_{q_1}\cdots(1+\gamma_s+\delta_s)_{q_s}}{(1+\gamma_1)_{q_1}\cdots(1+\gamma_s)_{q_s}\,q_1!\cdots q_s!}\,(-t)^n\left(\frac{1-y_1}{2}\right)^{q_1}\cdots\left(\frac{1-y_s}{2}\right)^{q_s}$$
$$=\sum_{m=0}^{\infty}\sum_{p_1=0}^{\infty}\sum_{p_2=0}^{\infty}\cdots\sum_{p_r=0}^{\infty}\frac{(\lambda)_{m+p_1+\cdots+p_r}}{m\,!}$$
$$\times\frac{(1+\alpha_1+\beta_1)_{p_1}\cdots(1+\alpha_r+\beta_r)_{p_r}}{(1+\alpha_1)_{p_1}\cdots(1+\alpha_r)_{p_r}\,p_1!\cdots p_r!}\,t^{m+p_1+\cdots+p_r}\left(\frac{x_1-1}{2}\right)^{p_1}\cdots\left(\frac{x_r-1}{2}\right)^{p_r}$$
$$\times\sum_{n=0}^{\infty}\sum_{q_1=0}^{\infty}\sum_{q_2=0}^{\infty}\cdots\sum_{q_s=0}^{\infty}\frac{(\lambda+m+p_1+\cdots+p_r)_{n+q_1+\cdots+q_s}}{n\,!}$$
$$\times\frac{(1+\gamma_1+\delta_1)_{q_1}\cdots(1+\gamma_s+\delta_s)_{q_s}}{(1+\gamma_1)_{q_1}\cdots(1+\gamma_s)_{q_s}\,q_1!\cdots q_s!}\,(-t)^{n+q_1+\cdots+q_s}\left(\frac{y_1-1}{2}\right)^{q_1}\cdots\left(\frac{y_s-1}{2}\right)^{q_s}$$
$$=\sum_{p_1=0}^{\infty}\cdots\sum_{p_r=0}^{\infty}\sum_{q_1=0}^{\infty}\cdots\sum_{q_s=0}^{\infty}\frac{(\lambda)_{p_1+\cdots+p_r+q_1+\cdots+q_s}(1+\alpha_1+\beta_1)_{p_1}\cdots(1+\alpha_r+\beta_r)_{p_r}}{(1+\alpha_1)_{p_1}\cdots(1+\alpha_r)_{p_r}\;p_1!\cdots p_r!}$$
$$\times\frac{(1+\gamma_1+\delta_1)_{q_1}\cdots(1+\gamma_s+\delta_s)_{q_s}}{(1+\gamma_1)_{q_1}\cdots(1+\gamma_s)_{q_s}\,q_1!\cdots q_s!}\left(\frac{(x_1-1)t}{2}\right)^{p_1}\cdots\left(\frac{(x_r-1)t}{2}\right)^{p_r}\left(\frac{(1-y_1)t}{2}\right)^{q_1}\cdots\left(\frac{(1-y_s)t}{2}\right)^{q_s}$$
$$\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda+p_1+\cdots+p_r+q_1+\cdots+q_s)_{m+n}t^m(-t)^n}{m\,!\,n\,!}$$

Since,

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda+p_1+\cdots+p_r+q_1+\cdots+q_s)_{m+n}t^m(-t)^n}{m\,!\,n\,!}=1$$

the above series yields the right-hand side of (11). This completes the proof of (11). The results (12) and (13) can be proved by the similar manner. In view of the following relationship between the Sister Celine's polynomials of two variables $f_{m,n}$ [5] and Jacobi and Laguerre polynomials respectively:

$$f_{m,n}\left(\begin{matrix}(-m:1,1)&:&(1+\alpha+\beta,1),\left(\frac{1}{2},1\right),(1,1)&;&(1+\gamma+\delta,1),\left(\frac{1}{2},1\right),(1,1)\\-&:&(\alpha+1,1),(-m,1),(m+1,1)&;&(\gamma+1:1),(-n,1),(n+1,1)\end{matrix};\frac{1-x}{2},\frac{1-y}{2}\right)$$

$$= \frac{(m\,!)^2 P_m^{(\alpha,\,\beta-m;\gamma,\delta-m)}(x,y)}{(\alpha+1)_m\,(\gamma+1)_m}, \tag{14}$$

$$f_{m,n}\left(\begin{matrix} - \;:\; (1+\alpha+\beta\,,1)\,,(\frac{1}{2}\,,1)\,,(1\,,1) \;;\; (1+\gamma+\delta\,,1)\,,(\frac{1}{2}\,,1)\,,(1\,,1) \\ - \;:\; (\alpha+1,\,1)\,,(m+1\,,1) \;;\; (\gamma+1:\,1)\,,(n+1\,,1) \end{matrix} ;\, \frac{1-x}{2}\,,\,\frac{1-y}{2}\right)$$

$$= \frac{m\,!\,n\,!\,P_m^{(\alpha,\,\beta-m)}(x)P_n^{(\gamma,\,\delta-n)}(y)}{(\alpha+1)_m\,(\gamma+1)_n}, \tag{15}$$

$$f_{m,n}\left(\begin{matrix} (-m:1,1) \;:\; (\frac{1}{2}\,,1)\,,(1\,,1) \;;\; (\frac{1}{2}\,,1)\,,(1\,,1) \\ - \;:\; (-m,1)\,,(m+1\,,1),(1+\alpha:\,1) \;;\; (-n,1)\,,(n+1\,,1),(1+\beta,\,1) \end{matrix} ;\, x\,,\,y\right)$$

$$= \frac{(m\,!)^2 L_m^{(\alpha,\,\beta)}(x,y)}{(1+\alpha)_m\,(1+\beta)_m} \tag{16}$$

and

$$f_{m,n}\left(\begin{matrix} - \;:\; (\frac{1}{2}\,,1)\,,(1\,,1) \;;\; (\frac{1}{2}\,,1)\,,(1\,,1) \;; \\ - \;:\; (m+1\,,1),(1+\alpha,\,1) \;;\; (n+1\,,1),(1+\beta,\,1) \;; \end{matrix}\, x\,,\,y\right) = \frac{m\,!\,n\,!\,L_m^{(\alpha)}(x)L_n^{(\beta)}(y)}{(1+\alpha)_m\,(1+\beta)_n}, \tag{17}$$

we get from (11), (12) and (13) respectively the following results :

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda)_{m+n}t^m(-t)^n}{m\,!\,n\,!}$$

$$f_{m,n}\left(\begin{matrix} (-m:1,1) \;:\; (1+\alpha_1+\beta_1\,,1)\,,(\frac{1}{2}\,,1)\,,(1\,,1) \;;\; (1+\alpha_2+\beta_2\,,1)\,,(\frac{1}{2}\,,1)\,,(1\,,1) \\ - \;:\; (\alpha_1+1,\,1)\,,(-m,1)\,,(m+1\,,1) \;;\; (\alpha_2+1:\,1)\,,(-n,1)\,,(n+1\,,1) \end{matrix} ;\, \frac{1-x_1}{2}\,,\,\frac{1-x_2}{2}\right)$$

$$\times f_{n,m}\left(\begin{matrix} (-n:1,1) \;:\; (1+\gamma_1+\delta_1\,,1)\,,(\frac{1}{2}\,,1)\,,(1\,,1) \;;\; (1+\gamma_2+\delta_2\,,1)\,,(\frac{1}{2}\,,1)\,,(1\,,1) \\ - \;:\; (\gamma_1+1,\,1)\,,(-n,1)\,,(n+1\,,1) \;;\; (\gamma_2+1:\,1)\,,(-m,1)\,,(m+1\,,1) \end{matrix} ;\, \frac{1-y_1}{2}\,,\,\frac{1-y_2}{2}\right)$$

$$= F\begin{matrix} 1:\;1;\;1\;;\;1\;;\;1 \\ 0:\;1;\;1\;;\;1\;;\;1 \end{matrix}\left[\begin{matrix} \lambda \;:\; 1+\alpha_1+\beta_1 \;;\; 1+\alpha_2+\beta_2 \;;\; 1+\gamma_1+\delta_1 \;;\; 1+\gamma_2+\delta_2 \;; \\ - \;:\; \alpha_1+1 \;;\; \alpha_2+1 \;;\; \gamma_1+1 \;;\; \gamma_2+1 \;; \end{matrix}\right.$$

$$\left.\frac{1}{2}(x_1-1)t,\,\frac{1}{2}(x_2-1)t\,,\,\frac{1}{2}(1-y_1)t,\,\frac{1}{2}(1-y_2)t\right] \tag{18}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda)_{m+n}t^m(-t)^n}{m\,!\,n\,!}f_{m,n}\left(\begin{matrix} - \;:\; (1+\alpha+\beta\,,1)\,,(\frac{1}{2}\,,1)\,,(1\,,1) \;;\; (1+\gamma+\delta\,,1)\,,(\frac{1}{2}\,,1)\,,(1\,,1) \\ - \;:\; (\alpha+1,\,1)\,,(m+1\,,1) \;;\; (\gamma+1:\,1)\,,(n+1\,,1) \end{matrix} ;\, \frac{1-x}{2}\,,\,\frac{1-y}{2}\right)$$

$$= F_2\left[\lambda\,,\,1+\alpha+\beta,\,1+\gamma+\delta\,;\,\alpha+1\,,\,\gamma+1;\;\tfrac{1}{2}\,(x-1)t\,,\;\tfrac{1}{2}\,(1-y)t\right] \tag{19}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda)_{m+n}t^m(-t)^n}{m\,!\,n\,!}$$

$$f_{m,n}\left(\begin{matrix} (-m:1,1) \;:\; (\frac{1}{2}\,,1)\,,(1\,,1) \;;\; (\frac{1}{2}\,,1)\,,(1\,,1) \;; \\ - \;:\; (-m,1)\,,(m+1\,,1),(\alpha_1+1,\,1) \;;\; (-n,1)\,,(n+1\,,1),(\alpha_2+1,\,1) \;; \end{matrix}\, x_1\,,\,x_2\right)$$

$$\times f_{n,m}\left(\begin{matrix} (-n:1,1) \;:\; (\frac{1}{2}\,,1)\,,(1\,,1) \;;\; (\frac{1}{2}\,,1)\,,(1\,,1) \;; \\ - \;:\; (-n,1)\,,(n+1\,,1),(\beta_1+1,\,1) \;;\; (-m,1)\,,(m+1\,,1),(\beta_2+1,\,1) \;; \end{matrix}\, y_1\,,\,y_2\right)$$

$$= \Psi_2^{(4)}\left[\lambda\,;\,1+\alpha_1,\,1+\alpha_2,\,1+\beta_1,\,1+\beta_2;\,-x_1t,\,-x_2t,\,y_1t,\,y_2t\right] \tag{20}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda)_{m+n}t^m(-t)^n}{m!\,n!}f_{m,n}\left(\begin{array}{ccccc} - & : & \left(\frac{1}{2},1\right),(1,1) & ; & \left(\frac{1}{2},1\right),(1,1) & ; \\ - & : & (m+1,1),(\alpha+1,1) & ; & (n+1,1),(\beta+1,1) & ; \end{array} x,\,y\right)$$

$$= \Psi_2\,(\lambda\,;\,\alpha+1,\,\beta+1\,;\,-xt,\,yt) \tag{21}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda)_{m+n}t^m(-t)^n}{m!\,n!}$$

$$f_{m,n}\left(\begin{array}{ccccc} (-m:1,1) & : & (1+\alpha_1+\beta_1,1),\left(\frac{1}{2},1\right),(1,1) & ; & (1+\alpha_2+\beta_2,1),\left(\frac{1}{2},1\right),(1,1) \\ - & : & (\alpha_1+1,1),(-m,1),(m+1,1) & ; & (\alpha_2+1:1),(-n,1),(n+1,1) \end{array} ;\ \frac{1-x_1}{2},\frac{1-x_2}{2}\right)$$

$$\times f_{n,m}\left(\begin{array}{ccccc} (-n:1,1) & : & \left(\frac{1}{2},1\right),(1,1) & ; & \left(\frac{1}{2},1\right),(1,1) & ; \\ - & : & (-n,1),(n+1,1),(\gamma_1+1,1) & ; & (-m,1),(m+1,1),(\gamma_2+1,1) & ; \end{array} y_1,\,y_2\right)$$

$$= F\begin{array}{c} 1:1;1;0;0 \\ 0:1;1;1;1 \end{array}\left[\begin{array}{ccccccccc} \lambda & : & 1+\alpha_1+\beta_1 & ; & 1+\alpha_2+\beta_2 & ; & - & ; & - & ; \\ - & : & \alpha_1+1 & ; & \alpha_2+1 & ; & \gamma_1+1 & ; & \gamma_2+1 & ; \end{array} \tfrac{1}{2}(x_1-1)t,\tfrac{1}{2}(x_2-1)t,y_1t,y_2t\right] \tag{22}$$

3. Special Cases

In (11) putting $r=s=2$, $x_1=1-x$, $x_2=x+1$, $y_1=1-y$, $y_2=y+1$, $\alpha_1=\alpha_2=\alpha$, $\beta_1=\beta_2=\beta$, $\gamma_1=\gamma_2=\gamma$, $\delta_1=\delta_2=\delta$ and using the result [6]

$$F_2\,[\alpha\,,\,\beta,\,\beta\,;\,\gamma,\gamma\,;x,-x]=\,_4F_3\,\left[\tfrac{1}{2}\alpha\,,\tfrac{1}{2}\alpha+\tfrac{1}{2},\beta,\gamma-\beta\,;\,\gamma\,,\,\tfrac{1}{2}\gamma\,,\tfrac{1}{2}\gamma+\tfrac{1}{2}\,;\,x^2\right], \tag{23}$$

we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{m!\,n!\,(\lambda)_{m+n}\,t^m(-t)^nP_m^{(\alpha,\beta-m;\,\alpha,\beta-m)}\,(1-x\,,\,x+1)P_n^{(\gamma,\delta-n;\gamma,\delta-n)}\,(1-y\,,\,y+1)}{(\alpha+1)_m\,(\alpha+1)_m(\gamma+1)_n(\,\gamma+1)_n}$$

$$= F\begin{array}{c} 2:2;2 \\ 0:3;3 \end{array}\left[\begin{array}{ccccc} \frac{1}{2}\lambda,\frac{1}{2}\lambda+\frac{1}{2} & : & 1+\alpha+\beta,\,-\beta & ; & 1+\gamma+\delta,\,-\delta & ; \\ - & : & \alpha+1,\frac{1}{2}(\alpha+1),\frac{1}{2}(\alpha+2) & ; & \gamma+1,\frac{1}{2}(\gamma+1),\frac{1}{2}(\gamma+2) & ; \end{array} \left(\frac{xt}{2}\right)^2,\left(\frac{yt}{2}\right)^2\right] \tag{24}$$

Similarly, in (11) putting $r=s=3$, $x_1=y_1=x$, $x_2=y_2=y$, $x_3=y_3=z$, $\alpha_1=\gamma_1=\alpha$, $\alpha_2=\gamma_2=\gamma$, $\alpha_3=\gamma_3=\varepsilon$, $\beta_1=\delta_1=\beta$, $\beta_2=\delta_2=\delta$, $\beta_3=\delta_3=\nu$ and using the result (23), we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{m!\,n!\,(\lambda)_{m+n}\,t^m(-t)^nP_m^{(\alpha,\beta-m;\,\gamma,\delta-m;\,\varepsilon,\nu-m)}\,(x,y,z)P_n^{(\alpha,\beta-n;\,\gamma,\delta-n;\,\varepsilon,\nu-n)}\,(x,y,z)}{(\alpha+1)_m\,(\gamma+1)_m(\varepsilon+1)_m(\alpha+1)_n(\gamma+1)_n(\,\varepsilon+1)_n}$$

$$= F\begin{array}{c} 2:2;2;2 \\ 0:3;3;3 \end{array}\left[\begin{array}{ccccc} \frac{1}{2}\lambda,\frac{1}{2}\lambda+\frac{1}{2} & : & 1+\alpha+\beta,\,-\beta & ; & 1+\gamma+\delta,\,-\delta & ; \\ - & : & \alpha+1,\frac{1}{2}(\alpha+1),\frac{1}{2}(\alpha+2) & ; & \gamma+1,\frac{1}{2}(\gamma+1),\frac{1}{2}(\gamma+2) & ; \end{array}\right.$$

$$\left.\begin{array}{cc} 1+\varepsilon+\nu\,,-\nu & ; \\ \varepsilon+1,\frac{1}{2}(\varepsilon+1),\frac{1}{2}(\varepsilon+2) & ; \end{array} \left(\frac{(x-1)t}{2}\right)^2,\left(\frac{(y-1)t}{2}\right)^2,\left(\frac{(z-1)t}{2}\right)^2\right] \tag{25}$$

In (12) putting $r=s=2$, $x_1=x_2=x$, $y_1=y_2=y$ and using the result [6]

$$\Psi_2\,(\alpha\,;\,\gamma,\,\gamma'\,;\,x,\,x)\;=\,_3F_3\left[\begin{array}{cc} \alpha\,,\,\frac{1}{2}(\gamma+\gamma'),\frac{1}{2}(\gamma+\gamma'-1) & ; \\ \gamma,\,\gamma',\,\gamma+\gamma'-1 & ; \end{array} 4x\right], \tag{26}$$

we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda)_{m+n}\ m!\ n!\ t^m\,(-t)^n}{(\alpha_1+1)_m\,(\alpha_2+1)_m\,(\beta_1+1)_n\,(\beta_2+1)_n}\ L_m^{(\alpha_1,\,\alpha_2)}(x,\,x)\ L_n^{(\beta_1,\,\beta_2)}(y,\,y)$$

$$= F\begin{matrix}1 & : & 2 & ; & 2\\ 0 & : & 3 & ; & 3\end{matrix}\left[\begin{matrix}\lambda & : & \frac{1}{2}(\alpha_1+\alpha_2+1),\frac{1}{2}(\alpha_1+\alpha_2+2) & ; & \frac{1}{2}(\beta_1+\beta_2+1),\frac{1}{2}(\beta_1+\beta_2+2) & ;\\ - & : & \alpha_1+1,\alpha_2+1,\alpha_1+\alpha_2+1 & ; & \beta_1+1,\beta_2+1,\beta_1+\beta_2+1 & ;\end{matrix}\ -4xt,\,4yt\right] \tag{27}$$

Similarly,in (12) putting $r=s=2$, $x_1=x$, $x_2=-x$, $y_1=y$, $y_2=-y$, $\alpha_1=\alpha_2=\alpha$, $\beta_1=\beta_2=\beta$ and using the result [6]

$$\Psi_2\,(\alpha\,;\,\gamma,\,\gamma\,;\,x,\,-x)\ =\ {}_2F_3\left[\begin{matrix}\frac{1}{2}\alpha\,,\ \frac{1}{2}(\alpha+1) & ;\\ \gamma,\ \frac{1}{2}\gamma,\ \frac{1}{2}(\gamma+1) & ;\end{matrix}\ -x^2\right], \tag{28}$$

we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda)_{m+n}\ m!\ n!\,t^m\,(-t)^n}{(\alpha+1)_m\,(\alpha+1)_m\,(\beta+1)_n\,(\beta+1)_n}\ L_m^{(\alpha,\,\alpha)}(x,\,-x)\ L_n^{(\beta,\,\beta)}(y,\,-y)$$

$$= F\begin{matrix}2 & : & 0 & ; & 0\\ 0 & : & 3 & ; & 3\end{matrix}\left[\begin{matrix}\frac{1}{2}\lambda,\frac{1}{2}\lambda+\frac{1}{2} & : & - & ; & - & ;\\ - & : & \alpha+1\,,\frac{1}{2}(\alpha+1),\frac{1}{2}(\alpha+2) & ; & \beta+1,\frac{1}{2}(\beta+1),\frac{1}{2}(\beta+2) & ;\end{matrix}\ -x^2\,,\,-y^2\right] \tag{29}$$

In (12) putting $r=s=1$, $\alpha_1=\beta_1=\alpha$, $\lambda=\alpha+1$ and using the result [6]

$$\Psi_2\,(\gamma\,;\,\gamma,\,\gamma\,;\,x,\,y)\ =\ e^{x+y}\ {}_0F_1\,(-\,;\,\gamma\,;\,xy), \tag{30}$$

we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha+1)_{m+n}\,t^m\,(-t)^n}{(\alpha+1)_m\,(\alpha+1)_n}\ L_m^{(\alpha)}(x)\ L_n^{(\alpha)}(y) =\ \exp\,(ty-tx)\ {}_0F_1(-\,;\,\alpha+1\,;\,-t^2xy), \tag{31}$$

which is well known generating function of Exton [1].

In (13) putting $(r=1,\,s=1)$, $(r=2,\ s=1)$ and $(r=1,\ s=2)$, we get respectively the following results:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda)_{m+n}\ t^m\ (-t)^n}{(\alpha+1)_m\,(\gamma+1)_n}P_m^{(\alpha,\beta-m)}(x)\ L_n^{(\gamma)}(y) =\ \Psi_1\ \left[\lambda\,,\,1+\alpha+\beta;\,\alpha+1\,,\,\gamma+1\,;\ \tfrac{1}{2}\,(x-1)t\,,\ yt\right] \tag{32}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda)_{m+n}\ t^m\ (-t)^n}{(\alpha_1+1)_m(\alpha_2+1)_m\,(\gamma_1+1)_n}P_m^{(\alpha_1,\beta_1-m\,;\,\alpha_2,\beta_2-m)}(x_1,\,x_2)\,L_n^{(\gamma_1)}(y_1)$$

$$=\ {}_3\Phi_A^{(1)}\ \left[\lambda\,,\,1+\alpha_1+\beta_1,1+\alpha_2+\beta_2;\,1+\alpha_1,\,1+\alpha_2\,,1+\gamma_1\,;\ \tfrac{1}{2}\,(x_1-1)t,\tfrac{1}{2}\,(x_2-1)t,\ y_1t\right] \tag{33}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda)_{m+n}\ t^m\ (-t)^n}{(\alpha_1+1)_m(\gamma_1+1)_n\,(\gamma_2+1)_n}P_m^{(\alpha_1,\beta_1-m\ )}(x_1)\,L_n^{(\gamma_1,\gamma_2)}(y_1,y_2)$$

$$=\ {}_3\Phi_A^{(2)}\ \left[\lambda\,,\,1+\alpha_1+\beta_1;\,1+\alpha_1,\,1+\gamma_1\,,1+\gamma_2\,;\ \tfrac{1}{2}\,(x_1-1)t,y_1t,\ y_2t\right]. \tag{34}$$

References

[1] H.Exton, *A multiple generating functions suggested by an extension of a theorem of Bailey*, Pure Appl. Math. Sci., 37(1993), 1-12.

[2] R.N.Jain, *The confluent hypergeometric functions of three variables*, Proc. Nat. Acad. Sci. India, 36(1966), 395-408.

[3] M.A.Khan and A.K.Shukla, *On Laguerre polynomials of several variables*, Bull. Cal. Math. Soc., 89(1997), 155-164.

[4] H.S.P.Shrivastava, *On Jacobi polynomials of several variables*, Integral Trans. Spec. Funct., 10(2000), 61-70.

[5] H.S.P.Shrivastava, *On Sister Celine's polynomials of several variables*, Math. And Comp. Appl., 9(2004), 309-320.

[6] H.M.Srivastava and P.W.Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press, New York, (1985).

[7] H.M.Srivastava and H.L.Manocha, *A Treatise on Generating Functions*, Halsted Press, New York, (1984).