International Journal of Mathematics And its Applications
Volume 3, Issue 4–B (2015), 59–70.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

$(1, 2)^{\star}$-$g^{\star}$-Closed Sets

Research Article

O. Ravi[1]*, M. Jeyaraman[2], M. Sajan Joseph[3] and R. Muthuraj[4]

1 Department of Mathematics, P. M. Thevar College, Usilampatti, Madurai Dt, Tamil Nadu, India.
2 PG and Research Department of Mathematics, Raja Dorai Singam Govt.Arts College, Sivagangai, Tamil Nadu, India.
3 Department of Mathematics, Arul Anandar College, Karumathur, Madurai Dt, Tamil Nadu, India.
4 PG and Research Department of Mathematics, H. H. The Rajah's College, Pudukottai, Tamil Nadu, India.

Abstract: The aim of this paper is to introduce a new class of sets namely $(1, 2)^{\star}$-$g^{\star}$-closed sets in bitopological spaces. This class lies between the class of $\tau_{1,2}$-closed sets and the class of $(1, 2)^{\star}$-g-closed sets. The complement of an $(1, 2)^{\star}$-$g^{\star}$-closed set is called an $(1, 2)^{\star}$-$g^{\star}$-open set. Moreover we introduce two new spaces namely, $(1, 2)^{\star}$-$T^{\star}_{g}$-spaces and $(1, 2)^{\star}$-${}_{g}T^{\star}_{g}$-spaces.

MSC: 54E55.

Keywords: $(1, 2)^{\star}$-$g^{\star}$-closed set, $(1, 2)^{\star}$-$g^{\#}$-closed set, $(1, 2)^{\star}$-αg-closed set, $(1, 2)^{\star}$-$T_{1/2}$-space, $(1, 2)^{\star}$-$T^{\star}_{g}$-space, $(1, 2)^{\star}$-${}_{g}T^{\star}_{g}$-space.

1. Introduction

In 1963 Levine [17] introduced the notion of semi-open sets. According to Cameron [7] this notion was Levine's most important contribution to the field of topology. The motivation behind the introduction of semi-open sets was a problem of Kelley which Levine has considered in [18], i.e., to show that cl(U) = cl(U ∩ D) for all open sets U and dense sets D. He proved that U is semi-open if and only if cl(U) = cl(U ∩ D) for all dense sets D and D is dense if and only if cl(U) = cl(U ∩ D) for all semi-open sets U. Since the advent of the notion of semi-open sets, many mathematicians worked on such sets and also introduced some other notions, among others, preopen sets [19], α-open sets [20] and β-open sets [1] (Andrijevic [2] called them semi-pre open sets). It has been shown in [10] recently that the notion of preopen sets and semi-open sets are important with respect to the digital plane.

Levine [16] also introduced the notion of g-closed sets and investigated its fundamental properties. This notion was shown to be productive and very useful. For example it is shown that g-closed sets can be used to characterize the extremally disconnected spaces and the submaximal spaces (see [8] and [9]). Moreover the study of g-closed sets led to some separation axioms between T_0 and T_1 which proved to be useful in computer science and digital topology.

Bhattacharyya and Lahiri [6], Arya and Nour [5], Sheik John [32], Veera Kumar [33] and Rajamani and Viswanathan [21] introduced sg-closed sets, gs-closed sets, ω-closed sets, $g^{\#}$-closed sets and αgs-closed sets respectively. Levine [16] introduced the notion of $T_{1/2}$-spaces which properly lie between T_1-spaces and T_0-spaces. Many authors studied properties of $T_{1/2}$-spaces: Dunham [11], Arenas et al. [4] etc.

* *E-mail: siingam@yahoo.com*

In this paper, we introduce a new class of sets namely $(1,2)^{\star}$-$g^{\star}$-closed sets in bitopological spaces. This class lies between the class of $\tau_{1,2}$-closed sets and the class of $(1,2)^{\star}$-g-closed sets. The complement of an $(1,2)^{\star}$-$g^{\star}$-closed set is called an $(1,2)^{\star}$-$g^{\star}$-open set. Moreover, we introduce two new spaces namely, $(1,2)^{\star}$-$T_g^{\star}$-spaces and $(1,2)^{\star}$-${}_gT_g^{\star}$-spaces.

2. Preliminaries

Throughout this paper, X, Y and Z denote bitopological spaces (X, τ_1, τ_2), (Y, σ_1, σ_2) and (Z, η_1, η_2) respectively.

Definition 2.1. *Let A be a subset of a bitopological space X. Then A is called $\tau_{1,2}$-open [15] if $A = P \cup Q$, for some $P \in \tau_1$ and $Q \in \tau_2$. The complement of $\tau_{1,2}$-open set is called $\tau_{1,2}$-closed. The family of all $\tau_{1,2}$-open (resp. $\tau_{1,2}$-closed) sets of X is denoted by $(1,2)^{\star}$-O(X) (resp. $(1,2)^{\star}$-C(X)).*

Definition 2.2 ([15])**.** *Let A be a subset of a bitopological space X. Then*

1. *the $\tau_{1,2}$-interior of A, denoted by $\tau_{1,2}$-int(A), is defined by $\cup$ { $U : U \subseteq A$ and U is $\tau_{1,2}$-open};*
2. *the $\tau_{1,2}$-closure of A, denoted by $\tau_{1,2}$-cl(A), is defined by $\cap$ { $U : A \subseteq U$ and U is $\tau_{1,2}$-closed}.*

Remark 2.3 ([15])**.** *Notice that $\tau_{1,2}$-open subsets of X need not necessarily form a topology.*

Definition 2.4. *Let A be a subset of a bitopological space X. Then A is called*

1. *$(1,2)^{\star}$-semi-open set [15] if $A \subseteq \tau_{1,2}$-cl($\tau_{1,2}$-int(A)).*
2. *$(1,2)^{\star}$-preopen set [15] if $A \subseteq \tau_{1,2}$-int($\tau_{1,2}$-cl(A)).*
3. *$(1,2)^{\star}$-α-open set [15] if $A \subseteq \tau_{1,2}$-int($\tau_{1,2}$-cl($\tau_{1,2}$-int(A))).*
4. *$(1,2)^{\star}$-β-open set [27] if $A \subseteq \tau_{1,2}$-cl($\tau_{1,2}$-int($\tau_{1,2}$-cl(A))).*
5. *$(1,2)^{\star}$-regular open set [25] if $A = \tau_{1,2}$-int($\tau_{1,2}$-cl(A)).*

The complements of the above mentioned open sets are called their respective closed sets.

The $(1,2)^{\star}$-preclosure [23] (resp. $(1,2)^{\star}$-semi-closure [23], $(1,2)^{\star}$-α-closure [23], $(1,2)^{\star}$-β-closure [27]) of a subset A of X, denoted by $(1,2)^{\star}$-pcl(A) (resp. $(1,2)^{\star}$-scl(A), $(1,2)^{\star}$-αcl(A), $(1,2)^{\star}$-βcl(A)) is defined to be the intersection of all $(1,2)^{\star}$-preclosed (resp. $(1,2)^{\star}$-semi-closed, $(1,2)^{\star}$-α-closed, $(1,2)^{\star}$-β-closed) sets of X containing A. It is known that $(1,2)^{\star}$-pcl(A) (resp. $(1,2)^{\star}$-scl(A), $(1,2)^{\star}$-αcl(A), $(1,2)^{\star}$-βcl(A)) is a $(1,2)^{\star}$-preclosed (resp. $(1,2)^{\star}$-semi-closed, $(1,2)^{\star}$-α-closed, $(1,2)^{\star}$-β-closed) set. For any subset A of an arbitrarily chosen bitopological space, the $(1,2)^{\star}$-semi-interior [23] (resp. $(1,2)^{\star}$-α-interior [23], $(1,2)^{\star}$-preinterior [23]) of A, denoted by $(1,2)^{\star}$-sint(A) (resp. $(1,2)^{\star}$-αint(A), $(1,2)^{\star}$-pint(A)), is defined to be the union of all $(1,2)^{\star}$-semi-open (resp. $(1,2)^{\star}$-α-open, $(1,2)^{\star}$-preopen) sets of X contained in A.

Definition 2.5. *Let A be a subset of a bitopological space X. Then A is called*

1. *a $(1,2)^{\star}$-generalized closed (briefly, $(1,2)^{\star}$-g-closed) set [30] if $\tau_{1,2}$-cl(A) $\subseteq U$ whenever $A \subseteq U$ and U is $\tau_{1,2}$-open in X. The complement of $(1,2)^{\star}$-g-closed set is called $(1,2)^{\star}$-g-open set.*
2. *a $(1,2)^{\star}$-semi-generalized closed (briefly, $(1,2)^{\star}$-sg-closed) set [3] if $(1,2)^{\star}$-scl(A) $\subseteq U$ whenever $A \subseteq U$ and U is $(1,2)^{\star}$-semi-open in X. The complement of $(1,2)^{\star}$-sg-closed set is called $(1,2)^{\star}$-sg-open set.*
3. *a $(1,2)^{\star}$-generalized semi-closed (briefly, $(1,2)^{\star}$-gs-closed) set [3] if $(1,2)^{\star}$-scl(A) $\subseteq U$ whenever $A \subseteq U$ and U is $\tau_{1,2}$-open in X. The complement of $(1,2)^{\star}$-gs-closed set is called $(1,2)^{\star}$-gs-open set.*

4. *an $(1,2)^\star$-α-generalized closed (briefly, $(1,2)^\star$-αg-closed) set [12] if $(1,2)^\star$-$\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_{1,2}$-open in X. The complement of $(1,2)^\star$-αg-closed set is called $(1,2)^\star$-αg-open set.*

5. *a $(1,2)^\star$-generalized semi-preclosed (briefly, $(1,2)^\star$-gsp-closed) set [12] if $(1,2)^\star$-$\beta cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_{1,2}$-open in X. The complement of $(1,2)^\star$-gsp-closed set is called $(1,2)^\star$-gsp-open set.*

6. *a $(1,2)^\star$-$g\alpha$-closed set [29] if $(1,2)^\star$-$\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $(1,2)^\star$-α-open in X. The complement of $(1,2)^\star$-$g\alpha$-closed set is called $(1,2)^\star$-$g\alpha$-open set.*

7. *a $(1,2)^\star$-regular generalized closed (briefly, $(1,2)^\star$-r-g-closed) set [26] if $\tau_{1,2}$-$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $(1,2)^\star$-regular open in X. The complement of $(1,2)^\star$-r-g-closed set is called $(1,2)^\star$-r-g-open set.*

8. *a $(1,2)^\star$-generalized preregular closed (briefly, $(1,2)^\star$-gpr-closed) set [31] if $(1,2)^\star$-$pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is $(1,2)^\star$-regular open in X. The complement of $(1,2)^\star$-gpr-closed set is called $(1,2)^\star$-gpr-open set.*

9. *a $(1,2)^\star$-generalized preclosed (briefly, $(1,2)^\star$-gp-closed) set [31] if $(1,2)^\star$-$pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_{1,2}$-open in X. The complement of $(1,2)^\star$-gp-closed set is called $(1,2)^\star$-gp-open set.*

10. *an $(1,2)^\star$-$\alpha^{\star\star}$-generalized closed (briefly, $(1,2)^\star$-$\alpha^{\star\star}g$-closed) set [31] if $(1,2)^\star$-$\alpha cl(A) \subseteq \tau_{1,2}$-$int(\tau_{1,2}$-$cl(U))$ whenever $A \subseteq U$ and U is $\tau_{1,2}$-open in X. The complement of $(1,2)^\star$-$\alpha^{\star\star}g$-closed set is called $(1,2)^\star$-$\alpha^{\star\star}g$-open set.*

11. *an $(1,2)^\star$-$g^{\#}$-closed set [22] if $\tau_{1,2}$-$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $(1,2)^\star$-αg-open in X. The complement of $(1,2)^\star$-$g^{\#}$-closed set is called $(1,2)^\star$-$g^{\#}$-open set.*

12. *an $(1,2)^\star$-$\ddot{g}$-closed set [14] if $\tau_{1,2}$-$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $(1,2)^\star$-sg-open in X. The complement of $(1,2)^\star$-$\ddot{g}$-closed set is called $(1,2)^\star$-$\ddot{g}$-open set.*

Remark 2.6. *The collection of all $(1,2)^\star$-gpr-closed (resp. $(1,2)^\star$-$\ddot{g}$-closed, $(1,2)^\star$-g-closed, $(1,2)^\star$-gs-closed, $(1,2)^\star$-gsp-closed, $(1,2)^\star$-αg-closed, $(1,2)^\star$-sg-closed, $(1,2)^\star$-α-closed, $(1,2)^\star$-semi-closed) sets is denoted by $(1,2)^\star$-$GPRC(X)$ (resp. $(1,2)^\star$-$\ddot{G}C(X)$, $(1,2)^\star$-$GC(X)$, $(1,2)^\star$-$GSC(X)$, $(1,2)^\star$-$GSPC(X)$, $(1,2)^\star$-$\alpha GC(X)$, $(1,2)^\star$-$SGC(X)$, $(1,2)^\star$-$\alpha C(X)$, $(1,2)^\star$-$SC(X)$).*

The collection of all $(1,2)^\star$-gpr-open (resp. $(1,2)^\star$-$\ddot{g}$-open, $(1,2)^\star$-g-open, $(1,2)^\star$-gs-open, $(1,2)^\star$-gsp-open, $(1,2)^\star$-αg-open, $(1,2)^\star$-sg-open, $(1,2)^\star$-α-open, $(1,2)^\star$-semi-open) sets is denoted by $(1,2)^\star$-$GPRO(X)$ (resp. $(1,2)^\star$-$\ddot{G}O(X)$, $(1,2)^\star$-$GO(X)$, $(1,2)^\star$-$GSO(X)$, $(1,2)^\star$-$GSPO(X)$, $(1,2)^\star$-$\alpha GO(X)$, $(1,2)^\star$-$SGO(X)$, $(1,2)^\star$-$\alpha O(X)$, $(1,2)^\star$-$SO(X)$).

We denote the power set of X by $P(X)$.

Definition 2.7. *A bitopological space X is called:*

1. *$(1,2)^\star$-$T_{1/2}$-space [28] if every $(1,2)^\star$-g-closed set in it is $\tau_{1,2}$-closed.*

2. *$(1,2)^\star$-T_b-space [24] if every $(1,2)^\star$-gs-closed set in it is $\tau_{1,2}$-closed.*

3. *$(1,2)^\star$-α-space [23] if every $(1,2)^\star$-α-closed set in it is $\tau_{1,2}$-closed.*

3. $(1,2)^\star$-$g^\star$-closed Sets

We introduce the following definitions.

Definition 3.1. *Let A be a subset of a bitopological space X. Then A is called*

1. *$(1,2)^\star$-$g^\star$-closed set if $\tau_{1,2}$-cl(A) $\subseteq U$ whenever $A \subseteq U$ and U is $(1,2)^\star$-g-open in X. The complement of $(1,2)^\star$-$g^\star$-closed set is called $(1,2)^\star$-$g^\star$-open. The family of all $(1,2)^\star$-$g^\star$-closed sets in X is denoted by $(1,2)^\star$-$G^\star C(X)$.*

2. *$(1,2)^\star$-$g^\star_\alpha$-closed set if $(1,2)^\star$-$\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $(1,2)^\star$-g-open in X. The family of all $(1,2)^\star$-$g^\star_\alpha$-closed sets in X is denoted by $(1,2)^\star$-$G^\star_\alpha C(X)$.*

Proposition 3.2. *Every $\tau_{1,2}$-closed set is $(1,2)^\star$-$g^\star$-closed.*

Proof. If A is any $\tau_{1,2}$-closed set in X and G is any $(1,2)^\star$-g-open set containing A, then G $\supseteq$ A $= \tau_{1,2}$-cl(A). Hence A is $(1,2)^\star$-$g^\star$-closed.

The converse of Proposition 3.2 need not be true as seen from the following example. □

Example 3.3. *Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, X\}$ and $\tau_2 = \{\phi, \{a, c\}, X\}$. Then the sets in $\{\phi, \{a\}, \{a, c\}, X\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, \{b\}, \{b, c\}, X\}$ are called $\tau_{1,2}$-closed. Then $\{a, b\}$ is $(1,2)^\star$-$g^\star$-closed set but not $\tau_{1,2}$-closed.*

Proposition 3.4. *Every $(1,2)^\star$-$g^\star$-closed set is $(1,2)^\star$-$g^\star_\alpha$-closed.*

Proof. If A is a $(1,2)^\star$-$g^\star$-closed subset of X and G is any $(1,2)^\star$-g-open set containing A, then G $\supseteq \tau_{1,2}$-cl(A) $\supseteq (1,2)^\star$-αcl(A). Hence A is $(1,2)^\star$-$g^\star_\alpha$-closed.

The converse of Proposition 3.4 need not be true as seen from the following example. □

Example 3.5. *Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{b\}, X\}$ and $\tau_2 = \{\phi, \{a, b\}, X\}$. Then the sets in $\{\phi, \{b\}, \{a, b\}, X\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi, \{c\}, \{a, c\}, X\}$ are called $\tau_{1,2}$-closed. Then $\{a\}$ is $(1,2)^\star$-$g^\star_\alpha$-closed but not $(1,2)^\star$-$g^\star$-closed in X.*

Remark 3.6. *The following examples show that $(1,2)^\star$-$g^\star$-closed sets are independent of $(1,2)^\star$-α-closed sets.*

Example 3.7. *In Example 3.5, $\{b, c\}$ is $(1,2)^\star$-$g^\star$-closed but not $(1,2)^\star$-α-closed and $\{a\}$ is $(1,2)^\star$-α-closed but not $(1,2)^\star$-$g^\star$-closed in X.*

Remark 3.8. *The following examples show that $(1,2)^\star$-$g^\star$-closed sets are independent of $(1,2)^\star$-semi-closed sets.*

Example 3.9. *In Example 3.5, $\{b, c\}$ is $(1,2)^\star$-$g^\star$-closed but not $(1,2)^\star$-semi-closed and $\{a\}$ is $(1,2)^\star$-semi-closed but not $(1,2)^\star$-$g^\star$-closed in X.*

Remark 3.10. *The following examples show that $(1,2)^\star$-$g^\star$-closed sets are independent of $(1,2)^\star$-pre-closed sets.*

Example 3.11. *In Example 3.5, $\{b, c\}$ is $(1,2)^\star$-$g^\star$-closed but not $(1,2)^\star$-pre-closed and $\{a\}$ is $(1,2)^\star$-pre-closed but not $(1,2)^\star$-$g^\star$-closed in X.*

Remark 3.12. *The following examples show that $(1,2)^\star$-$g^\star$-closed sets are independent of $(1,2)^\star$-β-closed sets.*

Example 3.13. *In Example 3.5, $\{b, c\}$ is $(1,2)^\star$-$g^\star$-closed but not $(1,2)^\star$-β-closed and $\{a\}$ is $(1,2)^\star$-β-closed but not $(1,2)^\star$-$g^\star$-closed in X.*

Remark 3.14. *The following examples show that $(1,2)^\star$-$g^\star$-closed sets are independent of $(1,2)^\star$-$g\alpha$-closed sets.*

Example 3.15. *In Example 3.5, $\{b, c\}$ is $(1,2)^\star$-$g^\star$-closed but not $(1,2)^\star$-$g\alpha$-closed and $\{a\}$ is $(1,2)^\star$-$g\alpha$-closed but not $(1,2)^\star$-$g^\star$-closed in X.*

Remark 3.16. *The following examples show that $(1,2)^\star$-$g^\star$-closed sets are independent of $(1,2)^\star$-sg-closed sets.*

Example 3.17. *In Example 3.5, $\{b,\ c\}$ is $(1,2)^{\star}$-$g^{\star}$-closed but not $(1,2)^{\star}$-sg-closed and $\{a\}$ is $(1,2)^{\star}$-sg-closed but not $(1,2)^{\star}$-$g^{\star}$-closed in X.*

Proposition 3.18. *Every $(1,2)^{\star}$-$g^{\star}$-closed set is $(1,2)^{\star}$-g-closed.*

Proof. If A is a $(1,2)^{\star}$-$g^{\star}$-closed subset of X and G is any $\tau_{1,2}$-open set containing A, since every $\tau_{1,2}$-open set is $(1,2)^{\star}$-g-open, we have G $\supseteq \tau_{1,2}$-cl(A). Hence A is $(1,2)^{\star}$-g-closed in X. □

The converse of Proposition 3.18 need not be true as seen from the following example.

Example 3.19. *Let $X = \{a,\ b,\ c\}$, $\tau_1 = \{\phi,\ X\}$ and $\tau_2 = \{\phi,\ \{c\},\ X\}$. Then the sets in $\{\phi,\ \{c\},\ X\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi,\ \{a,\ b\},\ X\}$ are called $\tau_{1,2}$-closed. Then $\{a\}$ is $(1,2)^{\star}$-g-closed but not $(1,2)^{\star}$-$g^{\star}$-closed set in X.*

Proposition 3.20. *Every $(1,2)^{\star}$-$g^{\star}$-closed set is $(1,2)^{\star}$-gs-closed.*

Proof. If A is a $(1,2)^{\star}$-$g^{\star}$-closed subset of X and G is any $\tau_{1,2}$-open set containing A, since every $\tau_{1,2}$-open set is $(1,2)^{\star}$-g-open, we have G $\supseteq \tau_{1,2}$-cl(A) $\supseteq (1,2)^{\star}$-scl(A). Hence A is $(1,2)^{\star}$-gs-closed in X. □

The converse of Proposition 3.20 need not be true as seen from the following example.

Example 3.21. *In Example 3.19, $\{a\}$ is $(1,2)^{\star}$-gs-closed but not $(1,2)^{\star}$-$g^{\star}$-closed set in X.*

Proposition 3.22. *Every $(1,2)^{\star}$-$g^{\star}$-closed set is $(1,2)^{\star}$-r-g-closed.*

Example 3.23. *In Example 3.5, $\{a\}$ is $(1,2)^{\star}$-r-g-closed but not $(1,2)^{\star}$-$g^{\star}$-closed set in X.*

Proposition 3.24. *Every $(1,2)^{\star}$-$g^{\star}$-closed set is $(1,2)^{\star}$-αg-closed.*

Proof. If A is a $(1,2)^{\star}$-$g^{\star}$-closed subset of X and G is any $\tau_{1,2}$-open set containing A, since every $\tau_{1,2}$-open set is $(1,2)^{\star}$-g-open, we have G $\supseteq \tau_{1,2}$-cl(A) $\supseteq (1,2)^{\star}$-αcl(A). Hence A is $(1,2)^{\star}$-αg-closed in X. □

The converse of Proposition 3.24 need not be true as seen from the following example.

Example 3.25. *Let $X = \{a,\ b,\ c\}$, $\tau_1 = \{\phi,\ \{c\},\ X\}$ and $\tau_2 = \{\phi,\ \{a,\ b\},\ X\}$. Then the sets in $\{\phi,\ \{c\},\ \{a,\ b\},\ X\}$ are called $\tau_{1,2}$-open and $\tau_{1,2}$-closed. Then $\{a,\ c\}$ is $(1,2)^{\star}$-αg-closed but not $(1,2)^{\star}$-$g^{\star}$-closed set in X.*

Proposition 3.26. *Every $(1,2)^{\star}$-$g^{\star}$-closed set is $(1,2)^{\star}$-gsp-closed.*

Proof. If A is a $(1,2)^{\star}$-$g^{\star}$-closed subset of X and G is any $(1,2)^{\star}$-regular open set containing A, since every $(1,2)^{\star}$-regular set is $(1,2)^{\star}$-g-open, we have G $\supseteq \tau_{1,2}$-cl(A) $\supseteq (1,2)^{\star}$-βcl(A). Hence A is $(1,2)^{\star}$-gsp-closed in X. □

The converse of Proposition 3.26 need not be true as seen from the following example.

Example 3.27. *Let $X = \{a,\ b,\ c\}$, $\tau_1 = \{\phi,\ X\}$ and $\tau_2 = \{\phi,\ \{b\},\ X\}$. Then the sets in $\{\phi,\ \{b\},\ X\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi,\ \{a,\ c\},\ X\}$ are called $\tau_{1,2}$-closed. Then $\{c\}$ is $(1,2)^{\star}$-gsp-closed but not $(1,2)^{\star}$-$g^{\star}$-closed set in X.*

Proposition 3.28. *Every $(1,2)^{\star}$-$g^{\star}$-closed set is $(1,2)^{\star}$-gp-closed.*

Proof. If A is a $(1,2)^{\star}$-$g^{\star}$-closed subset of X and G is any $\tau_{1,2}$-open set containing A, since every $\tau_{1,2}$-open set is $(1,2)^{\star}$-g-open, we have G $\supseteq \tau_{1,2}$-cl(A) $\supseteq (1,2)^{\star}$-pcl(A). Hence A is $(1,2)^{\star}$-gp-closed in X. □

The converse of Proposition 3.28 need not be true as seen from the following example.

Example 3.29. *Let* $X = \{a,\ b,\ c\}$, $\tau_1 = \{\phi,\ \{b\},\ X\}$ *and* $\tau_2 = \{\phi,\ \{a,\ b\},\ X\}$. *Then the sets in* $\{\phi,\ \{b\},\ \{a,\ b\},\ X\}$ *are called* $\tau_{1,2}$*-open and the sets in* $\{\phi,\ \{c\},\ \{a,\ c\},\ X\}$ *are called* $\tau_{1,2}$*-closed. Then* $\{a\}$ *is* $(1,2)^{\star}$*-gp-closed but not* $(1,2)^{\star}$*-*$g^{\star}$*-closed in* X.

Proposition 3.30. *Every* $(1,2)^{\star}$*-*$g^{\star}$*-closed set is* $(1,2)^{\star}$*-gpr-closed.*

Proof. If A is a $(1,2)^{\star}$-$g^{\star}$-closed subset of X and G is any $(1,2)^{\star}$-regular open set containing A, since every $(1,2)^{\star}$-regular open set is $(1,2)^{\star}$-g-open, we have G $\supseteq \tau_{1,2}$-cl(A) $\supseteq (1,2)^{\star}$-pcl(A). Hence A is $(1,2)^{\star}$-gpr-closed in X. ☐

The converse of Proposition 3.30 need not be true as seen from the following example.

Example 3.31. *Let* $X = \{a,\ b,\ c\}$, $\tau_1 = \{\phi,\ \{b\},\ X\}$ *and* $\tau_2 = \{\phi,\ \{a,\ b\},\ X\}$. *Then the sets in* $\{\phi,\ \{b\},\ \{a,\ b\},\ X\}$ *are called* $\tau_{1,2}$*-open and the sets in* $\{\phi,\ \{c\},\ \{a,\ c\},\ X\}$ *are called* $\tau_{1,2}$*-closed. Then* $\{a\}$ *is* $(1,2)^{\star}$*-gpr-closed but not* $(1,2)^{\star}$*-*$g^{\star}$*-closed in* X.

Proposition 3.32. *Every* $(1,2)^{\star}$*-*$g^{\star}$*-closed set is* $(1,2)^{\star}$*-*$\alpha^{\star\star}g$*-closed.*

Proof. If A is a $(1,2)^{\star}$-$g^{\star}$-closed subset of X and G is any $\tau_{1,2}$-open set containing A, since every $\tau_{1,2}$-open set is $(1,2)^{\star}$-g-open, we have G $\supseteq \tau_{1,2}$-cl(A) $\supseteq \tau_{1,2}$-int($\tau_{1,2}$-cl(A)). Hence A is $(1,2)^{\star}$-$\alpha^{\star\star}g$-closed in X. ☐

The converse of Proposition 3.32 need not be true as seen from the following example.

Example 3.33. *In Example 3.31,* $\{a\}$ *is* $(1,2)^{\star}$*-*$\alpha^{\star\star}g$*-closed but not* $(1,2)^{\star}$*-*$g^{\star}$*-closed in* X.

Proposition 3.34. *Every* $(1,2)^{\star}$*-*$g^{\#}$*-closed set is* $(1,2)^{\star}$*-*$g^{\star}$*-closed but not conversely.*

Proof. If A is (1,2)-$g^{\#}$-closed subset of X and G is $(1,2)^{\star}$-g-open set containing A, since every $(1,2)^{\star}$-g-open set is $(1,2)^{\star}$-αg-open, we have $G \supseteq \tau_{1,2}$-cl(A). Hence A is $(1,2)^{\star}$-g*-closed in X. ☐

Example 3.35. *Let* $X = \{a,\ b,\ c\}$, $\tau_1 = \{\phi,\ \{a\},\ X\}$ *and* $\tau_2 = \{\phi,\ \{a,\ c\},\ X\}$. *Then* $\{a,\ b\}$ *is* $(1,2)^{\star}$*-*$g^{\star}$*-closed set but not* $(1,2)$*-*$g^{\#}$*-closed.*

Remark 3.36. *From the above discussions and known results in [3, 12, 15, 22, 23, 25], we obtain the following diagrams, where* $A \to B$ *(resp.* $A \not\longleftrightarrow B$*) represents A implies B but not conversely (resp. A and B are independent of each other).*

Diagram - I

$$\begin{array}{ccccccc}
\tau_{1,2}\text{-}closed \longrightarrow & (1,2)^{\star}\text{-}g^{\#}\text{-}closed & \longrightarrow & (1,2)^{\star}\text{-}g^{\star}\text{-}closed & \longrightarrow & (1,2)^{\star}\text{-}g\text{-}closed \\
\searrow & & & \not\updownarrow & & \downarrow \\
 & (1,2)^{\star}\text{-}\alpha\text{-}closed & \longrightarrow & (1,2)^{\star}\text{-}g\alpha\text{-}closed & & (1,2)^{\star}\text{-}\alpha g\text{-}closed \\
 & \downarrow & & & \searrow & \downarrow \\
 & (1,2)^{\star}\text{-}semi\text{-}closed & \longrightarrow & (1,2)^{\star}\text{-}sg\text{-}closed & \longrightarrow & (1,2)^{\star}\text{-}gs\text{-}closed \\
 & \downarrow & & \swarrow & & \\
 & (1,2)^{\star}\text{-}\beta\text{-}closed & \longrightarrow & (1,2)^{\star}\text{-}gsp\text{-}closed & & \\
 & \nwarrow & & \nearrow & & \\
 & & (1,2)^{\star}\text{-}preclosed & & &
\end{array}$$

Diagram - II

$$\begin{array}{ccc}
(1,2)^{\star}\text{-}semi\text{-}closed \not\longleftrightarrow & (1,2)^{\star}\text{-}g^{\star}\text{-}closed \longrightarrow & (1,2)^{\star}\text{-}g\text{-}closed \\
 & \downarrow & \\
 & (1,2)^{\star}\text{-}gs\text{-}closed &
\end{array}$$

4. Properties of $(1,2)^\star$-$g^\star$-closed Sets

Definition 4.1. *The intersection of all $(1,2)^\star$-g-open subsets of X containing A is called the $(1,2)^\star$-g-kernel of A and denoted by $(1,2)^\star$-g-ker(A).*

Lemma 4.2. *A subset A of a bitopological space X is $(1,2)^\star$-$g^\star$-closed if and only if $\tau_{1,2}$-cl(A) $\subseteq$ $(1,2)^\star$-g-ker(A).*

Proof. Suppose that A is $(1,2)^\star$-$g^\star$-closed. Then $\tau_{1,2}$-cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is $(1,2)^\star$-g-open. Let x $\in$ $\tau_{1,2}$-cl(A). If x $\notin$ $(1,2)^\star$-g-ker(A), then there is a $(1,2)^\star$-g-open set U containing A such that x $\notin$ U. Since U is a $(1,2)^\star$-g-open set containing A, we have x $\notin$ $\tau_{1,2}$-cl(A) and this is a contradiction.
Conversely, let $\tau_{1,2}$-cl(A) $\subseteq$ $(1,2)^\star$-g-ker(A). If U is any $(1,2)^\star$-g-open set containing A, then $\tau_{1,2}$-cl(A) $\subseteq$ $(1,2)^\star$-g-ker(A) $\subseteq$ U. Therefore, A is $(1,2)^\star$-$g^\star$-closed. □

Proposition 4.3. *If a set A is $(1,2)^\star$-$g^\star$-closed in X, then $\tau_{1,2}$-cl(A) $-$ A contains no nonempty $(1,2)^\star$-g-closed set in X.*

Proof. Suppose that A is $(1,2)^\star$-$g^\star$-closed. Let F be a $(1,2)^\star$-g-closed subset of $\tau_{1,2}$-cl(A) $-$ A. Then A $\subseteq$ F^c. But A is $(1,2)^\star$-$g^\star$-closed, therefore $\tau_{1,2}$-cl(A) $\subseteq$ F^c. Consequently, F $\subseteq$ $(\tau_{1,2}\text{-cl(A)})^c$. We already have F $\subseteq$ $\tau_{1,2}$-cl(A). Thus F $\subseteq$ $\tau_{1,2}$-cl(A) $\cap$ $(\tau_{1,2}\text{-cl(A)})^c$ and hence F is empty. □

Proposition 4.4. *If A is $(1,2)^\star$-$g^\star$-closed in X and A $\subseteq$ B $\subseteq$ $\tau_{1,2}$-cl(A), then B is $(1,2)^\star$-$g^\star$-closed in X.*

Proof. Let U be $(1,2)^\star$-g-open set in X such that B $\subseteq$ U. Since A is $(1,2)^\star$-$g^\star$-closed, $\tau_{1,2}$-cl(A) $\subseteq$ U. Since $\tau_{1,2}$-cl(B) $\subseteq$ $\tau_{1,2}$-cl(A), we have $\tau_{1,2}$-cl(B) $\subseteq$ U. Hence B is $(1,2)^\star$-$g^\star$-closed set. □

Proposition 4.5. *If A is both $(1,2)^\star$-g-open and $(1,2)^\star$-$g^\star$-closed in X, then A is $\tau_{1,2}$-closed in X.*

Proof. Since A is $(1,2)^\star$-g-open and $(1,2)^\star$-$g^\star$-closed, $\tau_{1,2}$-cl(A) $\subseteq$ A and hence A is $\tau_{1,2}$-closed in X. □

Proposition 4.6. *For each $x \in X$, either $\{x\}$ is $(1,2)^\star$-g-closed or $\{x\}^c$ is $(1,2)^\star$-$g^\star$-closed in X.*

Proof. Suppose that {x} is not $(1,2)^\star$-g-closed in X. Then $\{x\}^c$ is not $(1,2)^\star$-g-open and the only $(1,2)^\star$-g-open set containing $\{x\}^c$ is the space X itself. Therefore $\tau_{1,2}$-cl($\{x\}^c$) $\subseteq$ X and so $\{x\}^c$ is $(1,2)^\star$-$g^\star$-closed in X. □

Theorem 4.7. *Let A be a $(1,2)^\star$-$g^\star$-closed set of a bitopological space X. Then,*

1. *If A is $(1,2)^\star$-regular open, then $(1,2)^\star$-pint(A) and $(1,2)^\star$-scl(A) are also $(1,2)^\star$-$g^\star$-closed sets.*
2. *If A is $(1,2)^\star$-regular closed, then $(1,2)^\star$-pcl(A) is also $(1,2)^\star$-$g^\star$-closed set.*

Proof. (1) Since A is $(1,2)^\star$-regular open in X, A = $\tau_{1,2}$-int($\tau_{1,2}$-cl(A)). Then $(1,2)^\star$-scl(A) = A $\cup$ $\tau_{1,2}$-int($\tau_{1,2}$-cl(A)) = A. Thus, $(1,2)^\star$-scl(A) is $(1,2)^\star$-$g^\star$-closed in X. Since $(1,2)^\star$-pint(A) = A $\cap$ $\tau_{1,2}$-int($\tau_{1,2}$-cl(A)) = A, $(1,2)^\star$-pint(A) is $(1,2)^\star$-$g^\star$-closed.
(2) Since A is $(1,2)^\star$-regular closed in X, A = $\tau_{1,2}$-cl($\tau_{1,2}$-int(A)). Then $(1,2)^\star$-pcl(A) = A $\cup$ $\tau_{1,2}$-cl($\tau_{1,2}$-int(A)) = A. Thus, $(1,2)^\star$-pcl(A) is $(1,2)^\star$-$g^\star$-closed in X. □

The converses of the statements in the Theorem 4.7 are not true as we can see in the following examples.

Example 4.8. *Let X = {a, b, c}, τ_1 = {ϕ, {a}, {a, b}, X} and τ_2 = {ϕ, {b}, {a, c}, X}. Then the sets in {ϕ, {a}, {b}, {a, b}, {a, c}, X} are called $\tau_{1,2}$-open and the sets in {ϕ, {b}, {c}, {a,c}, {b, c}, X} are called $\tau_{1,2}$-closed. Then $(1,2)^\star$-$G^\star C$(X) = {ϕ, {b}, {c}, {a, c}, {b, c}, X}. Then the set A = {c} is not $(1,2)^\star$-regular open. However A is $(1,2)^\star$-$g^\star$-closed and $(1,2)^\star$-scl(A) = {c} is a $(1,2)^\star$-$g^\star$-closed and $(1,2)^\star$-pint(A) = ϕ is also $(1,2)^\star$-$g^\star$-closed.*

Example 4.9. *Let $X = \{a,\ b,\ c\}$, $\tau_1 = \{\phi,\ X\}$ and $\tau_2 = \{\phi,\ \{a,\ b\},\ X\}$. Then the sets in $\{\phi,\ \{a,\ b\},\ X\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi,\ \{c\},\ X\}$ are called $\tau_{1,2}$-closed. Then $(1,2)^{\star}$-$G^{\star}C(X) = \{\phi,\ \{c\},\ \{a,\ c\},\ \{b,\ c\},\ X\}$. Then the set $A = \{c\}$ is not $(1,2)^{\star}$-regular closed. However A is a $(1,2)^{\star}$-$g^{\star}$-closed and $(1,2)^{\star}$-pcl(A) $= \{c\}$ is $(1,2)^{\star}$-$g^{\star}$-closed.*

5. $(1,2)^{\star}$-$g^{\star}$-closure

Definition 5.1. *For every set $A \subseteq X$, we define the $(1,2)^{\star}$-$g^{\star}$-closure of A to be the intersection of all $(1,2)^{\star}$-$g^{\star}$-closed sets containing A. That is $(1,2)^{\star}$-$g^{\star}$-cl(A) $= \cap \{F : A \subseteq F \in (1,2)^{\star}$-$G^{\star}C(X)\}$.*

Lemma 5.2. *For any $A \subseteq X$, $A \subseteq (1,2)^{\star}$-$g^{\star}$-cl($A$) $\subseteq \tau_{1,2}$-cl(A).*

Remark 5.3. *Both containment relations in Lemma 5.2 may be proper as seen from the following example.*

Example 5.4. *Let $X = \{a,\ b,\ c\}$, $\tau_1 = \{\phi,\ X\}$ and $\tau_2 = \{\phi,\ \{a,\ b\},\ X\}$. Then the sets in $\{\phi,\ \{a,\ b\},\ X\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi,\ \{c\},\ X\}$ are called $\tau_{1,2}$-closed. Let $A = \{a\}$. Then $(1,2)^{\star}$-$g^{\star}$-cl(A) $= \{a,\ c\}$ and so $A \subseteq (1,2)^{\star}$-$g^{\star}$-cl($A$) $\subseteq \tau_{1,2}$-cl(A).*

Lemma 5.5. *For any $A \subseteq X$, $(1,2)^{\star}$-$g^{\star}$-cl(A) $\subseteq (1,2)^{\star}$-$\ddot{g}$-cl(A), where $(1,2)^{\star}$-$\ddot{g}$-cl(A) is given by $(1,2)^{\star}$-$\ddot{g}$-cl(A) $= \cap \{F : A \subseteq F \in (1,2)^{\star}$-$\ddot{G}C(X)\}$.*

Remark 5.6. *Containment relation in the above Lemma 5.5 may be proper as shown from the following example.*

Example 5.7. *Let $X = \{a,\ b,\ c,\ d\}$, $\tau_1 = \{\phi,\ \{a\},\ \{a,\ b,\ c\},\ X\}$ and $\tau_2 = \{\phi,\ \{b,\ c\},\ X\}$. Then the sets in $\{\phi,\ \{a\},\ \{b,\ c\},\ \{a,\ b,\ c\},\ X\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi,\ \{d\},\ \{a,\ d\},\ \{b,\ c,\ d\},\ X\}$ are called $\tau_{1,2}$-closed. Then $(1,2)^{\star}$-$\ddot{G}C(X) = \{\phi,\ \{d\},\ \{a,\ d\},\ \{b,\ c,\ d\},\ X\}$ and $(1,2)^{\star}$-$G^{\star}C(X) = \{\phi,\ \{d\},\ \{a,\ d\},\ \{b,\ d\},\ \{c,\ d\},\ \{a,\ b,\ d\},\ \{a,\ c,\ d\},\ \{b,\ c,\ d\},\ X\}$. Let $A = \{b,\ d\}$. Then $(1,2)^{\star}$-$\ddot{g}$-cl(A) $= \{b,\ c,\ d\}$ and $(1,2)^{\star}$-$g^{\star}$-cl(A) $= \{b,\ d\}$. So, $(1,2)^{\star}$-$g^{\star}$-cl(A) $\subset (1,2)^{\star}$-$\ddot{g}$-cl(A).*

The following two Propositions are easy consequences from definitions.

Proposition 5.8. *For any $A \subseteq X$, the following hold:*

1. *$(1,2)^{\star}$-$g^{\star}$-cl(A) is the smallest $(1,2)^{\star}$-$g^{\star}$-closed set containing A.*
2. *A is $(1,2)^{\star}$-$g^{\star}$-closed if and only if $(1,2)^{\star}$-$g^{\star}$-cl(A) $= A$.*

Proposition 5.9. *For any two subsets A and B of X, the following hold:*

1. *If $A \subseteq B$, then $(1,2)^{\star}$-$g^{\star}$-cl(A) $\subseteq (1,2)^{\star}$-$g^{\star}$-cl(B).*
2. *$(1,2)^{\star}$-$g^{\star}$-cl($A \cap B$) $\subseteq (1,2)^{\star}$-$g^{\star}$-cl(A) $\cap (1,2)^{\star}$-$g^{\star}$-cl(B).*

6. $(1,2)^{\star}$-$g^{\star}$-open Sets

Definition 6.1. *Let A be a subset of a bitopological space X. Then A is called $(1,2)^{\star}$-$g^{\star}$-open in X if A^c is $(1,2)^{\star}$-$g^{\star}$-closed in X.*

The collection of all $(1,2)^{\star}$-$g^{\star}$-open sets in X is denoted by $(1,2)^{\star}$-G*O(X).

Proposition 6.2. *For any bitopological space X, the following assertions hold:*

1. *Every $\tau_{1,2}$-open set is $(1,2)^{\star}$-$g^{\star}$-open but not conversely.*

2. *Every* $(1,2)^\star$-$g^\star$*-open set is* $(1,2)^\star$-$g^\star_\alpha$*-open but not conversely.*

3. *Every* $(1,2)^\star$-$g^\star$*-open set is* $(1,2)^\star$-g*-open but not conversely.*

4. *Every* $(1,2)^\star$-$g^\star$*-open set is* $(1,2)^\star$-sg*-open but not conversely.*

5. *Every* $(1,2)^\star$-$g^\star$*-open set is* $(1,2)^\star$-αg*-open but not conversely.*

6. *Every* $(1,2)^\star$-$g^\star$*-open set is* $(1,2)^\star$-gs*-open but not conversely.*

7. *Every* $(1,2)^\star$-$g^\star$*-open set is* $(1,2)^\star$-gsp*-open but not conversely.*

Theorem 6.3. *A subset A of X is* $(1,2)^\star$-$g^\star$*-open if and only if* $F \subseteq \tau_{1,2}$*-int(A) whenever F is* $(1,2)^\star$-g*-closed and* $F \subseteq A$.

Proof. Suppose that F $\subseteq \tau_{1,2}$-int(A) such that F is $(1,2)^\star$-g-closed and F $\subseteq$ A. Let A$^c \subseteq$ U where U is $(1,2)^\star$-g-open. Then U$^c \subseteq$ A and U^c is $(1,2)^\star$-g-closed. Therefore U$^c \subseteq \tau_{1,2}$-int(A) by hypothesis. Since U$^c \subseteq \tau_{1,2}$-int(A), we have $(\tau_{1,2}$-int(A))^c $\subseteq$ U. i.e., $\tau_{1,2}$-cl(A^c) $\subseteq$ U, since $\tau_{1,2}$-cl(A^c) = $(\tau_{1,2}$-int(A))^c. Thus A^c is $(1,2)^\star$-$g^\star$-closed. i.e., A is $(1,2)^\star$-$g^\star$-open.
Conversely, suppose that A is $(1,2)^\star$-$g^\star$-open such that F $\subseteq$ A and F is $(1,2)^\star$-g-closed. Then F^c is $(1,2)^\star$-g-open and A^c $\subseteq$ F^c. Therefore, $\tau_{1,2}$-cl(A^c) $\subseteq$ F^c by definition of $(1,2)^\star$-$g^\star$-closedness and so F $\subseteq \tau_{1,2}$-int(A), since $\tau_{1,2}$-cl(A^c) = $(\tau_{1,2}$-int(A))^c. □

We introduce the following definition.

Definition 6.4. *For any* $A \subseteq X$, $(1,2)^\star$-$g^\star$*-int(A) is defined as the union of all* $(1,2)^\star$-$g^\star$*-open sets contained in A. i.e.,* $(1,2)^\star$-$g^\star$*-int(A)* $= \cup\, \{G : G \subseteq A$ *and G is* $(1,2)^\star$-$g^\star$*-open*$\}$.

Lemma 6.5. *For any* $A \subseteq X$, $\tau_{1,2}$*-int(A)* $\subseteq (1,2)^\star$-$g^\star$*-int(A)* $\subseteq A$.

The following two Propositions are easy consequences from definitions.

Proposition 6.6. *For any* $A \subseteq X$*, the following hold:*

1. $(1,2)^\star$-$g^\star$*-int(A) is the largest* $(1,2)^\star$-$g^\star$*-open set contained in A.*

2. *A is* $(1,2)^\star$-$g^\star$*-open if and only if* $(1,2)^\star$-$g^\star$*-int(A)* $= A$.

Proposition 6.7. *For any subsets A and B of X, the following hold:*

1. $(1,2)^\star$-$g^\star$*-int(*$A \cap B$*)* $\subseteq (1,2)^\star$-$g^\star$*-int(A)* $\cap\, (1,2)^\star$-$g^\star$*-int(B).*

2. $(1,2)^\star$-$g^\star$*-int(*$A \cup B$*)* $\supseteq (1,2)^\star$-$g^\star$*-int(A)* $\cup\, (1,2)^\star$-$g^\star$*-int(B).*

3. *If* $A \subseteq B$*, then* $(1,2)^\star$-$g^\star$*-int(A)* $\subseteq (1,2)^\star$-$g^\star$*-int(B).*

4. $(1,2)^\star$-$g^\star$*-int(X)* $= X$ *and* $(1,2)^\star$-$g^\star$*-int(*ϕ*)* $= \phi$.

7. Applications

As applications of $(1,2)^\star$-$g^\star$-closed sets, we introduce the notions called $(1,2)^\star$-T$^\star_g$-spaces and $(1,2)^\star$-${}_g$T$^\star_g$-spaces and obtain their properties and characterizations.

Definition 7.1. *A space X is called a* $(1,2)^\star$-$T^\star_g$*-space if every* $(1,2)^\star$-$g^\star$*-closed set in it is* $\tau_{1,2}$*-closed.*

Example 7.2. *Let $X = \{a,\ b,\ c\}$, $\tau_1 = \{\phi,\ X\}$ and $\tau_2 = \{\phi,\ \{b\},\ X\}$. Then the sets in $\{\phi,\ \{b\},\ X\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi,\ \{a,\ c\},\ X\}$ are called $\tau_{1,2}$-closed. Then $(1,2)^{\star}$-$G^{\star}C(X) = \{\phi,\ \{a,\ c\},\ X\}$. Thus X is a $(1,2)^{\star}$-$T_g^{\star}$-space.*

Example 7.3. *Let $X = \{a,\ b,\ c\}$, $\tau_1 = \{\phi,\ X\}$ and $\tau_2 = \{\phi,\ \{a,\ c\},\ X\}$. Then the sets in $\{\phi,\ \{a,\ c\},\ X\}$ are called $\tau_{1,2}$-open and the sets in $\{\phi,\ \{b\},\ X\}$ are called $\tau_{1,2}$-closed. Then $(1,2)^{\star}$-$G^{\star}C(X) = \{\phi,\ \{b\},\ \{a,\ b\},\ \{b,\ c\},\ X\}$. Thus X is not a $(1,2)^{\star}$-$T_g^{\star}$-space.*

Proposition 7.4. *Every $(1,2)^{\star}$-$T_{1/2}$-space is $(1,2)^{\star}$-$T_g^{\star}$-space but not conversely.*

Proof. Follows from Proposition 3.18. □

The converse of Proposition 7.4 need not be true as seen from the following example.

Example 7.5. *Let X, τ_1 and τ_2 be as in the Example 7.2. Then we have $(1,2)^{\star}$-$GC(X) = \{\phi,\ \{a\},\ \{c\},\ \{a,\ b\},\ \{a,\ c\},\ \{b,\ c\},\ X\}$. Thus X is not a $(1,2)^{\star}$-$T_{1/2}$-space.*

Proposition 7.6. *Every $(1,2)^{\star}$-T_b-space is $(1,2)^{\star}$-$T_g^{\star}$-space but not conversely.*

Proof. Follows from Proposition 3.20. □

The converse of Proposition 7.6 need not be true as seen from the following example.

Example 7.7. *Let X, τ_1 and τ_2 be as in the Example 7.2. Then we have $(1,2)^{\star}$-$GSC(X) = \{\phi,\ \{a\},\ \{c\},\ \{a,\ b\},\ \{a,\ c\},\ \{b,\ c\},\ X\}$. Thus X is not a $(1,2)^{\star}$-T_b-space.*

Remark 7.8. *We conclude from the next two examples that $(1,2)^{\star}$-$T_g^{\star}$-spaces and $(1,2)^{\star}$-α-spaces are independent.*

Example 7.9. *Let X, τ_1 and τ_2 be as in the Example 7.2. Then we have $(1,2)^{\star}$-$\alpha C(X) = \{\phi,\ \{a\},\ \{c\},\ \{a,\ c\},\ X\}$. Thus X is a $(1,2)^{\star}$-$T_g^{\star}$-space but not an $(1,2)^{\star}$-α-space.*

Example 7.10. *Let X, τ_1 and τ_2 be as in the Example 7.3. Then we have $(1,2)^{\star}$-$\alpha C(X) = \{\phi,\ \{b\},\ X\}$. Thus X is an $(1,2)^{\star}$-α-space but not a $(1,2)^{\star}$-$T_g^{\star}$-space.*

Theorem 7.11. *For a bitopological space X, the following properties are equivalent:*

1. *X is a $(1,2)^{\star}$-$T_g^{\star}$-space.*
2. *Every singleton subset of X is either $(1,2)^{\star}$-g-closed or $\tau_{1,2}$-open.*

Proof. (1) $\Rightarrow$ (2). Assume that for some x $\in$ X, the set {x} is not a $(1,2)^{\star}$-g-closed in X. Then the only $(1,2)^{\star}$-g-open set containing $\{x\}^c$ is X and so $\{x\}^c$ is $(1,2)^{\star}$-$g^{\star}$-closed in X. By assumption $\{x\}^c$ is $\tau_{1,2}$-closed in X or equivalently {x} is $\tau_{1,2}$-open.

(2) $\Rightarrow$ (1). Let A be a $(1,2)^{\star}$-$g^{\star}$-closed subset of X and let x $\in$ $\tau_{1,2}$-cl(A). By assumption {x} is either $(1,2)^{\star}$-g-closed or $\tau_{1,2}$-open.

Case (a): Suppose that {x} is $(1,2)^{\star}$-g-closed. If x $\notin$ A, then $\tau_{1,2}$-cl(A) $-$ A contains a nonempty $(1,2)^{\star}$-g-closed set {x}, which is a contradiction.

Case (b): Suppose that {x} is $\tau_{1,2}$-open. Since x $\in$ $\tau_{1,2}$-cl(A), {x} $\cap$ A $\neq$ ϕ and so x $\in$ A. Thus in both case, x $\in$ A and therefore $\tau_{1,2}$-cl(A) $\subseteq$ A or equivalently A is a $\tau_{1,2}$-closed set of X. □

Definition 7.12. *The space X is called a $(1,2)^{\star}$-${}_gT_g^{\star}$-space if every $(1,2)^{\star}$-g-closed set in it is $(1,2)^{\star}$-$g^{\star}$-closed.*

Example 7.13. *Let X, τ_1 and τ_2 be as in the Example 7.3. Then X is a $(1,2)^{\star}$-${}_gT_g^{\star}$-space and the space X in the Example 7.2 is not a $(1,2)^{\star}$-${}_gT_g^{\star}$-space.*

Proposition 7.14. *Every $(1,2)^{\star}$-$T_{1/2}$-space is $(1,2)^{\star}$-${}_gT_g^{\star}$-space but not conversely.*

The converse of Proposition 7.14 need not be true as seen from the following example.

Example 7.15. *Let X, τ_1 and τ_2 be as in the Example 7.3. Then X is a $(1,2)^{\star}$-${}_gT_g^{\star}$-space but not a $(1,2)^{\star}$-$T_{1/2}$-space.*

Remark 7.16. *$(1,2)^{\star}$-$T_g^{\star}$-spaces and $(1,2)^{\star}$-${}_gT_g^{\star}$-spaces are independent.*

Example 7.17. *The space X in the Example 7.3 is a $(1,2)^{\star}$-${}_gT_g^{\star}$-space but not a $(1,2)^{\star}$-$T_g^{\star}$-space and the space X in the Example 7.2 is a $(1,2)^{\star}$-$T_g^{\star}$-space but not a $(1,2)^{\star}$-${}_gT_g^{\star}$-space.*

Theorem 7.18. *A space X is $(1,2)^{\star}$-$T_{1/2}$ if and only if it is both $(1,2)^{\star}$-$T_g^{\star}$ and $(1,2)^{\star}$-${}_gT_g^{\star}$.*

Proof. Necessity. Follows from Propositions 7.4 and 7.14.

Sufficiency. Assume that X is both $(1,2)^{\star}$-$T_g^{\star}$ and $(1,2)^{\star}$-${}_gT_g^{\star}$. Let A be a $(1,2)^{\star}$-g-closed set of X. Then A is $(1,2)^{\star}$-$g^{\star}$-closed, since X is a $(1,2)^{\star}$-${}_gT_g^{\star}$. Again since X is a $(1,2)^{\star}$-$T_g^{\star}$, A is a $\tau_{1,2}$-closed set in X and so X is a $(1,2)^{\star}$-$T_{1/2}$. □

References

[1] M.E.Abd El-Monsef, S.N.El-Deeb and R.A.Mahmoud, *β-open sets and β-continuous mappings*, Bull. Fac. Sci. Assiut Univ., 12(1983), 77-90.

[2] D.Andrijevic, *Semi-preopen sets*, Mat. Vesnik, 38(1986), 24-32.

[3] J.Antony Rex Rodrigo, O.Ravi, A.Pandi and C.M.Santhana, *On $(1,2)^{\star}$-s-normal spaces and pre-$(1,2)^{\star}$-gs-closed functions*, International Journal of Algorithms, Computing and Mathematics, 4(1)(2011), 29-42.

[4] F.G.Arenas, J.Dontchev and M.Ganster, *On λ-sets and dual of generalized continuity*, Questions Answers Gen. Topology, 15 (1997), 3-13.

[5] S.P.Arya and T.M.Nour, *Characterizations of s-normal spaces*, Indian J. Pure. Appl. Math., 21(8)(1990), 717-719.

[6] P.Bhattacharyya and B.K.Lahiri, *Semi-generalized closed sets in topology*, Indian J. Math., 29(3)(1987), 375-382.

[7] D.E.Cameron, *Topology atlas*, http://gozips. uakron. deu/.

[8] J.Cao, M.Ganster and I.Reilly, *On sg-closed sets and $g\alpha$-closed sets*, Mem. Fac. Sci. Kochi Univ. Ser. A. Math., 20(1999), 1-5.

[9] J.Cao, M.Ganster and I.Reilly, *Submaximality, extremal disconnectedness and generalized closed sets*, Houston J. Math., 24(1998), 681-688.

[10] R.Devi, K.Bhuvaneswari and H.Maki, *Weak forms of $g\rho$-closed sets, where $\rho \in \{\alpha, \alpha^{\star}, \alpha^{\star\star}\}$ and digital plane*, Mem. Fac. Sci. Kochi Univ. Ser. A. Math., 25(2004), 37-54.

[11] W.Dunham, *$T_{1/2}$-spaces*, Kyungpook Math. J., 17(1977), 161-169.

[12] Z.Duszynski, M.Jeyaraman, M.S.Joseph, O.Ravi and M.L.Thivagar, *A new generalization of closed sets in bitopology*, South Asian Journal of Mathematics, 4(5)(2014), 215-224.

[13] J.C.Kelly, *Bitopological spaces*, Proc. London Math. Soc., 13(1963), 71-89.

[14] K.Kumaresan, *Another generalization of closed sets in bitopological spaces*, Ph. D Thesis, Alagappa University, Karaikudi, August (2012).

[15] M.Lellis Thivagar, O.Ravi and M.E.Abd El-Monsef, *Remarks on bitopological $(1,2)^{\star}$-quotient mappings*, J. Egypt Math. Soc., 16(1)(2008), 17-25.

[16] N.Levine, *Generalized closed sets in topology*, Rend. Circ. Math. Palermo, 19(2)(1970), 89-96.

[17] N.Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly, 70(1963), 36-41.

[18] N.Levine, *Some remarks on the closure operator in topological spaces*, Amer. Math. Monthly, 70(5)(1963), 553.

[19] A.S.Mashhour, M.E.Abd El-Monsef and S.N.El-Deeb, *On precontinuous and weak pre continuous mappings*, Proc. Math. and Phys. Soc. Egypt, 53(1982), 47-53.

[20] O.Njastad, *On some classes of nearly open sets*, Pacific J. Math., 15(1965), 961-970.

[21] M.Rajamani and K.Viswanathan, *On αgs-closed sets in topological spaces*, Acta Ciencia Indica, XXXM(3)(2004), 21-25.

[22] C.Rajan, *Further study of new bitopological generalized continuous functions*, Ph. D Thesis, Madurai Kamaraj University, Madurai, (2014).

[23] O.Ravi, M.L.Thivagar and E.Hatir, *Decomposition of $(1,2)^{\star}$-continuity and $(1,2)^{\star}$-α-continuity*, Miskolc Mathematical Notes., 10(2)(2009), 163-171.

[24] O.Ravi and M.L.Thivagar, *Remarks on λ-irresolute functions via $(1,2)^{\star}$-sets*, Advances in App. Math. Analysis, 5(1) (2010), 1-15.

[25] O.Ravi, E.Ekici and M.Lellis Thivagar, *On $(1,2)^{\star}$-sets and decompositions of bitopological $(1,2)^{\star}$-continuous mappings*, Kochi J. Math., 3(2008), 181-189.

[26] O.Ravi, K.Kayathri, M.L.Thivagar and M.Joseph Israel, *Mildly $(1,2)^{\star}$-normal spaces and some bitopological functions*, Mathematica Bohemica, 135(1)(2010), 1-15.

[27] O.Ravi, A.Pandi and R.Latha, *Contra-pre-$(1,2)^{\star}$-semi-continuous functions*, Bessel Journal of Mathematics (To appear).

[28] O.Ravi, S.Pious Missier and T.Salai Parkunan, *On bitopological $(1,2)^{\star}$-generalized homeomorphisms*, Int J. Contemp. Math. Sciences., 5(11)(2010), 543-557.

[29] O.Ravi, M.L.Thivagar and E.Ekici, *Decomposition of $(1,2)^{\star}$-continuity and complete $(1,2)^{\star}$-continuity in bitopological spaces*, Analele Universitatii Din Oradea. Fasc. Matematica Tom XV (2008), 29-37.

[30] O.Ravi, M.L.Thivagar and Jinjinli, *Remarks on extensions of $(1,2)^{\star}$-g-closed maps*, Archimedes J. Math., 1(2)(2011), 177-187.

[31] O.Ravi, A.Pandi, S.Pious Missier and T.Salai Parkunan, *Remarks on bitopological $(1,2)^{\star}$-$r\omega$-Homeomorphisms*, International Journal of Mathematical Archive, 2(4)(2011), 465-475.

[32] M.Sheik John, *A study on generalizations of closed sets and continuous maps in topological and bitopological spaces*, Ph.D Thesis, Bharathiar University, Coimbatore, September, (2002).

[33] M.K.R.S.Veera Kumar, *$g^{\#}$-closed sets in topological spaces*, Mem. Fac. Sci. Kochi Univ. (Math.)., 24(2003), 1-13.

[34] M.K.R.S.Veera Kumar, *Between closed sets and g-closed sets*, Mem. Fac. Sci. Kochi Univ. Ser. A. Math., 21(2000), 1-19.