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Irreducible Elements in $(\mathcal{Z}^+, \leq_C)$

Research Article

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Abstract: A convolution is a mapping $\mathcal{C}$ of the set Z^+ of positive integers into the set $\mathcal{P}(Z^+)$ of all subsets of Z^+ such that, for any $n \in Z^+$, each member of $C(n)$ is a divisor of n. If $D(n)$ is the set of all divisors of n, for any n, then D is called the Dirichlet's convolution. If $U(n)$ is the set of all Unitary(square free) divisors of n , for any n, then U is called unitary(square free) convolution. Corresponding to any general convolution C, we can define a binary relation $\leq_C$ on Z^+ by ' $m \leq_C n$ if and only if $m \in C(n)$ '. In this paper, we present irreducible elements in $(\mathcal{Z}^+, \leq_\mathcal{C})$, where $\leq_\mathcal{C}$ is the binary relation induced by the convolution $\mathcal{C}$.

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1. Introduction

A Convolution is a mapping $\mathcal{C}$ of the set $\mathcal{Z}^+$of positive integers into the set $\mathcal{P}(\mathcal{Z}^+)$ of subsets of $\mathcal{Z}^+$ such that, for any $n \in \mathcal{Z}^+$, $\mathcal{C}(n)$ is a nonempty set of divisors of n. If $\mathcal{C}(n)$ is the set of all divisors of n, for each $n \in \mathcal{Z}^+$, then $\mathcal{C}$ is the classical Dirichlet convolution [2]. If $\mathcal{C}(n)= \{d \ / \ d|n$ and $(d, \frac{n}{d}) = 1\}$, then $\mathcal{C}$ is the Unitary convolution[1]. As another example if $\mathcal{C}(n)=\{d \ / \ d|n$ and m^k doesnot divide d for any $m \in \mathcal{Z}^+\}$ then $\mathcal{C}$ is the k-free convolution. Corresponding to any convolution $\mathcal{C}$, we can define a binary relation $\leq_\mathcal{C}$ in a natural way by

$$m \leq_\mathcal{C} n \text{ if and only if } m \in \mathcal{C}(n).$$

$\leq_\mathcal{C}$ is a partial order on $\mathcal{Z}^+$ and is called partial order induced by the convolution $\mathcal{C}$ [5, 6]. In this paper, we present a characterization of join-irreducible elements and non existence of meet-irreducible elements in $(\mathcal{Z}^+, \leq_C)$, where $\mathcal{C}$ is a multiplicative convolution .

2. Preliminaries

Let us recall that a partial order on a non-empty set X is defined as a binary relation $\leq$ on X which is reflexive ($a \leq a$), transitive ($a \leq b, b \leq c \Longrightarrow a \leq c$) and antisymmetric ($a \leq b, b \leq a \Longrightarrow a = b$) and that a pair $(X, \leq)$ is called a partially ordered set(poset) if X is a non-empty set and $\leq$ is a partial order on X. For any $A \subseteq X$ and $x \in X$, x is called a lower(upper) bound of A if $x \leq a$(respectively $a \leq x$) for all $a \in A$. We have the usual notations of the greatest lower bound(glb) and least upper bound(lub) of A in X. If A is a finite subset $\{a_1, a_2, \cdots, a_n\}$, the glb of A(lub of A) is denoted

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by $a_1 \wedge a_2 \wedge \cdots \wedge a_n$ or $\bigwedge_{i=1}^{n} a_i$ (respectively by $a_1 \vee a_2 \vee \cdots \vee a_n$ or $\bigvee_{i=1}^{n} a_i$). A partially ordered set $(X, \leq)$ is called a meet semi lattice if $a \wedge b$ (=glb$\{a, b\}$) exists for all a and $b \in X$. $(X, \leq)$ is called a join semi lattice if $a \vee b$ (=lub$\{a, b\}$) exists for all a and $b \in X$. A poset $(X, \leq)$ is called a lattice if it is both a meet and join semi lattice. Equivalently, lattice can also be defined as an algebraic system $(X, \wedge, \vee)$, where $\wedge$ and $\vee$ are binary operations which are associative, commutative and idempotent and satisfying the absorption laws, namely $a \wedge (a \vee b) = a = a \vee (a \wedge b)$ for all $a, b \in X$; in this case the partial order $\leq$ on X is such that $a \wedge b$ and $a \vee b$ are respectively the glb and lub of $\{a, b\}$. The algebraic operations $\wedge$ and $\vee$ and the partial order $\leq$ are related by

$$a = a \wedge b \iff a \leq b \iff a \vee b = b.$$

Throughout the paper, $\mathcal{Z}^+$ and $\mathcal{N}$ denote the set of positive integers and the set of non-negative integers respectively.

Definition 2.1. *A mapping $\mathcal{C} : \mathcal{Z}^+ \longrightarrow \mathcal{P}(\mathcal{Z}^+)$ is called a convolution if the following are satisfied for any $n \in \mathcal{Z}^+$.*

(1). $\mathcal{C}(n)$ is a set of positive divisors of n

(2). $n \in \mathcal{C}(n)$

(3). $\mathcal{C}(n) = \bigcup_{m \in \mathcal{C}(n)} \mathcal{C}(m)$.

Definition 2.2. *For any convolution $\mathcal{C}$ and m and $n \in \mathcal{Z}^+$, we define*

$$m \leq n \text{ if and only if } m \in \mathcal{C}(n)$$

Then $\leq_{\mathcal{C}}$ is a partial order on $\mathcal{Z}^+$ and is called the partial order induced by $\mathcal{C}$ on $\mathcal{Z}^+$. In fact, for any mapping $\mathcal{C} : \mathcal{Z}^+ \longrightarrow \mathcal{P}(\mathcal{Z}^+)$ such that each member of $\mathcal{C}(n)$ is a divisor of n, $\leq_{\mathcal{C}}$ is a partial order on $\mathcal{Z}^+$ if and only if $\mathcal{C}$ is a convolution [6], as defined above.

Definition 2.3. *Let $(S, \wedge)$ be a meet semi lattice with smallest element 0 and let $0 \neq x \in S$. x is said to be join-irreducible if x is not the supremum of any two elements which are strictly less than x.*
In other words, x is said to be join irreducible if $x \neq 0$ and y and $z \in S$ and $x = y \vee z \Longrightarrow x = y$ or $x = z$

Definition 2.4. *A convolution $\mathcal{C}$ is said to be multiplicative if, for any relatively prime integers m and n,*

$$\mathcal{C}(mn) = \mathcal{C}(m)\mathcal{C}(n) := \{ab \mid a \in \mathcal{C}(m) \text{ and } b \in \mathcal{C}(n)\}.$$

It can be verified that a convolution $\mathcal{C}$ is multiplicative if and only if, for any distinct primes $p_1, p_2, \cdots, p_r$ and non-negative integers $a_1, a_2, \cdots, a_r$,

$$\mathcal{C}(\prod_{i=1}^{r} p_i^{a_i}) = \prod_{i=1}^{r} \mathcal{C}(p_i^{a_i}) := \{m_1 m_2 \cdots m_r \mid m_i \in \mathcal{C}(p_i^{a_i})\}[4].$$

Theorem 2.5. *Let $\mathcal{C}$ be a multiplicative convolution. If m and n are relatively prime positive integers, then $m \vee n$ exists in $(\mathcal{Z}^+, \leq_{\mathcal{C}})$ and is equal to the product mn [3, 4].*

Theorem 2.6. *Let $\mathcal{C}$ be multiplicative convolution such that $(\mathcal{Z}^+, \leq_C)$ is meet semi lattice and $x \in \mathcal{Z}^+$. Then x is join-irreducible in $(\mathcal{Z}^+, \leq_C)$ if and only if $x = p^a$ for some prime number p and a join-irreducible element in $(\mathcal{N}, \leq_C^p)$.*

Proof. We can assume that $x > 1$. Suppose that x is join-irreducible in $(\mathcal{Z}^+, \leq_C)$. We can choose a prime p dividing x. Then we can write

$$x = p^a.y, \text{ where } a \text{ and } y \in \mathcal{Z}^+ \text{ and } (p, y) = 1.$$

Then $(p^a, y) = 1$ and hence $p^a \vee y$ exists and is equal to $p^a.y = x$. Since x is join-irreducible and $y \neq x$ (since $p^a \neq 1$) it follows that $x = p^a$. Also, if b and $c \in \mathcal{N}$ such that $a = b \vee c$ in $(\mathcal{N}, \leq_C^p)$, then $x = p^a = p^b \vee p^c$ and hence $x = p^b$ or $x = p^c$ so that $a = b$ or $a = c$. Thus a is join-irreducible in $(\mathcal{N}, \leq_C^p)$, and $x = p^a$.

Conversely suppose that $x = p^a$, where p is a prime number and a is join-irreducible element in $(\mathcal{N}, \leq_C^p)$. Let y and $z \in \mathcal{Z}^+$ such that $x = y \vee z$. Then $y \leq_C x$ and $z \leq_C x$ and, in particular y and z are divisors of $x = p^a$. Therefore $y = p^b$ and $z = p^c$ for some b and $c \in \mathcal{N}$. Since $y \vee z$ exists in $(\mathcal{Z}^+, \leq_C)$, $b \vee c$ exists in $(\mathcal{N}, \leq_C^p)$ and $b \vee c = a$. Since a is join-irreducible, $a = b$ or $a = c$ so that $x = y$ or $x = z$. Thus x is join-irreducible. □

Example 2.7. *With respect to the Dirichlet convolution D , join-irreducible elements in $(\mathcal{Z}^+, \leq_D)$ are precisely of the form p^a where p is a prime and a is a positive integer, since the partial order $\leq_D^p$ on $\mathcal{N}$ is the usual order which is a total order and hence every non-zero element is join-irreducible.*

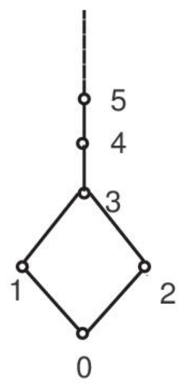
Example 2.8. *For any prime p and $a \in \mathcal{N}$, define*

$$\mathcal{C}(p^a) = \begin{cases} \{1, p^a\} & if \;\; a < 3 \\ \{1, p, p^2, \cdots, p^a\} & if \;\; a \geq 3 \end{cases}$$

and extend $\mathcal{C}$ multiplicative; that is,

$$\mathcal{C}(\prod_{i=1}^{r} p_i^{a_i}) \;\; = \;\; \prod_{i=1}^{r} \mathcal{C}(p_i^{a_i})$$

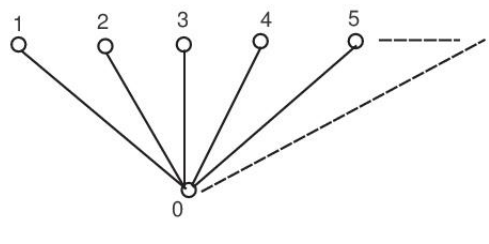
for any distinct primes $p_1, p_2, \cdots, p_r$ and non negative integers $a_1, a_2, \cdots, a_r$. Then the join-irreducible elements in $(\mathcal{Z}^+, \leq_C)$ are precisely of the form p^a, where p is prime and $3 \neq a \in \mathcal{Z}^+$. For, consider the Hasse diagram for $(\mathcal{N}, \leq_C^p)$ for any $p \in \mathcal{P}$.



Example 2.9. *Consider the Unitary convolution $\mathcal{U}$ defined by*

$$\mathcal{U}(n) = \{d \in \mathcal{D}(n) \mid (d, \tfrac{n}{d}) = 1\}$$

Here, the join-irreducible elements in $(\mathcal{Z}^+, \leq_\mathcal{U})$ are precisely those in $(\mathcal{Z}^+, \leq_\mathcal{D})$. The Hasse diagram for $(\mathcal{N}, \leq_\mathcal{U}^p)$ is given below for any $p \in \mathcal{P}$.



In fact, any element a having the unique immediate predecessor is join-irreducible in $(\mathcal{N}, \leq_C^p)$ and hence p^a is join-irreducible in $(\mathcal{Z}^+, \leq_C)$, for any multiplicative convolution $\mathcal{C}$. We can consider the concept dual to that of a join-irreducible element, namely the meet irreducibility. However, for any multiplicative convolution $\mathcal{C}$, $(\mathcal{Z}^+, \leq_C)$ have no meet-irreducible elements. Let us have the formal definition of meet irreducibility.

Definition 2.10. *An element x of a meet semi lattice $(S, \wedge)$ is said to be meet-irreducible if x is not the greatest element and, for any y and $z \in S$, $x = y \wedge z \Longrightarrow x = y$ or $x = z$.*

Theorem 2.11. *Let $\mathcal{C}$ be any multiplicative convolution such that $(\mathcal{Z}^+, \leq_C)$ is a meet semi lattice. Then there are no meet-irreducible elements in $(\mathcal{Z}^+, \leq_C)$.*

Proof. let n be an arbitrary element in $\mathcal{Z}^+$. Choose prime numbers p and q not dividing n and put

$$y = pn \text{ and } z = qn$$

Then $n = 1.n \in \mathcal{C}(p)\mathcal{C}(n) = \mathcal{C}(pn) = \mathcal{C}(y)$ and hence $n \leq_C y$ and, similarly $n \leq_C z$. If $m \leq_C y$ and $m \leq_C z$, then $m \in \mathcal{C}(y) \cap \mathcal{C}(z) = (\mathcal{C}(p)\mathcal{C}(n)) \cap (\mathcal{C}(q)\mathcal{C}(n)) = \mathcal{C}(n)$
and hence $m \leq_C n$. Therefore $n = y \wedge z$; but $n \neq y$ and $n \neq z$. Thus n is not meet-irreducible. Recall that an element x is called prime if, for any elements y and z,

$$y \wedge z \leq x \Longrightarrow y \leq x \text{ or } z \leq x.$$

Clearly any prime element is meet-irreducible and the converse is not true. For this reason, the prime elements are also called strongly meet-irreducible. □

The following is an immediate consequence of the above theorem.

Corollary 2.12. *If $\mathcal{C}$ is a multiplicative convolution such that $(\mathcal{Z}^+, \leq_C)$ is a meet semi lattice, then there are no prime elements in $(\mathcal{Z}^+, \leq_C)$.*

Unlike the meet-irreducible elements, $(\mathcal{Z}^+, \leq_C)$ may possess join-irreducible elements, for any multiplicative convolution $\mathcal{C}$. For example, with respect to the Dirichlet's convolution $\mathcal{D}$, every non-zero a is join- irreducible in $(\mathcal{N}, \leq_{\mathcal{D}}^p)$ for every prime p and hence p^a is join-irreducible in $(\mathcal{Z}^+, \leq_{\mathcal{D}})$ for all $p \in \mathcal{P}$ and $a \in \mathcal{Z}^+$. However, $(\mathcal{Z}^+, \leq_{\mathcal{D}})$ has no meet-irreducible elements.

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