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Solutions of Some Integrals in Hypergeometric Functions and their Generalization in Double Series Identities

Research Article

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Abstract: In this paper, we obtain solutions of some integrals in the form of hypergeometric functions. Further, we generalize these integrals in the form of double series identities involving bounded sequences. We also derive hypergeometric forms of these identities involving Gaussian hypergeometric function, Srivastava-Daoust double hypergeometric function and Kampé de Fériet double hypergeometric function.

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1. Introduction and Preliminaries

Pochhammer's symbol

The Pochhammer's symbol or Appell's symbol or shifted factorial or rising factorial or generalized factorial function is defined by

$$(b,\ k) = (b)_k = \frac{\Gamma(b+k)}{\Gamma(b)} = \begin{cases} b(b+1)(b+2)\cdots(b+k-1); & \text{if } \ k = 1,2,3,\cdots \\ 1 \qquad ; & \text{if } \ k = 0 \\ k! \qquad ; & \text{if } \ b = 1,\ k = 1,2,3,\cdots \end{cases} \tag{1}$$

where b is neither zero nor negative integer and the notation Γ stands for Gamma function.

Generalized Gaussian Hypergeometric Function[17,p.42(1)]

Generalized ordinary hypergeometric function of one variable is defined by

$${}_AF_B\left[\begin{matrix} a_1, a_2, \cdots, a_A\ ; \\ \\ b_1, b_2, \cdots, b_B\ ; \end{matrix}\ z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!}$$

or

$${}_AF_B\left[\begin{matrix} (a_A)\ ; \\ \\ (b_B)\ ; \end{matrix}\ z\right] \equiv {}_AF_B\left[\begin{matrix} (a_j)_{j=1}^{A}\ ; \\ \\ (b_j)_{j=1}^{B}\ ; \end{matrix}\ z\right] = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!} \tag{2}$$

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where denominator parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and $A,\ B$ are non-negative integers. If $A \leq B$, then series ${}_AF_B$ is always convergent for all finite values of z(real or complex). If $A = B+1$, then series ${}_AF_B$ is convergent when $|z| < 1$.

$$((a_A))_{2m} = 4^{Am}\Big(\Big(\frac{a_A}{2}\Big)\Big)_m\Big(\Big(\frac{1+a_A}{2}\Big)\Big)_m$$

$$((a_A))_{1+2m} = 4^{Am}\Big(\Big(\frac{1+a_A}{2}\Big)\Big)_m\Big(\Big(\frac{2+a_A}{2}\Big)\Big)_m\prod_{i=1}^{A}(a_i)$$

where $\quad m = 0, 1, 2, 3, \ldots$

Kampé de Fériet's General Double Hypergeometric Function[17,p.63(16); see also 16]

In 1921, Appell's four double hypergeometric functions F_1, F_2, F_3, F_4 and their confluent forms $\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1, \Xi_2$ were unified and generalized by Kampé de Fériet. We recall the definition of general double hypergeometric function of Kampé de Fériet in slightly modified notation of H.M.Srivastava and R.Panda:

$$F^{A:B;D}_{E:G;H}\left[\begin{array}{ll}(a_A):(b_B);(d_D)\ ; & \\ & x,\ y\\ (e_E):(g_G);(h_H)\ ; & \end{array}\right] = \sum_{m,n=0}^{\infty}\frac{((a_A))_{m+n}\,((b_B))_m\,((d_D))_n\,x^m\,y^n}{((e_E))_{m+n}\,((g_G))_m\,((h_H))_n\,m!\,n!} \tag{3}$$

where for convergence

(i) $A+B < E+G+1,\ A+D < E+H+1 \quad ; |x| < \infty,\quad |y| < \infty,$ or

(ii) $A+B = E+G+1,\ A+D = E+H+1,$ and

$$\begin{cases} |x|^{\frac{1}{(A-E)}} + |y|^{\frac{1}{(A-E)}} < 1 & ,\text{if } E < A \\ \\ \max\ \{|x|, |y|\} < 1 & ,\text{if } E \geq A \end{cases}$$

Srivastava-Daoust General Multiple Hypergeometric Function[16,p.37(21,22,23);17,pp.64-65(18,19,20)]

In 1969, H.M. Srivastava and M.C. Daoust defined extremely generalized hypergeometric function of n-variables (which is referred to in the literature as the generalized Lauricella function of several variables).

$$F^{A:B^{(1)};B^{(2)};\cdots;B^{(n)}}_{C:D^{(1)};D^{(2)};\cdots;D^{(n)}}\left(\begin{array}{ll}[(a_A):\theta^{(1)},\cdots,\theta^{(n)}]\ :\ [(b^{(1)}_{B^{(1)}}):\phi^{(1)}];\cdots;[(b^{(n)}_{B^{(n)}}):\phi^{(n)}]\ ; & \\ & z_1, z_2, \cdots, z_n\\ [(c_C):\psi^{(1)},\cdots,\psi^{(n)}]\ :\ [(d^{(1)}_{D^{(1)}}):\delta^{(1)}];\cdots;[(d^{(n)}_{D^{(n)}}):\delta^{(n)}]\ ; & \end{array}\right)$$

$$= \sum_{m_1,m_2,\cdots,m_n=0}^{\infty}\Omega(m_1,m_2,\cdots,m_n)\,\frac{z_1^{m_1}\,z_2^{m_2}\cdots z_n^{m_n}}{m_1!\ m_2!\cdots m_n!} \tag{4}$$

where for convenience

$$\Omega(m_1,m_2,\cdots,m_n) = \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+\cdots+m_n\theta_j^{(n)}}\prod_{j=1}^{B^{(1)}}(b_j^{(1)})_{m_1\phi_j^{(1)}}\cdots\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi_j^{(1)}+\cdots+m_n\psi_j^{(n)}}\prod_{j=1}^{D^{(1)}}(d_j^{(1)})_{m_1\delta_j^{(1)}}\cdots\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}}$$

the coefficients

$$\begin{cases}\theta_j^{(k)},\quad j=1,2,\cdots,A;\quad \phi_j^{(k)},\quad j=1,2,\cdots,B^{(k)};\quad \psi_j^{(k)},\quad j=1,2,\cdots,C\\ \\ \delta_j^{(k)},\quad j=1,2,\cdots,D^{(k)};\quad \text{for all}\quad k\in\{1,2,3,\cdots,n\}\end{cases}$$

are real and positive, and (a_A) abbreviates the array of A parameters $a_1, a_2, \cdots, a_A$, and $(b^{(k)}_{B^{(k)}})$ abbreviates the array of $B^{(k)}$ parameters ${b_j}^{(k)}$, $j = 1, 2, \cdots, B^{(k)}$; for all $k \in \{1, 2, 3, \cdots, n\}$, with same interpretations for (c_C) and $(d^{(k)}_{D^{(k)}})$ etc. The convergence conditions of above multiple series is given by Srivastava and Daoust[16;17].

Some Interesting Series Identities

We recall the following identities which are potentially useful in the series rearrangement techniques.

$$\sum_{m=0}^{\infty}\sum_{r=0}^{m-1}\Theta(m,\ r) = \sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Theta(m+r+1,\ r) \tag{5}$$

$$\sum_{m=0}^{\infty}\sum_{r=0}^{2m-1}\Theta(m,\ r) = \sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Theta(m+r+1,\ r) + \sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Theta(m+r+1,\ 2r+m+1) \tag{6}$$

where $\{\Theta(m,r)\}_{m,r=0}^{\infty}$ are suitably bounded double and triple sequences of essentially arbitrary(real or complex) parameters.

Some Useful Indefinite Integrals

When $m = 0, 1, 2, 3, \cdots,$ then

$$\int \sinh^{2m}\theta\, d\theta = \left\{\frac{-(\frac{1}{2})_m(-1)^m \sinh\theta\cosh\theta}{(1)_m}\sum_{r=0}^{m-1}\frac{(1)_r(-1)^r\sinh^{2r}\theta}{(\frac{3}{2})_r}\right\} + \left\{\frac{\theta\,(-1)^m\,(\frac{1}{2})_m}{(1)_m}\right\} + \text{Constant} \tag{7}$$

$$\int \cosh^{2m}\theta\, d\theta = \left\{\frac{(\frac{1}{2})_m \sinh\theta\cosh\theta}{(1)_m}\sum_{r=0}^{m-1}\frac{(1)_r\cosh^{2r}\theta}{(\frac{3}{2})_r}\right\} + \left\{\frac{\theta\,(\frac{1}{2})_m}{(1)_m}\right\} + \text{Constant} \tag{8}$$

$$\int \sinh^{2m+1}\theta\, d\theta = \frac{(1)_m\,(-1)^m\cosh\theta}{(\frac{3}{2})_m}\sum_{r=0}^{m}\frac{(\frac{1}{2})_r\,(-1)^r\sinh^{2r}\theta}{(1)_r} + \text{Constant} \tag{9}$$

$$\int \cosh^{2m+1}\theta\, d\theta = \frac{(1)_m\sinh\theta}{(\frac{3}{2})_m}\sum_{r=0}^{m}\frac{(\frac{1}{2})_r\cosh^{2r}\theta}{(1)_r} + \text{Constant} \tag{10}$$

Above formulas (7)-(10) can be verified for $m = 0, 1, \ldots$ and it is the convention that the empty sum $\sum\limits_{r=0}^{-1} F(r)$ is treated as zero.

2. A General Family of Double Series Identities

Theorem 2.1. *Let $\{\Omega_m\}_{m=0}^{\infty}$ be a suitably bounded sequence of arbitrary complex numbers, then*

$$\sum_{m=0}^{\infty}\Omega_m\left(\int_0^{\gamma}\sinh^{2m}\theta\, d\theta\right)\frac{y^m}{m!} = \frac{y\sinh\gamma\cosh\gamma}{2}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r+1}\frac{(\frac{3}{2})_{m+r}(1)_r\ (-y)^m(y\sinh^2\gamma)^r}{(2)_{m+r}(2)_{m+r}(\frac{3}{2})_r} + \gamma\sum_{m=0}^{\infty}\Omega_m\frac{(\frac{1}{2})_m(-y)^m}{(m!)^2} \tag{11}$$

provided that each of the series involved is absolutely convergent.

Theorem 2.2. *Let $\{\Omega_m\}_{m=0}^{\infty}$ be a suitably bounded sequence of arbitrary complex numbers, then*

$$\sum_{m=0}^{\infty}\Omega_m\left(\int_0^{\gamma}\cosh^{2m}\theta\, d\theta\right)\frac{y^m}{m!}$$

$$= \frac{y\sinh\gamma\cosh\gamma}{2}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r+1}\frac{(\frac{3}{2})_{m+r}(1)_r\ y^m(y\cosh^2\gamma)^r}{(2)_{m+r}(2)_{m+r}(\frac{3}{2})_r} + \gamma\sum_{m=0}^{\infty}\Omega_m\frac{(\frac{1}{2})_m y^m}{(m!)^2} \tag{12}$$

provided that each of the series involved is absolutely convergent.

Theorem 2.3. *Let $\{\Omega_m\}_{m=0}^{\infty}$ be a suitably bounded sequence of arbitrary complex numbers, then*

$$\sum_{m=0}^{\infty}\Omega_m\left(\int_0^{\gamma}\sinh^{4m}\theta\,\mathrm{d}\theta\right)\frac{y^m}{m!}=\gamma\sum_{m=0}^{\infty}\Omega_m\frac{(\frac{1}{4})_m(\frac{3}{4})_m y^m}{(\frac{1}{2})_m(m!)^2}$$
$$-\frac{3y\sinh\gamma\cosh\gamma}{8}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r+1}\frac{(\frac{5}{4})_{m+r}(\frac{7}{4})_{m+r}(1)_r\ y^m(-y\sinh^2\gamma)^r}{(2)_{m+r}(2)_{m+r}(\frac{3}{2})_{m+r}(\frac{3}{2})_r}$$
$$+\frac{y\sinh^3\gamma\cosh\gamma}{4}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r+1}\frac{(\frac{5}{4})_{m+r}(\frac{7}{4})_{m+r}(2)_{m+2r}\ (-y\sinh^2\gamma)^m(y\sinh^4\gamma)^r}{(2)_{m+r}(2)_{m+r}(\frac{3}{2})_{m+r}(\frac{5}{2})_{m+2r}} \tag{13}$$

provided that each of the series involved is absolutely convergent.

Theorem 2.4. *Let $\{\Omega_m\}_{m=0}^{\infty}$ be a suitably bounded sequence of arbitrary complex numbers, then*

$$\sum_{m=0}^{\infty}\Omega_m\left(\int_0^{\gamma}\cosh^{4m}\theta\,\mathrm{d}\theta\right)\frac{y^m}{m!}=\gamma\sum_{m=0}^{\infty}\Omega_m\frac{(\frac{1}{4})_m(\frac{3}{4})_m y^m}{(\frac{1}{2})_m(m!)^2}$$
$$+\frac{3y\sinh\gamma\cosh\gamma}{8}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r+1}\frac{(\frac{5}{4})_{m+r}(\frac{7}{4})_{m+r}(1)_r\ y^m(y\cosh^2\gamma)^r}{(2)_{m+r}(2)_{m+r}(\frac{3}{2})_{m+r}(\frac{3}{2})_r}$$
$$+\frac{y\sinh\gamma\cosh^3\gamma}{4}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r+1}\frac{(\frac{5}{4})_{m+r}(\frac{7}{4})_{m+r}(2)_{m+2r}\ (y\cosh^2\gamma)^m(y\cosh^4\gamma)^r}{(2)_{m+r}(2)_{m+r}(\frac{3}{2})_{m+r}(\frac{5}{2})_{m+2r}} \tag{14}$$

provided that each of the series involved is absolutely convergent.

Theorem 2.5. *Let $\{\Omega_m\}_{m=0}^{\infty}$ be a suitably bounded sequence of arbitrary complex numbers, then*

$$\sum_{m=0}^{\infty}\Omega_m\left(\int_0^{\gamma}\sinh^{m}\theta\,\mathrm{d}\theta\right)\frac{y^m}{m!}$$
$$=\gamma\sum_{m=0}^{\infty}\Omega_{2m}\frac{(-y^2)^m}{4^m\ (m!)^2}+\frac{y^2\sinh\gamma\cosh\gamma}{4}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{2m+2r+2}\frac{(1)_r\ (-y^2)^m(y^2\sinh^2\gamma)^r}{4^{m+r}(2)_{m+r}(2)_{m+r}(\frac{3}{2})_r}$$
$$+y\cosh\gamma\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{2m+2r+1}\frac{(\frac{1}{2})_r\ (-y^2)^m(y^2\sinh^2\gamma)^r}{4^{m+r}(\frac{3}{2})_{m+r}(\frac{3}{2})_{m+r}(1)_r}-y\sum_{m=0}^{\infty}\Omega_{2m+1}\frac{(-y^2)^m}{4^m\ (\frac{3}{2})_m(\frac{3}{2})_m} \tag{15}$$

provided that each of the series involved is absolutely convergent.

Theorem 2.6. *Let $\{\Omega_m\}_{m=0}^{\infty}$ be a suitably bounded sequence of arbitrary complex numbers, then*

$$\sum_{m=0}^{\infty}\Omega_m\left(\int_0^{\gamma}\cosh^{m}\theta\,\mathrm{d}\theta\right)\frac{y^m}{m!}$$
$$=\gamma\sum_{m=0}^{\infty}\Omega_{2m}\frac{y^{2m}}{4^m\ (m!)^2}+\frac{y^2\sinh\gamma\cosh\gamma}{4}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{2m+2r+2}\frac{(1)_r\ y^{2m}(y\cosh\gamma)^{2r}}{4^{m+r}(2)_{m+r}(2)_{m+r}(\frac{3}{2})_r}$$
$$+y\sinh\gamma\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{2m+2r+1}\frac{(\frac{1}{2})_r\ y^{2m}(y\cosh\gamma)^{2r}}{4^{m+r}(\frac{3}{2})_{m+r}(\frac{3}{2})_{m+r}(1)_r} \tag{16}$$

provided that each of the series involved is absolutely convergent.

Theorem 2.7. *Let $\{\Omega_m\}_{m=0}^{\infty}$ be a suitably bounded sequence of arbitrary complex numbers, then*

$$\sum_{m=0}^{\infty}\Omega_m\left(\int_0^{\gamma}\sinh^{2m+1}\theta\,\mathrm{d}\theta\right)\frac{y^m}{m!}=\cosh\gamma\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r}\frac{(\frac{1}{2})_r\ (-y)^m(y\sinh^2\gamma)^r}{(\frac{3}{2})_{m+r}(1)_r}-\sum_{m=0}^{\infty}\ \Omega_m\frac{(-y)^m}{(\frac{3}{2})_m} \tag{17}$$

provided that each of the series involved is absolutely convergent.

Theorem 2.8. *Let $\{\Omega_m\}_{m=0}^{\infty}$ be a suitably bounded sequence of arbitrary complex numbers, then*

$$\sum_{m=0}^{\infty}\Omega_m\left(\int_0^{\gamma}\cosh^{2m+1}\theta\mathrm{d}\theta\right)\frac{y^m}{m!}=\sinh\gamma\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r}\frac{(\frac{1}{2})_r\ y^m(y\cosh^2\gamma)^r}{(\frac{3}{2})_{m+r}(1)_r} \tag{18}$$

provided that each of the series involved is absolutely convergent.

3. Proof of Identities

Suppose left hand side of (11) is denoted by "T" and using the integral (7), then

$$T = -\sum_{m=0}^{\infty}\sum_{r=0}^{m-1}\Omega_m \frac{\sinh\gamma\cosh\gamma(\frac{1}{2})_m(1)_r(-1)^m(-1)^r\ y^m(\sinh^2\gamma)^r}{(1)_m(1)_m(\frac{3}{2})_r} + \gamma\sum_{m=0}^{\infty}\Omega_m\frac{(\frac{1}{2})_m(-1)^m y^m}{(m!)^2}$$

$$= -\sum_{m=0}^{\infty}\sum_{r=0}^{m}\Omega_{m+1}\frac{\sinh\gamma\cosh\gamma(\frac{1}{2})_{m+1}(1)_r\,(-1)^{m+1}(-1)^r\ y^{m+1}(\sinh^2\gamma)^r}{(1)_{m+1}(1)_{m+1}(\frac{3}{2})_r} + \gamma\sum_{m=0}^{\infty}\Omega_m\frac{(\frac{1}{2})_m(-y)^m}{(m!)^2}$$

$$= \frac{y\sinh\gamma\cos\gamma}{2}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\Omega_{m+r+1}\frac{(\frac{3}{2})_{m+r}(1)_r\ (-y)^m(y\sinh^2\gamma)^r}{(2)_{m+r}(2)_{m+r}(\frac{3}{2})_r} + \gamma\sum_{m=0}^{\infty}\Omega_m\frac{(\frac{1}{2})_m(-y)^m}{(m!)^2}$$

which is the right hand side of (11). On the same parallel lines of derivation of (11), we can derive (12) to (18) by means of series identities (5) and (6).

4. Hypergeometric Generalization of Double Series Identities

Setting $\Omega_m = \frac{(a_1)_m(a_2)_m(a_3)_m\cdots(a_A)_m}{(b_1)_m(b_2)_m(b_3)_m\cdots(b_B)_m} = \frac{((a_A))_m}{((b_B))_m}$ in theorems (11) to (18), using some algebraic properties of Pochhammer symbol and interpreting the multiple power series in hypergeometric notations given by (2) to (4), we get the analytical solutions of following integrals.

$$\int_0^{\gamma} {}_AF_B\left[\begin{matrix}(a_j)_{j=1}^A\ ; \\ \quad y\sinh^2\theta \\ (b_j)_{j=1}^B\ ;\end{matrix}\right]\mathrm{d}\theta = \gamma\ {}_{A+1}F_{B+1}\left[\begin{matrix}\frac{1}{2},(a_j)_{j=1}^A\ ; \\ \quad -y \\ 1,(b_j)_{j=1}^B\ ;\end{matrix}\right]$$
$$+\frac{y\sinh\gamma\cosh\gamma\prod_{i=1}^{A}(a_i)}{2\prod_{i=1}^{B}(b_i)}F^{A+1:1;2}_{B+2:0;1}\left[\begin{matrix}\frac{3}{2},(1+a_j)_{j=1}^A\ :1\ ;1,1\ ; \\ \quad -y,y\sinh^2\gamma \\ 2,2,(1+b_j)_{j=1}^B:\ -;\frac{3}{2}\ ;\end{matrix}\right] \tag{19}$$

$$\int_0^{\gamma} {}_AF_B\left[\begin{matrix}(a_j)_{j=1}^A\ ; \\ \quad y\cosh^2\theta \\ (b_j)_{j=1}^B\ ;\end{matrix}\right]\mathrm{d}\theta = \gamma\ {}_{A+1}F_{B+1}\left[\begin{matrix}\frac{1}{2},(a_j)_{j=1}^A\ ; \\ \quad y \\ 1,(b_j)_{j=1}^B\ ;\end{matrix}\right]$$
$$+\frac{y\sinh\gamma\ \cosh\gamma\prod_{i=1}^{A}(a_i)}{2\prod_{i=1}^{B}(b_i)}F^{A+1:1;2}_{B+2:0;1}\left[\begin{matrix}\frac{3}{2},(1+a_j)_{j=1}^A\ :1\ ;1\ ,\ 1\ ; \\ \quad y,y\cosh^2\gamma \\ 2,2,(1+b_j)_{j=1}^B:\ -;\ \frac{3}{2}\ ;\end{matrix}\right] \tag{20}$$

$$\int_0^{\gamma} {}_AF_B\left[\begin{matrix}(a_j)_{j=1}^A\ ; \\ \quad y\sinh^4\theta \\ (b_j)_{j=1}^B\ ;\end{matrix}\right]\mathrm{d}\theta = \gamma\ {}_{A+2}F_{B+2}\left[\begin{matrix}\frac{1}{4},\frac{3}{4},(a_j)_{j=1}^A\ ; \\ \quad y \\ 1,\frac{1}{2},(b_j)_{j=1}^B\ ;\end{matrix}\right]$$
$$-\frac{3y\ \sinh\gamma\cosh\gamma\prod_{i=1}^{A}(a_i)}{8\prod_{i=1}^{B}(b_i)}F^{A+2:1;2}_{B+3:0;1}\left[\begin{matrix}\frac{5}{4},\frac{7}{4},(1+a_j)_{j=1}^A\ :\ 1;\ 1,1\ ; \\ \quad y,\ -y\sinh^2\gamma \\ 2,2,\frac{3}{2},(1+b_j)_{j=1}^B:\ -;\ \frac{3}{2}\ ;\end{matrix}\right]$$

$$+\frac{y\sinh^3\gamma\,\cosh\gamma\prod\limits_{i=1}^{A}(a_i)}{4\prod\limits_{i=1}^{B}(b_i)}\,F^{A+3:1;1}_{B+4:0;0}\left(\begin{matrix}[(1+a_j){:}1,1]_{j=1}^{A},\ [\frac{5}{4}{:}1,1],\ [\frac{7}{4}{:}1,1],\ [2{:}1,2] & : & [1{:}1]\,;[1{:}1]\ ; & \\ & & & -y\sinh^2\gamma,\ y\sinh^4\gamma\\ [(1+b_j)\ {:}1,1]_{j=1}^{B},\ [2{:}1,1],\ [2{:}1,1],\ [\frac{3}{2}{:}1,1],\ [\frac{5}{2}{:}1,2] & : & —\,;—\ ; & \end{matrix}\right) \tag{21}$$

$$\int_0^{\gamma} {}_AF_B\left[\begin{matrix}(a_j)_{j=1}^{A}\ ; & \\ & y\cosh^4\theta\\ (b_j)_{j=1}^{B}\ ; & \end{matrix}\right]\mathrm{d}\theta=\gamma\ {}_{A+2}F_{B+2}\left[\begin{matrix}\frac{1}{4},\frac{3}{4},(a_j)_{j=1}^{A}\ ; & \\ & y\\ 1,\frac{1}{2},(b_j)_{j=1}^{B}\ ; & \end{matrix}\right]$$

$$+\frac{3y\,\sinh\gamma\cosh\gamma\prod\limits_{i=1}^{A}(a_i)}{8\prod\limits_{i=1}^{B}(b_i)}\,F^{A+2:1;2}_{B+3:0;1}\left[\begin{matrix}\frac{5}{4},\frac{7}{4},(1+a_j)_{j=1}^{A}\ :\ 1;1,1\ ; & \\ & y,\ y\cosh^2\gamma\\ 2,2,\frac{3}{2},(1+b_j)_{j=1}^{B}{:}\ —;\frac{3}{2}\ ; & \end{matrix}\right]$$

$$+\frac{y\sinh\gamma\cosh^3\gamma\prod\limits_{i=1}^{A}(a_i)}{4\prod\limits_{i=1}^{B}(b_i)}\,F^{A+3:1;1}_{B+4:0;0}\left(\begin{matrix}[(1+a_j){:}1,1]_{j=1}^{A},\ [\frac{5}{4}{:}1,1],\ [\frac{7}{4}{:}1,1],\ [2{:}1,2] & : & [1{:}1]\,;[1{:}1]\ ; & \\ & & & y\cosh^2\gamma,\ y\cosh^4\gamma\\ [(1+b_j)\ {:}1,1]_{j=1}^{B},\ [2{:}1,1],\ [2{:}1,1],\ [\frac{3}{2}{:}1,1],\ [\frac{5}{2}{:}1,2] & : & —\,;—\ ; & \end{matrix}\right) \tag{22}$$

$$\int_0^{\gamma} {}_AF_B\left[\begin{matrix}(a_j)_{j=1}^{A}\ ; & \\ & y\sinh\theta\\ (b_j)_{j=1}^{B}\ ; & \end{matrix}\right]\mathrm{d}\theta=\gamma\ {}_{2A}F_{2B+1}\left[\begin{matrix}(\frac{a_j}{2})_{j=1}^{A},\ (\frac{1+a_j}{2})_{j=1}^{A}\ ; & \\ & -\frac{y^2}{4^{(1+B-A)}}\\ 1,\ (\frac{b_j}{2})_{j=1}^{B},\ (\frac{1+b_j}{2})_{j=1}^{B}\ ; & \end{matrix}\right]+\frac{y\cosh\gamma\prod\limits_{i=1}^{A}(a_i)}{\prod\limits_{i=1}^{B}(b_i)}$$

$$\times F^{2A\ \ :1;1}_{2B+2:0;0}\left[\begin{matrix}(\frac{1+a_j}{2})_{j=1}^{A},\ (\frac{2+a_j}{2})_{j=1}^{A}\ :1\,;\frac{1}{2}\ ; & \\ & -\frac{y^2}{4^{(1+B-A)}},\frac{y^2\sinh^2\gamma}{4^{(1+B-A)}}\\ \frac{3}{2},\ \frac{3}{2},\ (\frac{1+b_j}{2})_{j=1}^{B},\ (\frac{2+b_j}{2})_{j=1}^{B}{:}—;\ —\ ; & \end{matrix}\right]+\frac{y^2\sinh\gamma\cosh\gamma\prod\limits_{i=1}^{A}(a_i)\prod\limits_{i=1}^{A}(1+a_i)}{4\prod\limits_{i=1}^{B}(b_i)\prod\limits_{i=1}^{B}(1+b_i)}$$

$$\times F^{2A\ \ :1;2}_{2B+2:0;1}\left[\begin{matrix}(\frac{2+a_j}{2})_{j=1}^{A},\ (\frac{3+a_j}{2})_{j=1}^{A}\ :1\ ;\ 1,1\ ; & \\ & -\frac{y^2}{4^{(1+B-A)}},\frac{y^2\sinh^2\gamma}{4^{(1+B-A)}}\\ 2,\ 2,\ (\frac{2+b_j}{2})_{j=1}^{B},\ (\frac{3+b_j}{2})_{j=1}^{B}{:}—;\ \frac{3}{2}\ ; & \end{matrix}\right]$$

$$-\frac{y\prod\limits_{i=1}^{A}(a_i)}{\prod\limits_{i=1}^{B}(b_i)}\ {}_{2A+1}F_{2B+2}\left[\begin{matrix}1,(\frac{1+a_j}{2})_{j=1}^{A},\ (\frac{2+a_j}{2})_{j=1}^{A}\ ; & \\ & -\frac{y^2}{4^{(1+B-A)}}\\ \frac{3}{2},\ \frac{3}{2},\ (\frac{1+b_j}{2})_{j=1}^{B},\ (\frac{2+b_j}{2})_{j=1}^{B}\ ; & \end{matrix}\right] \tag{23}$$

$$\int_0^{\gamma} {}_AF_B\left[\begin{matrix}(a_j)_{j=1}^{A}\ ; & \\ & y\cosh\theta\\ (b_j)_{j=1}^{B}\ ; & \end{matrix}\right]\mathrm{d}\theta=\gamma\ {}_{2A}F_{2B+1}\left[\begin{matrix}(\frac{a_j}{2})_{j=1}^{A},\ (\frac{1+a_j}{2})_{j=1}^{A}\ ; & \\ & \frac{y^2}{4^{(1+B-A)}}\\ 1,\ (\frac{b_j}{2})_{j=1}^{B},\ (\frac{1+b_j}{2})_{j=1}^{B}\ ; & \end{matrix}\right]+\frac{y\sinh\gamma\prod\limits_{i=1}^{A}(a_i)}{\prod\limits_{i=1}^{B}(b_i)}$$

$$\times\, F^{2A\ :1;1}_{2B+2:0;0}\left[\begin{matrix}(\frac{1+a_j}{2})^A_{j=1},\ (\frac{2+a_j}{2})^A_{j=1}\ :1\ ;\ \frac{1}{2}\ ; \\ \\ \frac{3}{2},\ \frac{3}{2},\ (\frac{1+b_j}{2})^B_{j=1},\ (\frac{2+b_j}{2})^B_{j=1}:\text{—};\text{—}\ ;\end{matrix}\ \frac{y^2}{4(1+B-A)},\ \frac{y^2\cosh^2\gamma}{4(1+B-A)}\right] + \frac{y^2\sinh\gamma\cosh\gamma\prod_{i=1}^{A}(a_i)\prod_{i=1}^{A}(1+a_i)}{4\prod_{i=1}^{B}(b_i)\prod_{i=1}^{B}(1+b_i)}$$

$$\times\, F^{2A\ :1;2}_{2B+2:0;1}\left[\begin{matrix}(\frac{2+a_j}{2})^A_{j=1},\ (\frac{3+a_j}{2})^A_{j=1}\ :1\ ;1,1\ ; \\ \\ 2,\ 2,\ (\frac{2+b_j}{2})^B_{j=1},\ (\frac{3+b_j}{2})^B_{j=1}:\text{—};\frac{3}{2}\ ;\end{matrix}\ \frac{y^2}{4(1+B-A)},\ \frac{y^2\cosh^2\gamma}{4(1+B-A)}\right] \tag{24}$$

$$\int_0^{\gamma}\sinh\theta\ {}_AF_B\left[\begin{matrix}(a_j)^A_{j=1}\ ; \\ \\ (b_j)^B_{j=1}\ ;\end{matrix}\ y\sinh^2\theta\right]\mathrm{d}\theta = -\ {}_{A+1}F_{B+1}\left[\begin{matrix}1,(a_j)^A_{j=1}\ ; \\ \\ \frac{3}{2},(b_j)^B_{j=1}\ ;\end{matrix}\ -y\right]$$
$$+\ \cosh\gamma\ F^{A\ :1;1}_{B+1:0;0}\left[\begin{matrix}(a_j)^A_{j=1}\ :1\ ;\ \frac{1}{2}\ ; \\ \\ \frac{3}{2},\ (b_j)^B_{j=1}:\ \text{—};\text{—}\ ;\end{matrix}\ -y,\ y\sinh^2\gamma\right] \tag{25}$$

$$\int_0^{\gamma}\cosh\theta\ {}_AF_B\left[\begin{matrix}(a_j)^A_{j=1}\ ; \\ \\ (b_j)^B_{j=1}\ ;\end{matrix}\ y\cosh^2\theta\right]\mathrm{d}\theta = \sinh\gamma\ F^{A\ :1;1}_{B+1:0;0}\left[\begin{matrix}(a_j)^A_{j=1}\ :1\ ;\ \frac{1}{2}\ ; \\ \\ \frac{3}{2},\ (b_j)^B_{j=1}:\ \text{—};\ \text{—}\ ;\end{matrix}\ y,\ y\cosh^2\gamma\right] \tag{26}$$

provided that each of the series as well as associated integrals involved are convergent.

5. Solutions of Some Integrals

Setting $A = 1, B = 0$ and $a_1 = \frac{1}{2}$ in (19) and (20) respectively, we get

$$\int_0^{\gamma}\frac{\mathrm{d}\theta}{\sqrt{(1-y\sinh^2\theta)}} = \gamma\ {}_2F_1\left[\begin{matrix}\frac{1}{2},\frac{1}{2}\ ; \\ \\ 1\ ;\end{matrix}\ -y\right]$$
$$+\ \frac{y\sinh\gamma\ \cosh\gamma}{4}\ F^{2:1;2}_{2:0;1}\left[\begin{matrix}\frac{3}{2},\frac{3}{2}:\ 1\ ;\ 1,1\ ; \\ \\ 2,2\ :\text{—};\ \frac{3}{2}\ ;\end{matrix}\ -y,y\sinh^2\gamma\right]\quad ;|y\sinh^2\theta| < 1 \tag{27}$$

$$\int_0^{\gamma}\frac{\mathrm{d}\theta}{\sqrt{(1-y\cosh^2\theta)}} = \gamma\ {}_2F_1\left[\begin{matrix}\frac{1}{2},\frac{1}{2}\ ; \\ \\ 1\ ;\end{matrix}\ y\right]$$
$$+\ \frac{y\sinh\gamma\ \cosh\gamma}{4}\ F^{2:1;2}_{2:0;1}\left[\begin{matrix}\frac{3}{2},\frac{3}{2}:\ 1;\ 1,1\ ; \\ \\ 2,2\ :\text{—};\ \frac{3}{2}\ ;\end{matrix}\ y,y\cosh^2\gamma\right]\quad ;|y\cosh^2\theta| < 1 \tag{28}$$

Putting $A = 1, B = 0$ and $a_1 = -\frac{1}{2}$ in (19) and (20) respectively, we get

$$\int_0^\gamma \sqrt{(1-y\sinh^2\theta)}\,\mathrm{d}\theta = \gamma\, {}_2F_1\left[\begin{matrix} \frac{1}{2}, -\frac{1}{2} \ ; \\ \\ 1 \ ; \end{matrix} \ -y\right]$$

$$-\frac{y\sinh\gamma\cosh\gamma}{4} F^{2:1;2}_{2:0;1}\left[\begin{matrix} \frac{1}{2}, \frac{3}{2}:1 \ ; \ 1,1 \ ; \\ \\ 2,2 :—; \ \frac{3}{2} \ ; \end{matrix} \ -y, y\sinh^2\gamma\right] \quad ;|y\sinh^2\theta| < 1 \tag{29}$$

$$\int_0^\gamma \sqrt{(1-y\cosh^2\theta)}\,\mathrm{d}\theta = \gamma {}_2F_1\left[\begin{matrix} \frac{1}{2}, -\frac{1}{2} \ ; \\ \\ 1 \ ; \end{matrix} \ y\right]$$

$$-\frac{y\sinh\gamma\cosh\gamma}{4} F^{2:1;2}_{2:0;1}\left[\begin{matrix} \frac{1}{2}, \frac{3}{2}:1 \ ; \ 1,1 \ ; \\ \\ 2,2 :—; \ \frac{3}{2} \ ; \end{matrix} \ y, y\cosh^2\gamma\right] \quad ;|y\cosh^2\theta| < 1 \tag{30}$$

Putting $A=1, B=0,\ a_1=\frac{1}{2},\ \gamma=\beta$ and $y=x^2$ in (21), we get

$$\int_0^\beta \frac{\mathrm{d}\theta}{\sqrt{(1-x^2\sinh^4\theta)}} = \beta\, {}_2F_1\left[\begin{matrix} \frac{1}{4}, \frac{3}{4} \ ; \\ \\ 1 \ ; \end{matrix} \ x^2\right] - \frac{3x^2\sinh\beta\cosh\beta}{16}$$

$$\times F^{2:1;2}_{2:0;1}\left[\begin{matrix} \frac{5}{4}, \frac{7}{4}:1 \ ; \ 1,1 \ ; \\ \\ 2,2 :—; \ \frac{3}{2} \ ; \end{matrix} \ x^2, -x^2\sinh^2\beta\right] - \frac{x^2\sinh^3\beta\cosh\beta}{8}$$

$$\times F^{3:1;1}_{3:0;0}\left(\begin{matrix} [\frac{5}{4}:1,1],\ [\frac{7}{4}:1,1],\ [2:1,2]:\ [1:1] \ ; \ \ [1:1] \ ; \\ \\ [2:1,1],\ [2:1,1],\ [\frac{5}{2}:1,2] \ : \ —— \ ; \ —— \ ; \end{matrix} \ -x^2\sinh^2\beta,\ x^2\sinh^4\beta\right) \quad ;|x^2\sinh^4\theta| < 1 \tag{31}$$

These solutions are not found in Ramanujan's notebooks [11–13], Five notebooks of B. C. Berndt [5–9], Three volumes of R. P. Agarwal [1–3] and other literature [4, 10, 14, 15] on special functions.

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