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Product Measure Spaces and Theorems of Fubini and Tonelli

Research Article

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Abstract: The product X×Y of measure spaces has as its measurable sub sets, the σ-algebra generated by the products A× B measurable sub sets of X and Y. Fubini's Theorem introduced by Guido Fubini in 1907 is a result which gives conditions under which it is possible to commute a double integral. It implies that two repeated integrals of a function of two variables are equal if the function is integrable. Tonelli's Theorem is a successor of the Fubini's Theorem. The conclusion of Tonelli's theorem is identical to that of Fubini's theorem, but the assumption that $|f|$ has a finite integral is replaced by the assumption that f is non-negative.

Keywords: Measure Spaces, Product of Measure Spaces, Theorems of Fubini and Tonelli.

1. Basics and Main Results

Definition 1.1. *Let $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},v)$ be any two measure spaces. If $A \subset X$ and $B \subset Y$ then $A\times B$ is called a rectangle of $X\times Y$. If $A \in \mathcal{A}$ and $B \in \mathcal{B}$ then $A\times B$ is called a measurable rectangle of $X\times Y$.*

Theorem 1.2. *Let $\mathcal{R}$ be the class of measurable rectangles of $Z = X\times Y$. For any $A\times B \in \mathcal{R}$, Define $\lambda(A\times B) = \mu(A)v(B)$, then $\mathcal{R}$ is a semi-algebra and λ is a measure on $\mathcal{R}$.*

Proof.

(1) Let $A \times B \in \mathcal{R}$ and $C \times D \in \mathcal{R}$ then $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D) \Rightarrow (A \times B) \cap (C \times D) \in \mathcal{R}$.

(2) $(A \times B)^C = A^C \times B^C \cup (A \times B^C) \cup (A^C \times B) \Rightarrow (A \times B)^C$ is a finite union of members of $\mathcal{R}$. Proves that $\mathcal{R}$ is semi-algebra.

(3) λ is obviously non-negative and $\lambda(\phi) = \lambda(\phi \times \phi) = \mu(\phi)v(\phi) = 0.0 = 0$.

(4) Let (E_n) be any sequence of disjoint measurable rectangles and suppose $\bigcup_1^\infty E_n = E$ is also a measurable rectangle.

Let $E_n = A_n \times B_n$, $E = A\times$ B where A and A_n are measurable subsets of X and B and B_n are measurable subsets of Y. Consider any $s \in A$ and $y \in B$, then $(s, y) \in A \times B = E = \bigcup_1^\infty E_n \Rightarrow (s, y) \in E_i$ for some $i \Rightarrow (s, y) \in A_i \times B_i$ for some $I \Rightarrow y \in B_i$ when $s \in A_i \Rightarrow B \subset \cup\{B_i/s \in A_i\}$.

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Suppose $z \subset \cup\{B_i/s \in A_i\}$, then $z \in B_i$ when $s \in A_i \Rightarrow (s,z) \in A_i \times B_i \Rightarrow (s,z) \in E_i \Rightarrow (s,z) \in \bigcup_1^\infty E_n = E = A \times B \Rightarrow s \in A$ and $z \in B$, shows that $\cup\{B_i/s \in A_i\} \subset B_i$. Therefore

$$B = \cup\{B_i/s \in A_i\} \quad \text{for any} \quad s \in A \tag{1}$$

Let $u \in A_i$ for some i and $B_i \neq \phi$ for some i. Take $v \in B_i$, this gives that $(u,v) \in A_i \times B_i = E_i \Rightarrow (u,v) \in \bigcup_1^\infty E_n = E = A \times B \Rightarrow u \in A$, shows if $u \notin A$ then

$$\text{Either} \quad u \notin A_i \quad \text{or if} \quad u \in A_i \quad \text{then} B_i = \phi \tag{2}$$

From (1) and (2) we get $v(B)C_A = \sum_{i=1}^\infty v(B_i)C_{A_i}$, by Monotone convergence theorem we have

$$\int v(B)C_A d\mu = \int (\sum_{i=1}^\infty v(B_i)C_{A_i})d\mu = \sum_{i=1}^\infty \int v(B_i)C_{A_i} d\mu$$
$$\Rightarrow v(B)\mu(A) = \sum_{i=1}^\infty v(B_i)\mu(A_i)$$
$$\Rightarrow \lambda(A \times B) = \sum_{i=1}^\infty \lambda(A_i \times B_i) \Rightarrow \lambda(E) = \sum_{i=1}^\infty \lambda(E_i)$$

Which proves that λ is a measure on R. □

Definition 1.3. *Let (X, A, μ) and (Y, B, v) be any measure spaces, $Z = X \times Y$, $\mathcal{R}$ be the class of measurable rectangles of Z, π be defined on $\mathcal{R}$ by $\pi(A \times B) = \mu(A)v(B)$. Then $\mathcal{R}$ is a semi algebra on Z and π is a measure on $\mathcal{R}$. Let a be the algebra generated by $\mathcal{R}$ and λ be the unique extension of π to a measure on a. Let $(Z, \overline{a}, \bar{\lambda})$ be the outer measure extension of (Z, a, λ). Then $(Z, \overline{a}, \bar{\lambda})$ is called the Product space of (X, A, μ) and (Y, B, v). The measure $\bar{\lambda}$ is called the Product measure of μ and v and is denoted by $\mu \times v$.*

Note 1.4. *(1) It is obvious that $(Z, \overline{a}, \bar{\lambda})$ is an extension of $(Z, \mathcal{R}, \pi)$. Hence if $A \times B \in R$ then*

$$(\mu \times v)(A \times B) = \pi(A \times B) \qquad [\text{Because } \mu x v \text{ is an extension of } \pi]$$
$$= \mu(A)v(B) \qquad [\text{By definition of } \pi]$$

(2) If μ and v both are finite then $\mu \times v$ is also finite.

(3) If μ and v are σ-finite then $\mu \times v$ is also σ-finite.

Remark 1.5. *If $\mathcal{F}$ be the any family of subsets of X and $A = \cup\{F/F \in \mathcal{F}\}$, $B = \cap\{F/F \in F\}$ then $C_A = \sup\{C_F/F \in \mathcal{F}\}$ and $C_B = \inf\{C_F/F \in \mathcal{F}\}$*

Definition 1.6. *Let $E \subset X \times Y$ and $x \in X$ then $E_x = \{y \in Y/(x,y) \in E\}$ is called the Cross-Section of E by x. If $y \in Y$ Then $E_y = \{x \in X/(x,y) \in E\}$ is called the Cross Section of E by y.*

Note 1.7. *Let E and E_α be any sub sets of $X \times Y$ and $x \in X$*

(1) $(\bigcup_\alpha E_\alpha)_x = \bigcup_\alpha (E_{\alpha x})$

(2) $(\bigcap_\alpha E_\alpha)_x = \bigcap_\alpha (E_{\alpha x})$

(3) $(E^c)_x = (E_x)^C$

(4) $C_{E_x(y)} = c_E(xy)$

Theorem 1.8. *Let $\mathcal{R}$ Be the class of measurable rectangles, $E \in R_{\sigma s}$ and $x \in X$, then E_x is measurable.*

Proof.

Case 1 : Let $E \in \mathcal{R}$, Then $E = A \times B$, where A is a measurable sub set of X and B is a measurable sub set of Y. Suppose $x \notin A$. If $E_x \neq \phi$ then $y \in E_x \Rightarrow (x,y) \in E = A \times B \Rightarrow x \in A$ which is a contradiction, Hence $E_x = \phi$. Let $x \in A$ consider any $y \in E_x$. Then $(x,y) \in E = A \times B \Rightarrow y \in B \Rightarrow E_x \subset B$, On the other hand if $z \in B$ then $(x,z) \in A \times B = E \Rightarrow z \in E_x \Rightarrow B \subset E_x$. Hence $B = E_x$. Thus we see that $E_x = \begin{cases} \phi, & \text{if } x \notin A; \\ B, & \text{if } x \in A. \end{cases}$. Hence E_x is measurable.

Case 2 : Let $E \in \mathcal{R}_\sigma$ then $E = \bigcup_1^\infty E_n$ when E_n are members of $\mathcal{R}$. Therefore $E_x = \left(\bigcup_1^\infty E_n\right)_x = \bigcup_1^\infty (E_{nx})$, by Case 1 E_{nx} are measurable for every n. It can imply that $\bigcup_1^\infty E_{nx}$ is measurable i.e. E_x is measurable.

Case 3 : Let $E \in R_{\sigma\delta}$. Then $E = \bigcap_1^\infty F_n$ where $F_n \in R_\sigma$, therefore $E_x = \left(\bigcap_1^\infty F_n\right)_x = \bigcap_1^\infty (F_{nx})$, By Case 2 F_{nx} is measurable for every $n \Rightarrow \bigcap_1^\infty F_{nx}$ is measurable, which means that E_x is measurable. □

Note 1.9. *Let $\mathcal{R}$ be the semi algebra of measurable rectangles of $Z = X \times Y$ and a be the algebra generated by $\mathcal{R}$ then $R_\sigma = a_\sigma$.*

Proof. Let $\{c_n\}$ be any sequence of members of a_σ. Suppose $n = 2$ Let $c_1, c_2 \in a \Rightarrow c_1 = \bigcup_{i=1}^m S_i$ and $c_2 = \bigcup_{j=1}^n T_j$ where S_i and $T_j \in \mathcal{R}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Then $c_1 \cap c_2 = c_1 \cap \left(\bigcup_{j=1}^n T_j\right) = \bigcup_{j=1}^n (c_1 \cap T_j) = \bigcup_{j=1}^n \left[T_j \cap \left(\bigcup_{i=1}^m S_i\right)\right] = \bigcup_{j=1}^n \left[\bigcup_{i=1}^m (T_j \cap S_i)\right] = \bigcup_{j=1}^n \bigcup_{i=1}^m (S_i \cap T_j) = \bigcup_{i=1}^m \bigcup_{j=1}^n S_{ij}$, where $S_{ij} = S_i \cap T_j$. $\mathcal{R}$ is closed for intersection hence $S_{ij} \in \mathcal{R}$. Thus $c_1 \cap c_2$ is a finite union of members of $\mathcal{R}$. Hence $c_1 \cap c_2 \in a_\sigma$. By induction it follows that $\bigcap_1^\infty C_n \in a_\sigma$. It follows that $\mathcal{R}_\sigma = a_\sigma$. □

Theorem 1.10. *Let $E \in \mathcal{R}_{\sigma\delta}$ and $(\mu \times v)(E) < \infty$, for $x \in X$ define $g(x) = v(E_x)$. Then g is a non negative measurable function on X and $\int_X g d\mu = (\mu \times v)(E)$.*

Proof.

Case 1 : Suppose $E \in \mathcal{R}$. Let $E = A \times B$, where A is a measurable sub set of X and B is a measurable sub set of Y, Let $x \in X$ then $E_x = \begin{cases} \phi, & \text{if } x \notin A; \\ B, & \text{if } x \in A. \end{cases}$. Therefore

$g(x) = v(E_x) = \begin{cases} 0, & \text{if } x \notin A; \\ v(B), & \text{if } x \in A. \end{cases} = v(B)C_A(x) \Rightarrow g = v(B)C_A \Rightarrow g$ is a non negative simple function. And $\int_X g d\mu = \int_X v(B)C_A d\mu = v(B)\mu(A) = (\mu \times v)(A \times B) = (\mu \times v)(E)$.

Case 2 : Suppose $E \in \mathcal{R}_{\sigma\delta}$, then E is a countable union of members of $\mathcal{R}$. Since every countable union of semi algebra can be written as a countable disjoint union of members of the given semi algebra, It follows that E is a countable disjoint union of members of $\mathcal{R}$. Let $E = \bigcup_1^\infty E_n$, where E_n is a disjoint sequence of members of $\mathcal{R}$. This gives that

$$E_x = \left(\bigcup_1^\infty E_n\right)_x = \bigcup_1^\infty (E_{nx}) \Rightarrow v(E_x) = \sum_1^\infty v(E_{nx}) \; \forall x \in X. \tag{3}$$

For a natural number n, define g_n on X by $g_n(x) = v(E_{nx})$ $x \in X$. By Case 1 g_n is a non negative measurable and

$$\int_X g d\mu = (\mu \times v)(E_n) \tag{4}$$

From (3) we get $g(x) = \sum_1^\infty g_n(x) \;\; \forall\, x \in X \Rightarrow g = \sum_1^\infty g_n$. By Monotone Convergence Theorem we get $\int_X g d\mu = \sum_{n=1}^\infty \int_X g_n d\mu = \sum_{i=1}^\infty (\mu \times v)(E_n) = (\mu \times v)\left(\bigcup_1^\infty E_n\right) = (\mu \times v)(E)$ [From (4)].

Case 3 : Assume that $E \in \mathcal{R}_{\sigma\delta}$, Then $E = \bigcap_1^\infty F_n$ where $F_n \in \mathcal{R}_\sigma$, since $\mathcal{R}_\sigma$ is closed for finite intersections, we can assume that $F_n \supset F_{n+1}$ for $n = 1, 2, \ldots$. Then by Caratheodory's Extension Theorem we can find $A \in a_\sigma \;\exists\; E \subset A$ and $(\mu \times v)(A) < (\mu \times v)(E) + 1$ $[\epsilon = 1]$. Define $D_n = A \cap F_n$, thus $D_n \in \mathcal{R}_\sigma$. Define $h_n(x) = v(D_{nx})$ for $x \in X$. By Case 2 the function h_n is non negative measurable and

$$\int_X h_n d\mu = (\mu \times v)(D_n) \tag{5}$$

$$\lim_{n\to\infty}(D_n) = \bigcap_1^\infty D_n = \bigcap_1^\infty (A \cap F_n) = A \cap \left(\bigcap_1^\infty F_n\right) = A \cap E = E \Rightarrow (D_n) \downarrow E \tag{6}$$

$(\mu \times v)(D_n) \leq (\mu \times v)(A) < (\mu \times v)(E) + 1 < \infty$

$$\Rightarrow (\mu \times v)(D_n) \to (\mu \times v)(E) \Rightarrow (\mu \times v)(E) = \lim_{n\to\infty}(\mu \times v)(D_n) \tag{7}$$

From (6) we have $(D_{nx}) \downarrow E_x \Rightarrow v(D_{nx}) \to v(E_x) \Rightarrow g(x) = \lim_{n\to\infty} v(D_{nx}) = \lim_{n\to\infty} h_n(x) \Rightarrow h_n \to g$. As (D_{nx}) is a decreasing sequence it is clear that (h_n) is a decreasing sequence. Thus $0 \leq h_n \leq h_1 \;\; \forall\, n$ [From (6)]

h_1 is integrable, hence by Dominated Convergence Theorem we get

$$\begin{aligned}\int_X g d\mu = \lim_{n\to\infty}\int_X h_n d\mu &= \lim_{n\to\infty}(\mu \times v)(D_n) && \text{[From (5)]}\\ &= (\mu \times v)(E) && \text{[From (7)]}\end{aligned}$$

□

Lemma 1.11. *Let E be a measurable null set with $(\mu \times v)(E) = 0$. Then for almost all x, E_x is measurable and $v(E_x) = 0$.*

Proof. We can find $F \in a_{\sigma\delta}$ and $E \subset F$ such that $(\mu \times v)(F) = (\mu \times v)(E)$ [For $\epsilon > 0$ there exist $A \in a_{\sigma\delta}$ such that $E \subset A$ and $\mu^*(A) = \mu^*(E)] \Rightarrow (\mu \times v)(E) = 0$. Since a is the algebra generated by $\mathcal{R}$ we have $a_{\sigma\delta} = \mathcal{R}_{\sigma\delta} \Rightarrow F \in \mathcal{R}_{\sigma\delta}$. Hence F_x is measurable and g defined by $g(x) = v(F_x)$ is non negative measurable and $\int g d\mu = (\mu \times v)(F) \Rightarrow \int g d\mu = 0 \Rightarrow g = 0$ a.e. $\Rightarrow v(F_x) = 0$ for almost all x. But $E_x \subset F_x$, hence E_x is measurable and $v(E_x) = 0$ for almost all x. □

Proposition 1.12. *Let E be any measurable set of finite measure with $(\mu \times v)(E) < \infty$. Then E_x is measurable for almost all x. If g s a non negative function such that $g(x) = v(E_x)$ whenever E_x is measurable then g is measurable (In fact Integrable) and $\int g d\mu = (\mu \times v)(E)$.*

Proof. Let $F \in \mathcal{R}_{\sigma\delta}$ such that $E \subset F$ and $(\mu \times v)(E) = (\mu \times v)(F)$. Define $G = F - E$. Then G is measurable and $(\mu \times v)(G) = (\mu \times v)(F) - (\mu \times v)(E) = 0$. By the above Lemma G_x is measurable and $v(G_x) = 0$ for almost all x. Then from $G = F - E$ we get $G_x = F_x - E_x \Rightarrow v(G_x) = v(F_x) - v(E_x) \Rightarrow v(F_x) = v(E_x)$ for almost all x. Let h be defined by $h(x) = v(F_x)$ then h is non negative measurable and $\int h d\mu = (\mu \times v)(F)$. But $g(x) = v(E_x) = v(F_x) = h(x)$ for almost all x.

$\Rightarrow g =$ h a.e. Hence g is measurable and $\int g = \int h = (\mu \times v)(F) = (\mu \times v)(E) \Rightarrow \int g d\mu = (\mu \times v)(E)$. □

Theorem 1.13 (Fubinis Theorem)**.** *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, v)$ be two complete measure spaces and $Z = X \times Y$. Suppose f be any integrable function on Z then*

(1) For almost all $x \in X$, the function f_x defined on Y by $f_x(y) = f(x,y)$ is integrable on Y.

(1') For almost all $y \in Y$, the function f_y defined on X by $f_y(x) = f(x,y)$ is integrable on X.

(2) $\int_Y f dv$ is integrable on X.

(2') $\int_X f d\mu$ is integrable on Y.

(3) $\int_X \int_Y f dv d\mu = \int_Z f d(\mu \times v) = \int_Y \int_X f d\mu dv$.

Proof. Because of symmetry it is enough to prove (1), (2) and first part of (3).

First suppose that f is non negative.

Case 1 : Let $f = \mathcal{C}_E$ where E be any measurable set of finite measure, i.e. $(\mu \times v)(E) < \infty$, this gives $f_x = (\mathcal{C}_E)_x = \mathcal{C}_{Ex} \Rightarrow \int_Y f_x dv = \int_Y \mathcal{C}_{Ex} dv = v(E_x)$

Let $g(x) = v(E_x)$, by the proceeding theorem g is non negative integrable and $\int g d\mu = (\mu \times v)(E)$. But $g(x) = v(E_x) = \int_Y f_x dv \Rightarrow \int_X g d\mu = \int_X (\int_Y f_x dv) d\mu \Rightarrow \int_X (\int_Y f dv) d\mu = (\mu \times v)(E) = \int_Z \mathcal{C}_E d(\mu \times v) = \int_Z f d(\mu \times v)$.

g is integrable implies that $g(x)$ is finite for almost all x $\Rightarrow \int_Y f_x dv$ is finite for almost all x $\Rightarrow f_x$ is integrable for almost all x.

Further g is integrable means $\int_Y f dv$ is integrable.

Case 2 : Since integral is a linear operator, it follows from Case 1 that the result holds for all non negative simple functions which vanish outside set of finite measure.

Case 3 : Let f be any non negative integrable function. Let (ϕ_n) be an increasing sequence of non negative simple functions such that each ϕ_n vanish outside a set of finite measure and $(\phi_n) \uparrow f$. Then this gives $(\phi_{nx}) \uparrow f_x$ and by M.C.T. we get

$$\int_Z f d(\mu \times v) = \lim_{n\to\infty} \int (\phi_n) d(\mu \times v) \tag{8}$$

$$\& \int_Y f_x dv = \lim_{n\to\infty} \int_Y (\phi_{nx}) dv \tag{9}$$

Let $g_n = \int_Y (\phi_n) dv$. Then g_n is non negative and measurable and

$$\int_X g_n d\mu = \int_X \left(\int_Y \phi_n dv \right) d\mu. \tag{10}$$

$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \int_Y (\phi_{nx}) dv = \int_Y f_x dv$ [From (9)] i.e. $g_n \uparrow \int_Y f_x dv = h$ (say). By M.C. T. we get

$$\begin{aligned} \int_X (h) d\mu &= \int_X \left(\int_Y f dv \right) d\mu \\ &= \lim_{n\to\infty} \int_X (g_n) d\mu = \lim_{n\to\infty} \int_X \int_Y (\phi_{nx}) dv d\mu \quad \text{[From (10)]} \\ &= \lim_{n\to\infty} \int_Z (\phi_{nx}) d(\mu \times v) \quad \text{[By Case 2]} \\ &= \int_Z (f) d(\mu \times v) \quad \text{[By (8)]} \end{aligned}$$

$\Rightarrow \int_X \int_Y f dv d\mu = \int_Z f d(\mu x v) = \int_Y \int_X f d\mu dv$. □

Theorem 1.14 (Tonelli's Theorem)**.** *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, v)$ be two σ-finite measure spaces and f be any non negative measurable function on $Z = X \times Y$. Then*

(1) For almost all $x \in X$, the function f_x defined on Y by $f_x(y) = f(x,y)$ is non negative measurable

(1') For almost all $y \in Y$, the function f_y defined on X by $f_y(x) = f(x,y)$is non negative measurable.

(2) $\int_Y f dv$ *is non negative measurable on* X.

(2') $\int_X f d\mu$ *is non negative measurable on* Y.

(3) $\int_X (\int_Y f dv) d\mu = \int_Z f d(\mu \times v) = \int_Y (\int_X f d\mu) dv$.

Proof. Because of symmetry it is enough to prove (1), (2)and First part of (3).

Case 1 : suppose $f = \mathcal{C}_E$ where E is a measurable set with $(\mu \times v)(E) < \infty$. Then for almost all $x \in X$, E_x is measurable. Let $g(x) = v(E_x)$, whenever E_x is measurable and $g(x) = 0$ otherwise. Then g is a non negative measurable function and $\int_X g d\mu = (\mu \times v)(E)$. Since E_x is measurable for almost all x and $f_x = (\mathcal{C}_{\dot{E}})_x = \mathcal{C}_{E_x}$. It follows that f_x is non negative measurable for all x. Further $\int_Y f_x dv = \int_Y C_{E_x} dv = v(E_x) \Rightarrow \int_Y f_x dv = g(x)$ for almost all x.

$\Rightarrow \int_Y f dv$ is also non negative measurable and $\int_X (\int_Y f dv) d\mu = \int_X g d\mu = (\mu \times v)(E) = \int_Z \mathcal{C}_E d(\mu \times v) = \int_Z f d(\mu \times v)$.

Case 2 : Since integral is a linear operator, therefore the theorem holds for all non negative simple functions which vanish outside the set of finite measure.

Case 3 : Let f be any non negative measurable function. Since μ and v are σ-finite we see that $\mu \times v$ is also $\sigma-$finite hence there exists an increasing sequence (ϕ_n) of non negative simple functions such that $\phi_n \uparrow f$ and each ϕ_n vanishes outside a set of finite measure. By M.C. T. we get

$$\int_Z f d(\mu \times v) = \lim_{n \to \infty} \int_Z (\phi_n) d(\mu \times v). \tag{11}$$

As $\phi_n \uparrow f$ it follows $0 \leq \phi_{nx} \uparrow f_x$ again by M.C.T. we obtain

$$\int_Y f_x dv = \lim_{n \to \infty} \int_Y (\phi_{nx}) dv \tag{12}$$

Define $g_n(x) = \int_Y (\phi_{nx}) dv$ for $x \in X$. Then (g_n) is an increasing sequence of non negative measurable functions and

$$\begin{aligned} \lim_{n \to \infty} g_n(x) &= \lim_{n \to \infty} \int_Y (\phi_{nx}) dv \quad \text{[From (12)]} \\ &= \lim_{n \to \infty} \int_Y (f_n) dv = h \text{ say} \end{aligned}$$

Then $g_n(x) \uparrow h$. By M.C.T. $\int_X h d\mu = \int_X (\int_Y f_x dv) d\mu = \lim_{n \to \infty} \int_X (g_n(x)) d\mu = \lim_{n \to \infty} \int_X (\int_Y \phi_n dv) d\mu = \lim_{n \to \infty} \int_Z (\phi_n)(\mu \times v) = \int_Z f d(\mu \times v)$ [From (11)]. This proves the theorem. □

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