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Tietze's Extension Theorem for Intuitionistic Fuzzy $\zeta-$Basically Disconnected Spaces in Katetov-Tong's Sense

Research Article

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Abstract: In this paper we initiate the concept of $\zeta-$basically disconnectedness in intuitionistic fuzzy topological spaces. We also apply these notions of $\zeta-$basically disconnectedness to analyse and prove Tietze extension theorem.

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1. Introduction

The concept of fuzzy sets was introduced by Zadeh [14]. Fuzzy basically disconnected spaces was discussed and studied in [12]. Bruce Hutton [7] constructed an interesting L-fuzzy topological space called L-fuzzy unit interval which plays the same role in fuzzy topology . Using the concept of L-fuzzy unit interval, Tomasz Kubiaz [10, 11] extended the Urysohn lemma and Tietze extension theorem for L-fuzzy normal spaces. Atanassov [1] generalised intuitionistic fuzzy sets using the notion of fuzzy sets. On the other hand Coker [5] introduced the notion of an intuitionistic fuzzy topological spaces. In this paper we introduce and study the concept of an intuitionistic fuzzy ζ-space and intuitionistic fuzzy ζ-basically disconnected space. An approach to Tietze's extension theorem for intuitionistic fuzzy ζ-basically disconnected spaces has been established based on Kotetov and Tong [8, 9, 13].

2. Preliminaries

Definition 2.1 ([5]). *An intuitionistic fuzzy set (IFS, in short) A in X is an object having the form $A = \{x, \mu_A(x), \upsilon_A(x)/x \in X\}$ where the functions $\mu_A : X \to I$ and $\upsilon_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\upsilon_A(x)$) of each element $x \in X$ to the set A on a nonempty set X and $0 \leq \mu_A(x) + \upsilon_A(x) \leq 1$ for each $x \in X$. Obviously every fuzzy set A on a nonempty set X is an IFS's A and B be in the form $A = \{x, \mu_A(x), 1 - \mu_A(x)/x \in X\}$.*

Definition 2.2 ([5]). *Let X be a nonempty set and the IFS's A and B be in the form $A = \{x, \mu_A(x), \upsilon_A(x)/x \in X\}$, $B = \{x, \mu_B(x), \upsilon_B(x)/x \in X\}$ and let $A = \{A_j : j \in J\}$ be an arbitrary family of IFS's in X. Then we define*

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(i) $A \subseteq B$ *if and only if* $\mu_A(x) \leq \mu_B(x)$ *and* $\upsilon_A(x) \geq \upsilon_B(x)$ *for all* $x \in X$.

(ii) $A = B$ *if and only if* $A \subseteq B$ *and* $B \subseteq A$.

(iii) $\overline{A} = \{x, \upsilon_A(x), \mu_A(x)/x \in X\}$. $\overline{A}$ *is the complement of* A.

(iv) $A \cap B = \{x, \mu_A(x) \cap \mu_B(x), \upsilon_A(x) \cup \upsilon_B(x)/x \in X\}$.

(v) $A \cup B = \{x, \mu_A(x) \cup \mu_B(x), \upsilon_A(x) \cap \upsilon_B(x)/x \in X\}$.

(vi) $1_{\sim} = \{\langle x, 1, 0\rangle\, x \in X\}$ *and* $0_{\sim} = \{\langle x, 0, 1\rangle\, x \in X\}$.

Definition 2.3 ([5])**.** *An intuitionistic fuzzy topology (IFT, in short) on a nonempty set X is a family* τ *of an intuitionistic fuzzy set (IFS, in short) in X satisfying the following axioms:*

(i) $0_{\sim}, 1_{\sim} \in \tau$.

(ii) $A_1 \cap A_2 \in \tau$ *for any* $A_1, A_2 \in \tau$.

(iii) $\cup A_j \in \tau$ *for any* $A_j : j \in J \subseteq \tau$.

In this paper we denote intuitionistic fuzzy topological space (IFTS, in short) by (X, τ), (Y, κ) or X, Y. Each IFS which belongs to τ is called an intuitionistic fuzzy open set (IFOS, in short) in X. The complement $\overline{A}$ of an IFOS A in X is called an intuitionistic fuzzy closed set (IFCS, in short). An IFS X is called intuitionistic fuzzy clopen (IF clopen) iff it is both intutionistic fuzzy open and intuitionistic fuzzy closed.

Definition 2.4 ([5])**.** *Let* (X, τ) *be an IFTS and* $A = \{x, \mu_A(x), \upsilon_A(x)\}$ *be an IFS in X. Then the fuzzy interior and closure of A are denoted by*

(i) $cl(A) = \bigcap\{K\text{: } K \text{ is an IFCS in } X \text{ and } A \subseteq K\}$.

(ii) $int(A) = \bigcup\{G\text{: } G \text{ is an IFOS in } X \text{ and } G \subseteq A\}$.

Note that, for any IFS A in (X, τ), we have $cl(\overline{A}) = \overline{\text{int}(A)}$ and $\text{int}(\overline{A}) = \overline{cl(A)}$.

Definition 2.5 ([6])**.** *Let a and b be two real numbers in [0,1] satisfying the inequality* $a + b \leq 1$. *Then the pair* $< a, b >$ *is called an intuitionistic fuzzy pair. Let* $< a_1, b_1 >, < a_2, b_2 >$ *be two intuitionistic fuzzy pairs. Then*

(i) $< a_1, b_1 > \leq < a_2, b_2 >$ *if and only if* $a_1 \leq a_2$ *and* $b_1 \geq b_2$.

(ii) $< a_1, b_1 > = < a_2, b_2 >$ *if and only if* $a_1 = a_2$ *and* $b_1 = b_2$.

(iii) If $\{< a_i, b_i >: i \in J\}$ *is a family of intuitionistic fuzzy pairs, then* $\cup < a_i, b_i > = < \cup a_i, \cap b_i >$ *and* $\cap < a_i, b_i > = < \cap a_i, \cup b_i >$.

(iv) The complement of an intuitionistic fuzzy pair $< a, b >$ *is the intuitionistic fuzzy pair defined by* $\overline{< a, b >} = < b, a >$

(v) $1^{\sim} = < 1, 0 >$ *and* $0^{\sim} = < 0, 1 >$.

Definition 2.6 ([4])**.** *Let X be a nonempty set and* $A \subset X$. *The characteristic function of A is denoted and defined by*

$$\chi_A(x) = \begin{cases} 1, x \in A \\ 0, x \notin A \end{cases}.$$

Definition 2.7 ([2]). *Let (X, τ) be a fuzzy topological space and be a fuzzy set in X. Then λ is called fuzzy G_δ if $\lambda = \bigwedge_{i=1}^{\infty} \lambda_i$ where each $\lambda_i \in T$. The complement of fuzzy G_δ is F_σ.*

Definition 2.8 ([3]). *A fuzzy bitopological space (X, τ_1, τ_2) is said to be pairwise fuzzy basically disconnected if τ_1-closure of each τ_2-fuzzy open, τ_2-fuzzy F_σ is τ_2-fuzzy open and τ_2-closure of each τ_1-fuzzy open, τ_1-fuzzy F_σ is τ_1-fuzzy open.*

Definition 2.9 ([5]). *An IFTS X is called fuzzy compact if and only if every fuzzy open cover has a finite subcover.*

3. An Intuitionistic Fuzzy $\zeta-$basically Disconnected Spaces

Definition 3.1. *Let (X, τ) be an intuitionistic fuzzy noncompact space. Let Ψ be a collection of all intuitionistic fuzzy sets which are both intuitionistic fuzzy closed set and intuitionistic fuzzy compact set in X. Let $U^+ = \{A \in \Psi : A \cap U \neq 0_\sim, U \in \tau\}$ and $V^- = \{A \in \Psi : A \cap V = 0_\sim : V$ is an intuitionistic fuzzy compact set in $X\}$. Then the collection $\zeta = \{A : \overline{A} \in U^+\} \cup \{B : \overline{B} \in V^-\}$ is said to be intuitionistic fuzzy $\zeta-$structure on X and the pair (X, ζ) is said to be an intuitionistic fuzzy $\zeta-$space.*

Notation 3.2. *Each member of an intuitionistic fuzzy $\zeta-$structure is an intuitionistic fuzzy $\zeta-$open set. The complement of an intuitionistic fuzzy $\zeta-$open set is an intuitionistic fuzzy $\zeta-$closed set.*

Example 3.3. *Let $X = \{a, b\}$ be a nonempty set. Let $G_1 =< x, (0.4, 0.4), (0.4, 0.4) >$ and $G_2 =< x, (0.5, 0.5), (0.4, 0.4) >$ be IFS of X. Then the family $\tau = \{0_\sim, 1_!, G_1, G_2\}$ be IFT on X. Thus (X, τ) be an intuitionistic fuzzy noncompact space. Now, $\Psi = \{0_\sim, 1_\sim, \overline{G_1}, \overline{G_2}\}$ is the collection of all IFS which are both intuitionistic fuzzy closed set and intuitionistic fuzzy compact set in X. Consider $U^+ = \{\overline{G_1}, \overline{G_2}, 1_\sim\}$ and $V^- = \{\overline{0_\sim}, \overline{G_1}, \overline{G_2}, 1_\sim\}$. Then $\zeta = \{0_\sim, 1_\sim, \overline{G_1}, \overline{G_2}\}$ is an intuitionistic fuzzy $\zeta-$structure. Thus (X, ζ) is an intuitionistic fuzzy $\zeta-$space.*

Definition 3.4. *Let (X, ζ) be an intuitionistic $\zeta-$fuzzy space. For an IFS A on X, the intuitionistic fuzzy $\zeta-$closure of A and the intuitionistic fuzzy $\zeta-$interior of A are defined by $IF\zeta cl(A) = \cap\{B : B =< x, \mu_B, \upsilon_B >$ is an $IF\zeta CS$ in X and $A \subseteq B\}$, $IF\zeta\text{int}(A) = \cup\{B : B =< x, \mu_B, \upsilon_B >$ is an $IF\zeta OS$ in X and $B \subseteq A\}$.*

Remark 3.5. *Let (X, ζ) be an intuitionistic fuzzy $\zeta-$space. Then for any IFS A in X,*

(i) $IF\zeta\text{int}(A) \subseteq A \subseteq IF\zeta cl(A)$

(ii) $IF\zeta cl(\overline{A}) = \overline{IF\zeta\text{int}(A)}$, $IF\text{int}(\overline{A}) = \overline{IF\zeta cl(A)}$.

Definition 3.6. *Let (X, ζ) be an intuitionistic fuzzy $\zeta-$space. An IFS A is said to be an intuitionistic fuzzy ζG_δ set (in short, $IF\zeta G_\delta S$) if $A = \cap_{i=1}^{\infty} A_i$, where each $A_i \in \zeta$. The complement of intuitionistic fuzzy ζG_δ set is said to be an intuitionistic fuzzy ζF_δ set (in short, $IF\zeta F_\delta S$) set.*

Notation 3.7. *Let (X, ζ) be an intuitionistic fuzzy $\zeta-$space.*

(i) An intuitionistic fuzzy ζ open F_δ (in short, $IF\zeta OF_\delta$) set is an IFS which is both $IF\zeta OS$ and $IF\zeta F_\delta$.

(ii) An intuitionistic fuzzy ζ closed G_δ (in short, $IF\zeta CG_\delta$) set is an IFS which is both $IF\zeta CS$ and $IF\zeta G_\delta$.

(iii) An intuitionistic fuzzy ζ closed open $G_\delta F_\sigma$ (in short, $IF\zeta COGF$) set is an intuitionistic fuzzy set which is both $IF\zeta OF_\sigma$ and $IF\zeta CG_\sigma$.

Definition 3.8. *Let (X, ζ) be an intuitionistic fuzzy $\zeta-$space. Then (X, τ) is said to be an intuitionistic fuzzy $\zeta-$basically disconnected space, if the intuitionistic fuzzy ζ closure of every $IF\zeta OF_\sigma$ set is an $IF\zeta OS$.*

Example 3.9. *In the above example 3.3, (X,ζ) is intuitionistic fuzzy $\zeta-$basically disconnected space.*

Theorem 3.10. *Let (X,ζ) be an intuitionistic fuzzy $\zeta-$space. Then the following statements are equivalent.*

(i) (X,ζ) is an intuitionistic fuzzy $\zeta-$basically disconnected space.

(ii) For each $IF\zeta CG_\delta$ set A, we have $IF\zeta \text{int}(A)$ is an $IF\zeta CS$.

(iii) For each $IF\zeta OF_\sigma$ set A, we have $IF\zeta cl(IF\zeta \text{int}(\overline{A})) = \overline{IF\zeta cl(A)}$.

(iv) For an $IF\zeta OF_\sigma$ set A and for any IFS B with $IF\zeta cl(A) = \overline{B}$, we have $IF\zeta cl(B) = \overline{IF\zeta cl(A)}$.

Proof. $(i) \Rightarrow (ii)$. Let A be an $IF\zeta CG_\delta$ set. Then $\bar{A}$ is an $IF\zeta OF_\sigma$ set. By assumption (i) $IF\zeta Cl(\bar{A})$ is an $IF\zeta OS$. Now, $IF\zeta cl(\overline{A}) = \overline{IF\zeta \text{int}(A)}$. Hence $IF\zeta \text{int}(A)$ is an $IF\zeta CS$.
$(ii) \Rightarrow (iii)$. Let A be an $IF\zeta OF_\sigma$ set. Then $\bar{A}$ is an $IF\zeta CG_\delta$ set. By assumption (ii) $IF\zeta \text{int}(\overline{A})$ is an $IF\zeta CS$. Consider $IF\zeta cl(IF\zeta \text{int}(\overline{A})) = IF\zeta \text{int}(\overline{A}) = \overline{IF\zeta cl(A)}$.
$(iii) \Rightarrow (iv)$. Let A be an $IF\zeta OF_\sigma$ set and for any IFS B such that $IF\zeta cl(A) = \overline{B}$. By (iii), $IF\zeta cl(IF\zeta \text{int}(\overline{A})) = \overline{IF\zeta cl(A)} = IF\zeta \text{int}(\overline{A})$. That is, $IF\zeta cl(B) = IF\zeta \text{int}(\overline{A})$.
$(iv) \Rightarrow (i)$. Let A be any $IF\zeta OF_\sigma$ set. Let $\overline{IF\zeta cl(A)} = B$. By (iv), it follows that $IF\zeta cl(B) = \overline{IF\zeta cl(A)}$. That is, $\overline{IF\zeta cl(A)}$ is an $IF\zeta CS$. This implies that $IF\zeta cl(A)$ is an $IF\zeta OS$. Hence (X,ζ) is intuitionistic fuzzy $\zeta-$basically disconnected space. □

Theorem 3.11. *Let (X,ζ) be an intuitionistic fuzzy ζ space. Then (X,ζ) is intuitionistic fuzzy $\zeta-$basically disconnected space if and only if for each $IF\zeta OF_\sigma$ set A and $IF\zeta CG_\delta$ set B such that $A \subseteq B$, $IF\zeta cl(A) \subseteq IF\zeta \text{int}(B)$.*

Proof. Let A be an $IF\zeta OF_\sigma$ set and B be $IF\zeta CG_\delta$ set such that $A \subseteq B$. Then by (ii) of Theorem 3.10, $IF\zeta \text{int}(B)$ is an $IF\zeta CS$. Also, since A is an $IF\zeta OF_\sigma$ set, $IF\zeta cl(A) \subseteq IF\zeta \text{int}(B)$.
Conversely, Let B be any $IF\zeta CG_\delta$ set. Then $IF\zeta int(B)$ is an $IF\zeta OS$ and $IF\zeta int(B) \subseteq B$. By assumption, $IF\zeta cl(IF\zeta \text{int}(B)) \subseteq IF\zeta \text{int}(B)$. Also we know that $IF\zeta cl(IF\zeta \text{int}(B)) \subseteq IF\zeta \text{int}(B)$. This implies that $IF\zeta cl(IF\zeta \text{int}(B)) = IF\zeta \text{int}(B)$. Therefore, $IF\zeta \text{int}(B)$ is an $IF\zeta CS$. Hence by (ii) of Theorem 3.10, it follows that (X,ζ) is intuitionistic fuzzy $\zeta-$basically disconnected space. □

Remark 3.12. *Let (X,ζ) be an intuitionistic fuzzy $\zeta-$basically disconnected space. Let $\{A_i, \overline{B_i}/i \in N\}$ be collection such that A_i's are $IF\zeta OF_\sigma$ and B_i's are $IF\zeta CG_\sigma$ sets. If $A_i \subseteq A \subseteq B_j$ and $A_i \subseteq A \subseteq B_j$ for all $i,j \in N$, then there exists an $IF\zeta COGF$ set C such that $IF\zeta cl(A_i) \subseteq C \subseteq IF\zeta \text{int}(B_j)$ for all $i,j \in N$.*

Proof. By Theorem 3.11, $IF\zeta cl(A_i) \subseteq IF\zeta cl(A) \cap IF\zeta \text{int}(B) \subseteq IF\zeta \text{int}(B_j)$ for all $i,j \in N$. Letting $C = IF\zeta cl(A) \cap IF\zeta \text{int}(B)$ in the above, we have C is an $IF\zeta COGF$ set satisfying the required conditions. □

Theorem 3.13. *Let (X,ζ) be an intuitionistic fuzzy $\zeta-$basically disconnected space. Let $\{A_q\}_{q\in Q}$ and $\{B_q\}_{q\in Q}$ be monotone increasing collections of $IF\zeta OF_\sigma$ an sets and $IF\zeta CG_\delta$ of (X,τ). Suppose that $A_{q1} \subseteq B_{q2}$ whenever $q_1 < q_2$ (Q is the set of all rational numbers). Then there exists a monotone increasing collection $\{C_q\}_{q\in Q}$ of an $IF\zeta COGF$ sets of (X,τ) such that $IF\zeta cl(A_{q1}) \subseteq C_{q2}$ and $C_{q1} \subseteq IF\zeta \text{int}(B_{q2})$ whenever $q_1 < q_2$.*

Proof. Let us arrange all rational numbers into a sequence $\{q_n\}$ (without repetitions). For every $n \geq 2$, we shall define inductively a collection $\{C_{qi}/1 \leq i < n\} \subset \Omega^X$ such that $IF\zeta cl(A_q) \subseteq C_{qi}$, $C_{qi} \subseteq IF\zeta \text{int}(B_q)$ if $q_i < q$, for all $i < n$. By Theorem 3.11 the countable collections $\{IF\zeta cl(A_q)\}$ and $\{IF\zeta int(B_q)\}$ satisfy $IF\zeta cl(A_{q1}) \subseteq IF\zeta \text{int}(B_{q2})$ if $q_1 < q_2$. By Remark 3.12, there exists an $IF\zeta COGF$ set D_1 such that $IF\zeta cl(A_{q1}) \subseteq D_1 \subseteq IF\zeta \text{int}(B_{q2})$.

Letting $C_{q1} = D_1$, we get (S_2). Assume that $IFSs\ C_{qi}$, are already defined for $i < n$ and satisfy (S_n). Define $E = \cup\{C_{qi}/i < n, q_i < q_n\} \cup A_{qn}$ and $F = \cap\{C_{qj}/j < n, q_j < q_n\} \cup B_{qn}$. Then $IF\zeta cl(C_{qi}) \subseteq IF\zeta cl(E) \subseteq IF\zeta \text{int}(C_{qj})$ and $IF\zeta cl(C_{qi}) \subseteq IF\zeta \text{int}(F) \subseteq IF\zeta \text{int}(C_{qj})$ whenever $q_i < q_n < q_j (i, j < n)$, as well as $A_q \subseteq IF\zeta cl(E) \subseteq B'_q)$ and $A_q \subseteq IF\zeta \text{int}(F) \subseteq B'_q)$ whenever $q < q_n < q'$. This shows that the countable collections $\{C_{qi}/i < n, q_i < q_n\} \cup A_q/q < q_n\}$ and $\{C_{qj}/j < n, q_j < q_n\} \cup \{B_q/q > q_n\}$ together with E and F fulfil the conditions of Remark 3.12. Hence, there exists an $IF\zeta COGF$ set D_n such that if $IF\zeta cl(D_n) \subseteq B_q$, $q_n < q$, $A_q \subseteq IF\zeta \text{int}(D_n)$, $q < q_n$, $IF\zeta cl(C_{qi}) \subseteq IF\zeta \text{int}(D_n)$ if $q_i < q_n$, $IF\zeta cl(D_n) \subseteq IF\zeta \text{int}(C_{qi})$ if $q_n < q_j$ where $1 \leq i, j \leq n-1$. Letting $C_{qn} = D_n$ we obtain IFSs $C_{q1}, C_{q2}, \dots C_{qn}$ that satisfy (S_{n+1}). Therefore, the collection $\{C_{qi}/i = 1, 2, \dots\}$ has the required property. □

4. Tietze's Extension Theorem for an Intuitionistic Fuzzy Basically Disconnected Space

Notation 4.1. *The family of all IFSs in $\Re$ is denoted by $\zeta_{\Re}$.*

Definition 4.2. *An intuitionistic fuzzy real line is the set of all monotone decreasing IFS $A \in \zeta_{\Re}$ satisfying $\cup\{A(t) : t \in \Re\} = 1_{\sim}$ and $\cap\{A(t) : t \in \Re\} = 0_{\sim}$ after the identification of an IFSs $A, B \in \Re_I(I)$ if and only if $A(t-) = B(t-)$ and $A(t+) = B(t+)$ for all $t \in \Re$ where $A(t-) = \cap\{A(s) : s < t\}$ and $A(t+) = \cup\{A(s) : s > t\}$. The intuitionistic fuzzy unit interval $I(I)$ is a subset of $\Re(I)$ such that $[A] \in I(I)$ if the membership and nonmembership of an IFS line $\Re(I)A \in \zeta_R$ are defined by $\mu_A(t) = \begin{cases} 1, & t < 0; \\ 0, & t > 1. \end{cases}$ and $\upsilon_A(t) = \begin{cases} 0, & t < 0; \\ 1, & t > 1. \end{cases}$ respectively.*
The natural intuitionistic fuzzy topology on $\Re(I)$ is generated from the subbasis $\{L_t, R_t : s < t\}$ where $\text{Lt}, \text{Rt} : \Re(\text{I}) \to \text{I}(\text{I})$ *are given by $L_t[A] = \overline{A(t-)}$ and $R_t[A] = A(t+)$ respectively.*

Definition 4.3. *Let (X, ζ) be intuitionistic fuzzy $\zeta-$space. A function $f : X \to \Re(I)$ is said to be lower (resp. upper) intuitionistic fuzzy $\zeta-$continuous function if $f^{-1}(\Re_t)(f^{-1}(L_t))$ is an $IF\zeta OF_\sigma$ set, for each $t \in \Re$.*

Notation 4.4. *Let X be any nonempty set and $A \in \zeta^X$. Then for $x \in X$, $< \mu_A(x), \upsilon_A(x) >$ is denoted by $A^\sim$.*

Definition 4.5. *Let X be any nonempty set. An intuitionistic fuzzy characteristic function of an IFS $A \in \zeta^X$ is a map $\psi_A : X \to I(I)$ is defined by $\psi_A(x) = A^\sim$, for each $x \in X$, $t \in \Re$.*

Theorem 4.6. *Let (X, τ) be intuitionistic fuzzy ζ space and let A be an IFS in X. Let $f : X \to \Re(I)$ be such that $f(x)(t) = \begin{cases} 1^\sim, t < 0 \\ A^\sim, 0 \leq t \leq 1 \\ 0^\sim, t > 1 \end{cases}$ for all $x \in X$ and $t \in \Re$. Then f is an lower (resp.upper) intuitionistic fuzzy $\zeta-$continuous function if and only if A is $IF\zeta OF_\sigma$ set.*

Proof. $f^{-1}(\Re_t) = \begin{cases} 1^\sim, t < 0 \\ A^\sim, 0 \leq t \leq 1 \\ 0^\sim, t > 1 \end{cases}$ implies that f is an lower intuitionistic fuzzy $\zeta-$continuous function if and only if A is an $IF\zeta OF_\sigma$ set.
$f^{-1}(\overline{L_t}) = \begin{cases} 1^\sim, t < 0 \\ A^\sim, 0 \leq t \leq 1 \\ 0^\sim, t > 1 \end{cases}$ implies that f is an upper intuitionistic fuzzy $\zeta-$continuous function if and only if A is an $IF\zeta CG_\delta$ set . Hence the proof is complete. □

Remark 4.7. *Let (X, ζ) be intuitionistic fuzzy $\zeta-$space. Let ψ_A be an intuitionistic fuzzy characteristic function of an IFS A in X. Then ψ_A is a lower (resp. upper) intuitionistic fuzzy ζ continuous function if and only if A is an $IF\zeta OF_\sigma$ set.*

Definition 4.8. *Let (X,ζ) be intuitionistic fuzzy $\zeta-$space. A function $f : X \to \Re(I)$ is said to be a strongly intuitionistic fuzzy $\zeta-$continuous function if $f^{-1}(\Re_t)$ is an $IF\zeta OF_\sigma$ and $f^{-1}(\overline{L_t})$ is both $IF\zeta OF_sigma$ and $IF\zeta CG_\delta$ for each $t \in \Re$.*

Notation 4.9. *The collection of all strongly intuitionistic fuzzy $\zeta-$continuous functions in an intuitionistic fuzzy ζ space (X,ζ) with values in $I(I)\zeta_s$.*

Theorem 4.10 (Intuitionistic Fuzzy $\zeta-$insertion Theorem)**.** *Let (X,ζ) be an intuitionistic fuzzy $\zeta-$space. Then the following statements are equivalent.*

(i) (X,ζ) is an intuitionistic fuzzy $\zeta-$basically disconnected space.

(ii) If $g,h : X \to R(I)$, g is lower intuitionistic fuzzy $\zeta-$continuous function, h is upper intuitionistic fuzzy $\zeta-$continuous function and $g \subseteq$, then there exists an $f \in \zeta_s$ such that $g \subseteq f \subseteq h$.

(iii) If A and B are $IF\zeta OF_\sigma$ sets such that $B \subseteq A$, then there exists strongly intuitionistic fuzzy $\zeta-$continuous function $f : X \to \Re(I)$ such that $B \subseteq f^{-1}(\overline{L_1}) \subseteq f^{-1}(R_0) \subseteq A$.

Proof. $(i) \Rightarrow (ii)$ Define $A_r = h^{-1}(L_r)$ and $B_r = g^{-1}(\overline{R_r})$, for all $r \in Q$ (Q is the set of all rationals). Clearly, $\{A_r\}_{r\in Q}$ and $\{B_r\}_{r\in Q}$ are monotone increasing families of an B sets and $IF\zeta CG_\delta$ sets of (X,ζ). Moreover $A_r \subseteq B_s$ if $r < s$. By Theorem 3.13, there exists a monotone increasing family $\{C_r\}_{r\in Q}$ of an $IF\zeta COGF$ sets of such (X,ζ) that $IF\zeta cl(A_r) \subseteq C_S$ and $C_r \subseteq IF\zeta\text{int}(B_S)$ whenever $r < s$ $(r,s \in Q)$. Letting $V_t = \cap_{r<t}\overline{C_r}$ for $t \in \Re$, we define a monotone decreasing family $\{V_t/t \in R\} \subseteq \zeta_x$. Moreover we have $IF\zeta cl(V_t) \subseteq IF\zeta\text{int}(Vs)$ whenever $s < t$. We have,

$$\begin{aligned}\bigcup_{t\in R} V_t &= \bigcup_{t\in R}\bigcap_{r<t}\overline{C_r} \supseteq \bigcup_{t\in R}\bigcap_{r<t}\overline{B_r} = \bigcup_{t\in R}\bigcap_{r<t} g^{-1}(R_r)\\ &= \bigcup_{t\in R} g^{-1}(\overline{L_t}) = g^{-1}(\bigcup_{t\in R}\overline{L_t}) = 1_\sim\end{aligned}$$

Similarly, $\bigcap_{t\in R} V_t = 0_\sim$. Now define a function $f : X \to \Re(I)$ possessing required conditions. Let $f(x)(t) = Vt(x)$, for all $x \in X$ and $t \in \Re$. By the above discussion, it follows that f is well defined. To prove f is a strongly intuitionistic fuzzy $\zeta-$continuous function. Observe that $\bigcup_{s>t} V_S = \bigcup_{s>t} IF\zeta\text{int}(V_S)$ and $\bigcap_{s<t} V_S = \bigcap_{s<t} IF\zeta cl(V_S)$. Then $f^{-1}(R_t) = \bigcup_{s>t} V_S = \bigcup_{s>t} IF\zeta\text{int}(V_S)$ is an $IF\zeta COGF$ and $f^{-1}(\overline{L_t}) = \bigcap_{s<t} V_S = \bigcap_{s<t} IF\zeta cl(V_s)$ is an $IF\zeta COGF$ set. Therefore, f is strongly intuitionistic fuzzy $\zeta-$continuous function. To conclude the proof it remains to show that $g \subseteq f \subseteq h$. That is $g^{-1}(\overline{L_t}) \subseteq f^{-1}(\overline{L_t}) \subseteq h^{-1}(\overline{L_t})$ and $g^{-1}(R_t) \subseteq f^{-1}(R_t) \subseteq h^{-1}(R_t)$ for each $t \in \Re$. We have,

$$\begin{aligned}g^{-1}(\overline{L_t}) &= \bigcap_{s<t} g^{-1}(\overline{L_s}) = \bigcap_{s<t}\bigcap_{r<s} g^{-1}(R_r)\\ &= \bigcap_{s<t}\bigcap_{r<s}\overline{B_r} \subseteq \bigcap_{s<t}\bigcap_{r<s}\overline{C_r} = \bigcap_{s<t} V_s = f^{-1}(\overline{L_t}).\end{aligned}$$

And

$$\begin{aligned}f^{-1}(\overline{L_t}) &= \bigcap_{s<t} V_s = \bigcap_{s<t}\bigcap_{r<s}\overline{C_r} \subseteq \bigcap_{s<t}\bigcap_{r<s}\overline{A_r}\\ &= \bigcap_{s<t}\bigcap_{r<s} h^{-1}(\overline{L_r}) = \bigcap_{s<t} h^{-1}(\overline{L_s}) = h^{-1}(\overline{L_t})\end{aligned}$$

Similarly,

$$\begin{aligned}g^{-1}(R_t) &= \bigcup_{s>t} g^{-1}(R_s) = \bigcup_{s<t}\bigcup_{r<s} g^{-1}(R_r)\\ &= \bigcup_{s>t}\bigcup_{r>s}\overline{B_r} \subseteq \bigcup_{s>t}\bigcap_{r<s}\overline{C_r} = \bigcap_{s>t} V_s = f^{-1}(R_t)\end{aligned}$$

And

$$f^{-1}(R_t) = \bigcup_{s>t} V_s = \bigcup_{s>t}\bigcap_{r<s} \overline{C_r} \subseteq \bigcup_{s>t}\bigcup_{r>s} \overline{A_r}$$
$$= \bigcup_{s>t}\bigcup_{r>s} h^{-1}(\overline{L_r}) = \bigcup_{s.t} h^{-1}(R_s) = h^{-1}(R_t)$$

Hence the condition (ii) is proved.

$(ii) \Rightarrow (iii)$ Let $\bar{A}$ be an $IF\zeta OF_\sigma$ set and B be an $IF\zeta CG_\delta$ set such that $B \subseteq A$. Then $\psi_B \subseteq \psi_A$, where ψ_A, ψ_B are lower and upper intuitionistic fuzzy $\zeta-$continuous functions respectively. By (ii), there exists a strongly intuitionistic fuzzy $\zeta-$continuous function $f : X \to I(I)$ such that $\psi_B \subseteq f \subseteq \psi_A$. Clearly $f(x) \in I(I)$ for all $x \in X$ and $B = \psi_B^{-1}(\overline{L_1}) \subseteq f^{-1}(\overline{L_1}) \subseteq f^{-1}(R_0) \subseteq \psi A^{-1}(R_0) = A$. Therefore, $B \subseteq f^{-1}(\overline{L_1}) \subseteq f^{-1}(R_0) \subseteq \psi A^{-1}(R_0) = A$.

$(iii) \Rightarrow (i)$. Since $f^{-1}(\overline{L_1})$ and $f^{-1}(R_0)$ are $IF\zeta COGF$ sets and by Theorem 3.11, (X, τ) is an intuitionistic fuzzy ζ basically disconnected space. □

Notation 4.11. *Let X be any nonempty set. Let $A \subset X$. Then an IFS ψ_A^* is of the form $< x, \psi_A(x), 1 - \psi_A(x)$.*

Theorem 4.12 (Tietze's Extension Theorem)**.** *Let (X, ζ) is an intuitionistic fuzzy $\zeta-$basically disconnected space. Let $A \subset X$ such that ψ_A^* is an $IF\zeta OF_\sigma$ set in X. Let $f : (A, \zeta/A) \to I(I)$ be a strongly intuitionistic fuzzy $\zeta-$continuous function. Then f has strongly intuitionistic fuzzy $\zeta-$continuous extension over (X, τ).*

Proof. Let $g, h : X \to I(I)$ be such that $g = f = h$ on A and $g(x) = 0_\sim$, $h(x) = 1_\sim$ if $x \notin A$. For every $t \in \Re$, we have,

$$g^{-1}(R_t) = \begin{cases} B_t \cap \psi_A^*, t \geq 0 \\ 1_\sim, t < 0 \end{cases}$$

where B_t is an $IF\zeta COGF$ set such that $\mathrm{B_t}/A = f^{-1}(R_t)$ and

$$h^{-1}(L_t) = \begin{cases} c_t \cap \psi_A^*, t \leq 1 \\ 1_\sim, t > 1 \end{cases}.$$

Where C_t is an $IF\zeta COGF$ set such that $\mathrm{C_t}/A = f^{-1}(L_t)$. Thus g is lower intuitionistic fuzzy $\zeta-$continuous function and h is upper intuitionistic fuzzy $\zeta-$continuous function with $g \subseteq h$. By Theorem 4.10, there is a strongly fuzzy $\zeta-$continuous function $F : X \to I(I)$ such that $g \subseteq F \subseteq h$. Hence $F \equiv f$ on A. □

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