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Solution and Ulam - Hyers Stability of an Additive - Quadratic Functional Equation in Banach Space: Hyers Direct and Fixed Point Methods

Research Article

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Abstract: In this paper, the authors establish the general solution and generalized Ulam - Hyers stability of an additive quadratic functional equation

$$\begin{aligned} f(x+2y+3z)+f(x-2y+3z)+f(x+2y-3z)+f(x-2y-3z) \\ =4f(x)+8[f(y)+f(-y)]+18[f(z)+f(-z)] \end{aligned}$$

in Banach spaces, using the Hyers direct and fixed point methods.

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1. Introduction

The investigation of stability problems for functional equations is related to the famous Ulam problem [34] (in 1940), concerning the stability of group homomorphisms, which was first solved by D. H. Hyers [14], in 1941. This stability problem was further generalized by several authors [2, 11, 29, 31, 33]. We cite also other pertinent research works [1, 3–8, 10, 13, 15, 21, 23, 28, 32]. The general solution and the generalized Hyers-Ulam stability for the **quadratic- additive type functional equation**

$$f(x+ay)+af(x-y)=f(x-ay)+af(x+y) \tag{1}$$

for any positive integer a with $a \neq -1, 0, 1$ was discussed by K.W. Jun and H.M. Kim [18]. Also, A. Najati and M.B.Moghimi [26] investigated the generalized Hyers-Ulam-Rassias stability for the **quadratic additive functional equation** of the form

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+2f(2x)-4f(x). \tag{2}$$

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Infact, M.E. Gordji et. al., [12] discussed the generalized Hyers- Ulam stability of the additive - quadratic functional equation

$$f(2x+y)+f(2x-y)=f(x+y)+f(x-y)+2f(2x)-2f(x) \tag{3}$$

in fuzzy Banach spaces. The general solution and generalized Ulam - Hyers stability of a mixed type additive quadratic(AQ)-functional equation

$$g(x+y)+g(x-y)=2g(x)+g(y)+g(-y) \tag{4}$$

was investigated by M. Arunkumar and J.M. Rassias [6]. Also, the general solution in vector space and generalized Ulam - Hyers stability of mixed type additive quadratic functional equation

$$f(2x\pm y\pm z)=2f(-x\mp y\mp z)-2f(\mp y\mp z)+f(\pm y\pm z)+3f(x)-f(-x) \tag{5}$$

in Random Normed Space was discussed by S. Murthy et.al., [25]. Several other mixed type additive - quadratic functional equations were introduced and investigated in [9, 16, 17, 19, 20, 22, 27, 35].

In this paper, the authors establish the general solution and generalized Ulam - Hyers stability of an additive quadratic functional equation

$$\begin{aligned} f(x+2y+3z)+f(x-2y+3z)+f(x+2y-3z)+f(x-2y-3z)\\ =4f(x)+8[f(y)+f(-y)]+18[f(z)+f(-z)] \end{aligned} \tag{6}$$

in Banach spaces, using the Hyers direct and fixed point methods.

In Section 2, the general solution of the functional equation (6) is given.

In Sections 3 and 4, the generalized Ulam - Hyers stability of the functional equation (6) using direct method and fixed point method is proved, respectively.

2. General Solution of the Functional Equation(6)

In this section, the general solution of the functional equation (6) is given. Through out this section, let us assume X and Y be vector spaces.

Lemma 2.1. *An odd function $f: X \to Y$ satisfies the additive functional equation*

$$f(x+y)=f(x)+f(y) \tag{7}$$

for all $x, y \in X$, if and only if $f: X \to Y$ satisfies the functional equation (6) for all $x, y, z \in X$.

Proof. Let $f: X \to Y$ satisfy the functional equation (7). Setting $x=y=0$ in (7), we get $f(0)=0$. Replacing y by x and y by $2x$ in (7), we obtain

$$f(2x)=2f(x) \quad \text{and} \quad f(3x)=3f(x) \tag{8}$$

for all $x \in X$. In general for any positive integer a, we have $f(ax)=af(x)$.

Replacing y by $y+z$ in (7) and using (7), we get

$$f(x+y+z)=f(x)+f(y)+f(z) \tag{9}$$

for all $x, y, z \in X$. Again replacing (x, y, z) by $(x, 2y, 3z)$ in (9) and using (8), we obtain

$$f(x + 2y + 3z) = f(x) + 2f(y) + 3f(z) \tag{10}$$

for all $x, y, z \in X$. Setting y by $-y$ in (10), we have

$$f(x - 2y + 3z) = f(x) + 2f(-y) + 3f(z) \tag{11}$$

for all $x, y, z \in X$. Again setting z by $-z$ in (10), we get

$$f(x + 2y - 3z) = f(x) + 2f(y) + 3f(-z) \tag{12}$$

for all $x, y, z \in X$. Putting (y, z) by $(-y, -z)$ in (10), we obtain

$$f(x - 2y - 3z) = f(x) + 2f(-y) + 3f(-z) \tag{13}$$

for all $x, y, z \in X$. Adding (10), (11), (12) and (13), we arrive

$$\begin{aligned} &f(x + 2y + 3z) + f(x - 2y + 3z) + f(x + 2y - 3z) + f(x - 2y - 3z) \\ &= 4f(x) + 4f(y) + 4f(-y) + 6f(z) + 6f(-z) \end{aligned} \tag{14}$$

for all $x, y, z \in X$. Adding $4f(y) + 12f(z)$ on both sides of (14), we have

$$\begin{aligned} &f(x + 2y + 3z) + f(x - 2y + 3z) + f(x + 2y - 3z) + f(x - 2y - 3z) + 4f(y) + 12f(z) \\ &= 4f(x) + 4f(y) + 4f(-y) + 6f(z) + 6f(-z) + 4f(y) + 12f(z) \end{aligned} \tag{15}$$

for all $x, y, z \in X$. It follows from (15) that

$$\begin{aligned} &f(x + 2y + 3z) + f(x - 2y + 3z) + f(x + 2y - 3z) + f(x - 2y - 3z) \\ &= 4f(x) + 4f(y) + 4f(-y) + 6f(z) + 6f(-z) + 4f(y) + 12f(z) - 4f(y) - 12f(z) \end{aligned} \tag{16}$$

for all $x, y, z \in X$. Using oddness of f in (16), we have demonstrated our result.

Conversely, let $f : X \to Y$ satisfy the functional equation (6). Setting $x = y = z = 0$ in (6), we get $f(0) = 0$. Using oddness of f in (6), we have

$$f(x + 2y + 3z) + f(x - 2y + 3z) + f(x + 2y - 3z) + f(x - 2y - 3z) = 4f(x) \tag{17}$$

for all $x, y, z \in X$. Replacing (y, z) by $\left(\frac{y}{2}, 0\right)$ in (17), we get

$$f(x + y) + f(x - y) = 2f(x) \tag{18}$$

for all $x, y \in X$. By Theorem 2.1 of [4], our result is demonstrated. □

Lemma 2.2. *An even function $f : X \to Y$ satisfies the quadratic functional equation*

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{19}$$

for all $x, y \in X$, if and only if $f : X \to Y$ satisfies the functional equation (6) for all $x, y, z \in X$.

Proof. Let $f : X \to Y$ satisfy the functional equation (19). Setting (x, y) by $(0, 0)$ in (7), we obtain $f(0) = 0$. Replacing y by x and y by $2x$ in (7), we get

$$f(2x) = 4f(x) \quad \text{and} \quad f(3x) = 9f(x) \tag{20}$$

for all $x \in X$. In general for any positive integer a, we have

$$f(ax) = a^2 f(x) \tag{21}$$

for all $x \in X$. Replacing y by $2y + 3z$ in (19), we get

$$f(x+2y+3z) + f(x-2y-3z) = 2f(x) + 2f(2y+3z) \tag{22}$$

for all $x, y, z \in X$. Again replacing y by $-2y + 3z$ in (19), we obtain

$$f(x-2y+3z) + f(x+2y-3z) = 2f(x) + 2f(-2y+3z) \tag{23}$$

for all $x, y, z \in X$. Adding (22) and (23), we arrive

$$\begin{aligned} &f(x+2y+3z) + f(x-2y+3z) + f(x+2y-3z) + f(x-2y-3z) \\ &= 4f(x) + 2f(2y+3z) + 2f(-2y+3z) \end{aligned} \tag{24}$$

for all $x, y, z \in X$. Using (19) in (24) and using evenness of f, we have

$$\begin{aligned} &f(x+2y+3z) + f(x-2y+3z) + f(x+2y-3z) + f(x-2y-3z) \\ &= 4f(x) + 2[2f(2y) + 2f(3z) - f(2y-3z)] + 2[2f(-2y) + 2f(3z) - f(-2y-3z)] \\ &= 4f(x) + 16[f(y) + f(-y)] + 72f(z) - 2[f(2y-3z) + f(2y+3z)] \\ &= 4f(x) + 32f(y) + 72f(z) - 2[2f(2y) + 2f(3z)] \\ &= 4f(x) + 32f(y) + 72f(z) - 16f(y) - 36f(z)] \\ &= 4f(x) + 16f(y) + 32f(z) \\ &= 4f(x) + 8[f(y) + f(-y)] + 18[f(z) + f(-z)] \end{aligned} \tag{25}$$

for all $x, y, z \in X$.

Conversely, assume $f : X \to Y$ satisfies the functional equation (6). Setting (x, y, z) by $(0, 0, 0)$ in (6), we obtain $f(0) = 0$. Replacing z by 0 and using evenness of f in (6), we have

$$f(x+2y) + f(x-2y) = 2f(x) + 8f(y) \tag{26}$$

for all $x, y \in X$. Setting x by 0 in (26) and using evenness of f, we get

$$f(2y) = 4f(y) \tag{27}$$

for all $y \in X$. Replacing y by $\frac{y}{2}$ in (26) and using (27), we arrive (19) as desired. □

3. Stability Results: Hyers Direct Method

In this section, the generalized Ulam - Hyers stability of functional equation (6), using the Hyers direct method, is provided. Now, let us consider X and Y to be a normed space and a Banach space, respectively. Define a mapping $Df : X \to Y$ by

$$\begin{aligned} Df(x,y,z) &= f(x+2y+3z) + f(x-2y+3z) + f(x+2y-3z) + f(x-2y-3z) \\ &- 4f(x) - 8[f(y)+f(-y)] - 18[f(z)+f(-z)] \end{aligned}$$

for all $x, y, z \in X$.

Theorem 3.1. *Let $j \in \{-1, 1\}$ and $\alpha, \beta : X^3 \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\alpha\left(6^{nj}x, 6^{nj}y, 6^{nj}z\right)}{6^{nj}} = 0 \tag{28}$$

for all $x, y, z \in X$. Let $f_a : X \to Y$ be an odd function satisfying the inequality

$$\|Df_a(x,y,z)\| \le \alpha(x,y,z) \tag{29}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ which satisfies (6) and

$$\|f_a(x) - A(x)\| \le \frac{1}{12} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\beta(6^{kj}x)}{6^{kj}} \tag{30}$$

where $\beta(6^{kj}x)$ and $A(x)$ are defined by the following two formulas

$$\beta(6^{kj}x) = 2\alpha(6^{kj}x, 6^{kj}x, 6^{kj}x) + \alpha(6^{kj}x, 0, 6^{kj}x) \tag{31}$$

$$A(x) = \lim_{n\to\infty} \frac{f_a(6^{nj}x)}{6^{nj}} \tag{32}$$

respectively, for all $x \in X$.

Proof. Assume $j = 1$. Replacing (x,y,z) by (x,x,x) in (29) and using oddness of f_a, we get

$$\|f_a(6x) + f_a(2x) - f_a(4x) - 4f_a(x)\| \le \alpha(x,x,x) \tag{33}$$

for all $x \in X$. Again replacing (x,y,z) by $(x,0,x)$ in (29) and using oddness of f_a, we obtain

$$\|2f_a(4x) - 2f_a(2x) - 4f_a(x)\| \le \alpha(x,0,x) \tag{34}$$

for all $x \in X$. It follows from (33) and (34) and the triangle inequality that

$$\begin{aligned} \|2f_a(6x) - 12f_a(x)\| &\le 2\|f_a(6x) + f_a(2x) - f_a(4x) - 4f_a(x)\| + \|2f_a(4x) - 2f_a(2x) - 4f_a(x)\| \\ &\le 2\alpha(x,x,x) + \alpha(x,0,x) \end{aligned} \tag{35}$$

for all $x \in X$. Divide the above inequality by 12, we obtain

$$\left\| \frac{f_a(6x)}{6} - f_a(x) \right\| \le \frac{\beta(x)}{12} \tag{36}$$

where

$$\beta(x) = 2\alpha\,(x,x,x) + \alpha\,(x,0,x)$$

for all $x \in X$. Now, replacing x by $6x$ and dividing by 6 in (36), we get

$$\left\| \frac{f_a(6^2x)}{6^2} - \frac{f_a(6x)}{6} \right\| \leq \frac{\beta(6x)}{12 \cdot 6} \tag{37}$$

for all $x \in X$. From (36) and (37), we obtain

$$\begin{aligned} \left\| \frac{f_a(6^2x)}{6^2} - f_a(x) \right\| &\leq \left\| \frac{f_a(6x)}{6} - f_a(x) \right\| + \left\| \frac{f_a(6^2x)}{6^2} - \frac{f_a(6x)}{6} \right\| \\ &\leq \frac{1}{12}\left[\beta(x) + \frac{\beta(6x)}{6}\right] \end{aligned} \tag{38}$$

for all $x \in X$. Proceeding further and using induction on a positive integer n, we get

$$\begin{aligned} \left\| \frac{f_a(6^nx)}{6^n} - f_a(x) \right\| &\leq \frac{1}{12}\sum_{k=0}^{n-1} \frac{\beta(6^kx)}{6^k} \\ &\leq \frac{1}{12}\sum_{k=0}^{\infty} \frac{\beta(6^kx)}{6^k} \end{aligned} \tag{39}$$

for all $x \in X$. In order to prove the convergence of the sequence

$$\left\{ \frac{f_a(6^nx)}{6^n} \right\},$$

first replace x by 6^mx and then divide by 6^m in (39), for any $m, n > 0$, and thus we deduce

$$\begin{aligned} \left\| \frac{f_a(6^{n+m}x)}{6^{(n+m)}} - \frac{f_a(6^mx)}{6^m} \right\| &= \frac{1}{6^m}\left\| \frac{f_a(6^n \cdot 6^mx)}{6^n} - f_a(6^mx) \right\| \\ &\leq \frac{1}{12}\sum_{k=0}^{n-1} \frac{\beta(6^{k+m}x)}{6^{k+m}} \\ &\leq \frac{1}{12}\sum_{k=0}^{\infty} \frac{\beta(6^{k+m}x)}{6^{k+m}} \\ &\to 0 \quad as \quad m \to \infty \end{aligned}$$

for all $x \in X$. Hence the sequence $\left\{ \frac{f_a(6^nx)}{6^n} \right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $A : X \to Y$ such that

$$A(x) = \lim_{n\to\infty} \frac{f_a(6^nx)}{6^n}, \quad \forall\ x \in X.$$

Letting $n \to \infty$ in (39), we see that (30) holds for all $x \in X$. Claim that A satisfies (6). In fact, replacing (x, y, z) by $(6^nx, 6^ny, 6^nz)$ and dividing by 6^n in (29), we obtain

$$\frac{1}{6^n}\left\| Df_a(6^nx, 6^ny, 6^nz) \right\| \leq \frac{1}{6^n}\alpha(6^nx, 6^ny, 6^nz)$$

for all $x, y, z \in X$. Letting $n \to \infty$ in the above inequality and using the definition of A, we see that

$$DA(x, y, z) = 0.$$

Hence A satisfies (6) for all $x, y, z \in X$. To show A is unique, let B be another additive mapping satisfying (6) and (30), then

$$\begin{aligned}\|A(x) - B(x)\| &= \frac{1}{6^n}\|A(6^n x) - B(6^n x)\| \\ &\leq \frac{1}{6^n}\{\|A(6^n x) - f_a(6^n x)\| + \|f_a(6^n x) - B(6^n x)\|\} \\ &\leq \frac{1}{6}\sum_{k=0}^{\infty}\frac{\beta(6^{k+n}x)}{6^{(k+n)}} \\ &\to 0 \quad as \ \ n \to \infty\end{aligned}$$

for all $x \in X$. Hence $A = B$ is unique.

For $j = -1$, we can prove a similar stability result. This completes the proof of the theorem. □

The following Corollary is an immediate consequence of Theorem 3.1 concerning the stability of (6).

Corollary 3.2. *Let λ and s be nonnegative real numbers. Let an odd function $f_a : X \to Y$ satisfy the inequality*

$$\|Df_a(x, y, z)\| \leq \begin{cases} \lambda, & \\ \lambda\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s < 1 \quad or \quad s > 1; \\ \lambda||x||^s||y||^s||z||^s, & 3s < 1 \quad or \quad 3s > 1; \\ \lambda\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s < 1 \quad or \quad 3s > 1; \end{cases} \tag{40}$$

for all $x, y, z \in X$. Then there exists a unique additive function $A : X \to Y$ such that

$$\|f_a(x) - A(x)\| \leq \begin{cases} \frac{3\lambda}{10}, \\ \frac{4\lambda||x||^s}{|6 - 6^s|}, \\ \frac{\lambda||x||^{3s}}{|6 - 6^{3s}|}, \\ \frac{5\lambda||x||^{3s}}{|6 - 6^{3s}|} \end{cases} \tag{41}$$

for all $x \in X$.

Now we provide an example to illustrate that the functional equation (6) is not stable for $s = 1$ in Condition (ii) of Corollary 3.2.

Example 3.3. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\phi(x) = \begin{cases} \mu x, & if \ |x| < 1 \\ \mu, & otherwise \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f_a : \mathbb{R} \to \mathbb{R}$ by

$$f_a(x) = \sum_{n=0}^{\infty}\frac{\phi(6^n x)}{6^n} \qquad for \ all \quad x \in \mathbb{R}.$$

Then f_a satisfies the functional inequality

$$|Df_a(x, y, z)| \leq 72\mu\,(|x| + |y| + |z|) \tag{42}$$

for all $x, y, z \in \mathbb{R}$. Then there is no an additive mapping $A : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ such that

$$|f_a(x) - A(x)| \leq \beta|x| \qquad for \ all \quad x \in \mathbb{R}. \tag{43}$$

Proof. Now

$$|f_a(x)| \leq \sum_{n=0}^{\infty} \frac{|\phi(6^n x)|}{|6^n|} = \sum_{n=0}^{\infty} \frac{\mu}{6^n} = \frac{1}{1-\frac{1}{6}}\mu = \frac{6\mu}{5}.$$

Therefore we see that f_a is bounded. We are now going to prove that f_a satisfies (42).

If $x = y = z = 0$ then (42) is trivial. If $|x| + |y| + |z| \geq 1$ then the left hand side of (42) is less than 72μ. Now suppose that $0 < |x| + |y| + |z| < 1$. Then there exists a positive integer k such that

$$\frac{1}{6^k} \leq |x| + |y| + |z| < \frac{1}{6^{k-1}}, \tag{44}$$

so that $6^{k-1}|x| < 1,\ 6^{k-1}|y| < 1, 6^{k-1}|z| < 1$ and consequently

$$\begin{aligned} &6^{k-1}(x+2y+3z), 6^{k-1}(x-2y+3z), 6^{k-1}(x+2y-3z), 6^{k-1}(x-2y-3z),\\ &\qquad 6^{k-1}(x), 6^{k-1}(-x), 6^{k-1}(y), 6^{k-1}(-y), 6^{k-1}(z), 6^{k-1}(-z) \in (-1,1). \end{aligned}$$

Therefore for each $n = 0, 1, \ldots, k-1$, we have

$$\begin{aligned} &6^n(x+2y+3z), 6^n(x-2y+3z), 6^n(x+2y-3z), 6^n(x-2y-3z),\\ &\qquad 6^n(x), 6^n(-x), 6^n(y), 6^n(-y), 6^n(z), 6^n(-z) \in (-1,1). \end{aligned}$$

and

$$\begin{aligned} &\phi\left(6^n(x+2y+3z)\right) + \phi\left(6^n(x-2y+3z)\right) + \phi\left(6^n(x+2y-3z)\right) + \phi\left(6^n(x-2y-3z)\right)\\ &\qquad - 4\phi(6^n x) - 8[\phi(6^n y) + \phi(-6^n y)] - 18[\phi(6^n z) + \phi(-6^n z)] = 0 \end{aligned}$$

for $n = 0, 1, \ldots, k-1$. From the definition of f_a and (44), we obtain that

$$\begin{aligned} &\left|Df_a(x,y,z)\right|\\ &= \sum_{n=0}^{\infty} \frac{1}{6^n}\Big|\phi\left(6^n(x+2y+3z)\right) + \phi\left(6^n(x-2y+3z)\right) + \phi\left(6^n(x+2y-3z)\right) + \phi\left(6^n(x-2y-3z)\right)\\ &\qquad - 4\phi(6^n x) - 8[\phi(6^n y) + \phi(-6^n y)] - 18[\phi(6^n z) + \phi(-6^n z)]\Big|\\ &= \sum_{n=k}^{\infty} \frac{1}{6^n}\Big|\phi\left(6^n(x+2y+3z)\right) + \phi\left(6^n(x-2y+3z)\right) + \phi\left(6^n(x+2y-3z)\right) + \phi\left(6^n(x-2y-3z)\right)\\ &\qquad - 4\phi(6^n x) - 8[\phi(6^n y) + \phi(-6^n y)] - 18[\phi(6^n z) + \phi(-6^n z)]\Big|\\ &\leq \sum_{n=k}^{\infty} \frac{1}{6^n} 60\mu = 60\ \mu \times \frac{1}{6^k} \times \frac{6}{5} \leq 72\mu\left(|x| + |y| + |z|\right). \end{aligned}$$

Thus f_a satisfies (42) for all $x, y, z \in \mathbb{R}$ with $0 < |x| + |y| + |z| < 1$.

We claim that the additive functional equation (6) is not stable for $s = 1$ in condition (ii) of Corollary 3.2. Suppose on the contrary that there exists an additive mapping $A : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ satisfying (43). Since f_a is bounded and continuous for all $x \in \mathbb{R}$, A is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.1, A must have the form $A(x) = cx$ for any x in $\mathbb{R}$. Thus we obtain that

$$|f_a(x)| \leq (\beta + |c|)\,|x|. \tag{45}$$

But we can choose a positive integer m with $m\mu > \beta + |c|$.

If $x \in \left(0, \frac{1}{6^{m-1}}\right)$, then $6^n x \in (0,1)$ for all $n = 0, 1, \ldots, m-1$. For this x, we get

$$f_a(x) = \sum_{n=0}^{\infty} \frac{\alpha(6^n x)}{6^n} \geq \sum_{n=0}^{m-1} \frac{\mu(6^n x)}{6^n} = m\mu x > (\beta + |c|)\, x$$

which contradicts (45). Therefore the additive functional equation (6) is not stable in the sense of Ulam, Hyers and Rassias if $s = 1$, assumed in the inequality (40). □

A counter example to illustrate the non stability in Condition (iii) of Corollary 3.2:

Example 3.4. *Let s be such that $0 < s < \frac{1}{3}$. Then there is a function $f_a : \mathbb{R} \to \mathbb{R}$ and a constant $\lambda > 0$ satisfying*

$$|Df_a(x, y, z)| \leq \lambda |x|^{\frac{s}{3}} \; |y|^{\frac{s}{3}} \; |z|^{\frac{1-2s}{3}} \tag{46}$$

for all $x, y, z \in \mathbb{R}$ and

$$\sup_{x \neq 0} \frac{|f_a\,(x) - A\,(x)|}{|x|} = +\infty \tag{47}$$

for every additive mapping $A : \mathbb{R} \to \mathbb{R}$.

Proof. If we take

$$f_a\,(x) = \begin{cases} x \ln |x|\,, & if \quad x \neq 0, \\ 0, & if \quad x = 0. \end{cases}$$

Then from the relation (47), it follows that

$$\begin{aligned} \sup_{x \neq 0} \frac{|f_a\,(x) - A\,(x)|}{|x|} &\geq \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \frac{|f_a\,(n) - A\,(n)|}{|n|} \\ &= \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \frac{|n \ln |n| - n \; A\,(1)|}{|n|} \\ &= \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \; |\ln |n| - A\,(1)| \; = \infty. \end{aligned}$$

We have to prove (46) is true.

Case (i): If $x, y, z > 0$ than $x + 2z + 3z > 0, x - 2z + 3z > 0$,

$x + 2z - 3z > 0, x - 2z - 3z > 0$ and therefore (46) becomes,

$$\begin{aligned} &|f_a(x + 2y + 3z) + f_a(x - 2y + 3z) + f_a(x + 2y - 3z) + f_a(x - 2y - 3z) \\ &\qquad -4f_a(x) - 8[f_a(y) + f_a(-y)] - 18[f_a(z) + f_a(-z)]| \,. \\ &= |(x + 2y + 3z) \ln |x + 2y + 3z| + (x - 2y + 3z) \ln |x - 2y + 3z| + (x + 2y - 3z) \ln |x + 2y - 3z| \\ &\qquad +(x - 2y - 3z) \ln |x - 2y - 3z| - 4x \ln |x| - 8[y \ln |y| - y \ln |-y|] - 18[z \ln |z| - z \ln |-z|]| \,. \end{aligned}$$

Set $x = u, y = v, z = w$ it follows that

$$\begin{aligned}
&|f_a(x+2y+3z) + f_a(x-2y+3z) + f_a(x+2y-3z) + f_a(x-2y-3z) \\
&\quad -4f_a(x) - 8[f_a(y) + f_a(-y)] - 18[f_a(z) + f_a(-z)]| \\
&= |(x+2y+3z)\ln|x+2y+3z| + (x-2y+3z)\ln|x-2y+3z| + (x+2y-3z)\ln|x+2y-3z| \\
&\quad +(x-2y-3z)\ln|x-2y-3z| - 4x\ln|x| - 8[y\ln|y| - y\ln|-y|] - 18[z\ln|z| - z\ln|-z|]|. \\
&= |(u+2v+3w)\ln|u+2v+3w| + (u-2v+3w)\ln|u-2v+3w| + (u+2v-3w)\ln|u+2v-3w| \\
&\quad +(u-2v-3w)\ln|u-2v-3w| - 4u\ln|u| - 8[v\ln|v| - v\ln|-v|] - 18[w\ln|w| - w\ln|-w|]|. \\
&|f_a(u+2v+3w) + f_a(u-2v+3w) + f_a(u+2v-3w) + f_a(u-2v-3w) \\
&\quad -4f_a(u) - 8[f_a(v) + f_a(-v)] - 18[f_a(w) + f_a(-w)]|. \\
&\quad \leq \lambda|u|^{\frac{s}{3}}\ |v|^{\frac{s}{3}}\ |w|^{\frac{1-2s}{3}} = \lambda|x|^{\frac{s}{3}}\ |y|^{\frac{s}{3}}\ |z|^{\frac{1-2s}{3}}.
\end{aligned}$$

Case (ii): If $x, y, z < 0$ than $x + 2z + 3z > 0, x - 2z + 3z > 0,$
$x + 2z - 3z > 0, x - 2z - 3z > 0$ and therefore (46) becomes,

$$\begin{aligned}
&|f_a(x+2y+3z) + f_a(x-2y+3z) + f_a(x+2y-3z) + f_a(x-2y-3z) \\
&\quad -4f_a(x) - 8[f_a(y) + f_a(-y)] - 18[f_a(z) + f_a(-z)]|. \\
&= |(x+2y+3z)\ln|x+2y+3z| + (x-2y+3z)\ln|x-2y+3z| + (x+2y-3z)\ln|x+2y-3z| \\
&\quad +(x-2y-3z)\ln|x-2y-3z| - 4x\ln|x| - 8[y\ln|y| - y\ln|-y|] - 18[z\ln|z| - z\ln|-z|]|.
\end{aligned}$$

Set $x = -u, y = -v, z = -w$ it follows that

$$\begin{aligned}
&|f_a(x+2y+3z) + f_a(x-2y+3z) + f_a(x+2y-3z) + f_a(x-2y-3z) \\
&\quad -4f_a(x) - 8[f_a(y) + f_a(-y)] - 18[f_a(z) + f_a(-z)]| \\
&= |(x+2y+3z)\ln|x+2y+3z| + (x-2y+3z)\ln|x-2y+3z| + (x+2y-3z)\ln|x+2y-3z| \\
&\quad +(x-2y-3z)\ln|x-2y-3z| - 4x\ln|x| - 8[y\ln|y| - y\ln|-y|] - 18[z\ln|z| - z\ln|-z|]|. \\
&= |(-u-2v-3w)\ln|-u-2v-3w| + (-u+2v-3w)\ln|-u+2v-3w| \\
&\quad +(-u-2v+3w)\ln|-u-2v+3w| + (-u+2v+3w)\ln|-u+2v+3w| \\
&\quad +4u\ln|-u| - 8[-v\ln|-v| + v\ln|v|] - 18[-w\ln|-w| + w\ln|w|]|. \\
&|f_a(-u-2v-3w) + f_a(-u+2v-3w) + f_a(-u-2v+3w) + f_a(-u+2v+3w) \\
&\quad -4f_a(-u) - 8[f_a(-v) + f_a(v)] - 18[f_a(-w) + f_a(w)]|. \\
&\leq \lambda|-u|^{\frac{s}{3}}\ |-v|^{\frac{s}{3}}\ |-w|^{\frac{1-2s}{3}} = \lambda|x|^{\frac{s}{3}}\ |y|^{\frac{s}{3}}\ |z|^{\frac{1-2s}{3}}.
\end{aligned}$$

Case (iii): If $x > 0, y < 0, z < 0$ than $x + 2z + 3z < 0, x - 2z + 3z < 0,$
$x + 2z - 3z < 0, x - 2z - 3z < 0$ and therefore (46) becomes,

$$\begin{aligned}
&|f_a(x+2y+3z) + f_a(x-2y+3z) + f_a(x+2y-3z) + f_a(x-2y-3z) \\
&\quad -4f_a(x) - 8[f_a(y) + f_a(-y)] - 18[f_a(z) + f_a(-z)]|. \\
&= |(x+2y+3z)\ln|x+2y+3z| + (x-2y+3z)\ln|x-2y+3z| + (x+2y-3z)\ln|x+2y-3z| \\
&\quad +(x-2y-3z)\ln|x-2y-3z| - 4x\ln|x| - 8[y\ln|y| - y\ln|-y|] - 18[z\ln|z| - z\ln|-z|]|.
\end{aligned}$$

Set $x = u, y = -v, z = -w$ it follows that

$$\begin{aligned}
&|f_a(x+2y+3z) + f_a(x-2y+3z) + f_a(x+2y-3z) + f_a(x-2y-3z) \\
&\qquad -4f_a(x) - 8[f_a(y) + f_a(-y)] - 18[f_a(z) + f_a(-z)]| \\
&\quad = |(x+2y+3z)\ln|x+2y+3z| + (x-2y+3z)\ln|x-2y+3z| + (x+2y-3z)\ln|x+2y-3z| \\
&\quad +(x-2y-3z)\ln|x-2y-3z| - 4x\ln|x| - 8[y\ln|y| - y\ln|-y|] - 18[z\ln|z| - z\ln|-z|]|\,. \\
&\quad = |(u-2v-3w)\ln|u-2v-3w| + (u+2v-3w)\ln|u+2v-3w| + (u-2v+3w)\ln|u-2v+3w| \\
&\quad +(u+2v+3w)\ln|u+2v+3w| - 4u\ln|u| - 8[-v\ln|-v| + v\ln|v|] - 18[-w\ln|-w| + w\ln|w|]|\,. \\
&|f_a(u-2v-3w) + f_a(u+2v-3w) + f_a(u-2v+3w) + f_a(u+2v+3w) \\
&\qquad -4f_a(u) - 8[f_a(-v) + f_a(v)] - 18[f_a(-w) + f_a(w)]|\,. \\
&\quad \leq \lambda |u|^{\frac{s}{3}}\ |-v|^{\frac{s}{3}}\ |-w|^{\frac{1-2s}{3}} = \lambda |x|^{\frac{s}{3}}\ |y|^{\frac{s}{3}}\ |z|^{\frac{1-2s}{3}}.
\end{aligned}$$

Case (iv): If $x < 0, y > 0, z > 0$ than $x + 2z + 3z < 0, x - 2z + 3z < 0$, $x + 2z - 3z < 0, x - 2z - 3z < 0$ and therefore (46) becomes,

$$\begin{aligned}
&|f_a(x+2y+3z) + f_a(x-2y+3z) + f_a(x+2y-3z) + f_a(x-2y-3z) \\
&\qquad -4f_a(x) - 8[f_a(y) + f_a(-y)] - 18[f_a(z) + f_a(-z)]|\,. \\
&\quad = |(x+2y+3z)\ln|x+2y+3z| + (x-2y+3z)\ln|x-2y+3z| + (x+2y-3z)\ln|x+2y-3z| \\
&\quad +(x-2y-3z)\ln|x-2y-3z| - 4x\ln|x| - 8[y\ln|y| - y\ln|-y|] - 18[z\ln|z| - z\ln|-z|]|\,.
\end{aligned}$$

Set $x = -u, y = v, z = w$ it follows that

$$\begin{aligned}
&|f_a(x+2y+3z) + f_a(x-2y+3z) + f_a(x+2y-3z) + f_a(x-2y-3z) \\
&\qquad -4f_a(x) - 8[f_a(y) + f_a(-y)] - 18[f_a(z) + f_a(-z)]| \\
&\quad = |(x+2y+3z)\ln|x+2y+3z| + (x-2y+3z)\ln|x-2y+3z| + (x+2y-3z)\ln|x+2y-3z| \\
&\quad +(x-2y-3z)\ln|x-2y-3z| - 4x\ln|x| - 8[y\ln|y| - y\ln|-y|] - 18[z\ln|z| - z\ln|-z|]|\,. \\
&= |(-u+2v+3w)\ln|-u+2v+3w| + (-u-2v+3w)\ln|-u-2v+3w| \\
&\quad +(-u+2v-3w)\ln|-u+2v-3w| + (-u-2v-3w)\ln|-u-2v-3w| \\
&\quad +4u\ln|u| - 8[v\ln|v| - v\ln|-v|] - 18[w\ln|w| - w\ln|-w|]|\,. \\
&|f_a(-u+2v+3w) + f_a(-u-2v+3w) + f_a(-u+2v-3w) + f_a(-u-2v-3w) \\
&\qquad -4f_a(-u) - 8[f_a(v) + f_a(-v)] - 18[f_a(w) + f_a(-w)]|\,. \\
&\quad \leq \lambda |-u|^{\frac{s}{3}}\ |v|^{\frac{s}{3}}\ |w|^{\frac{1-2s}{3}} = \lambda |x|^{\frac{s}{3}}\ |y|^{\frac{s}{3}}\ |z|^{\frac{1-2s}{3}}.
\end{aligned}$$

Case (v): If $x = y = z = 0$ in (46) then it is trivial. □

Now we will provide an example to illustrate that the functional equation (6) is not stable for $s = \frac{1}{3}$ in Condition (iv) of Corollary 3.2.

Example 3.5. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\phi(x) = \begin{cases} \mu x, & \text{if } |x| < \frac{1}{3} \\ \frac{\mu}{3}, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f_a : \mathbb{R} \to \mathbb{R}$ by

$$f_a(x) = \sum_{n=0}^{\infty} \frac{\phi(6^n x)}{6^n} \qquad for\ all \quad x \in \mathbb{R}.$$

Then f_a satisfies the functional inequality

$$|Df_a(x,y,z)| \le 24\mu \left(|x|^{\frac{1}{3}}|y|^{\frac{1}{3}}|z|^{\frac{1}{3}} + |x| + |y| + |z|\right), \tag{48}$$

for all $x, y, z \in \mathbb{R}$. Then there is no an additive mapping $A : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ such that

$$|f_a(x) - A(x)| \le \beta |x| \qquad for\ all \quad x \in \mathbb{R}. \tag{49}$$

Proof. Now

$$|f_a(x)| \le \sum_{n=0}^{\infty} \frac{|\phi(6^n x)|}{|6^n|} = \sum_{n=0}^{\infty} \frac{\mu}{3} \times \frac{1}{6^n} = \frac{2\mu}{5}.$$

Therefore we see that f_a is bounded. We are now going to prove that f_a satisfies (48).

If $x = y = z = 0$ then (48) is trivial. If $|x|^{\frac{1}{3}}|y|^{\frac{1}{3}}|z|^{\frac{1}{3}} + |x| + |y| + |z| \ge \dfrac{1}{6}$, then the left hand side of (48) is less than 24μ. Now suppose that $0 < |x|^{\frac{1}{3}}|y|^{\frac{1}{3}}|z|^{\frac{1}{3}} + |x| + |y| + |z| < \dfrac{1}{6}$. Then there exists a positive integer k such that

$$\frac{1}{6^k} \le |x|^{\frac{1}{3}}|y|^{\frac{1}{3}}|z|^{\frac{1}{3}} + |x| + |y| + |z| < \frac{1}{6^{k+1}}, \tag{50}$$

so that $6^{k-1}|x|^{\frac{1}{3}}|y|^{\frac{1}{3}}|z|^{\frac{1}{3}} < \frac{1}{6}, 6^{k-1}|x| < \frac{1}{6},\ 6^{k-1}|y| < \frac{1}{6}, 6^{k-1}|z| < \frac{1}{6}$ and consequently

$$\begin{aligned}6^{k-1}(x+2y+3z), 6^{k-1}(x-2y+3z), 6^{k-1}(x+2y-3z), 6^{k-1}(x-2y-3z),\\ 6^{k-1}(x), 6^{k-1}(-x), 6^{k-1}(y), 6^{k-1}(-y), 6^{k-1}(z), 6^{k-1}(-z) \in \left(\frac{-1}{6}, \frac{1}{6}\right).\end{aligned}$$

Therefore for each $n = 0, 1, \ldots, k-1$, we have

$$\begin{aligned}6^n(x+2y+3z), 6^n(x-2y+3z), 6^n(x+2y-3z), 6^n(x-2y-3z),\\ 6^n(x), 6^n(-x), 6^n(y), 6^n(-y), 6^n(z), 6^n(-z) \in \left(\frac{-1}{6}, \frac{1}{6}\right)\end{aligned}$$

and

$$\begin{aligned}&\phi\left(6^n(x+2y+3z)\right) + \phi\left(6^n(x-2y+3z)\right) + \phi\left(6^n(x+2y-3z)\right) + \phi\left(6^n(x-2y-3z)\right)\\ &\quad - 4\phi(6^n x) - 8[\phi(6^n y) + \phi(-6^n y)] - 18[\phi(6^n z) + \phi(-6^n z)] = 0\end{aligned}$$

for $n = 0, 1, \ldots, k-1$. From the definition of f_a and (50), we obtain that

$$\begin{aligned}&\left|Df_a(x,y,z)\right|\\ &= \sum_{n=0}^{\infty} \frac{1}{6^n}\Big|\phi\left(6^n(x+2y+3z)\right) + \phi\left(6^n(x-2y+3z)\right) + \phi\left(6^n(x+2y-3z)\right) + \phi\left(6^n(x-2y-3z)\right)\\ &\quad - 4\phi(6^n x) - 8[\phi(6^n y) + \phi(-6^n y)] - 18[\phi(6^n z) + \phi(-6^n z)]\Big|\\ &= \sum_{n=k}^{\infty} \frac{1}{6^n}\Big|\phi\left(6^n(x+2y+3z)\right) + \phi\left(6^n(x-2y+3z)\right) + \phi\left(6^n(x+2y-3z)\right) + \phi\left(6^n(x-2y-3z)\right)\\ &\quad - 4\phi(6^n x) - 8[\phi(6^n y) + \phi(-6^n y)] - 18[\phi(6^n z) + \phi(-6^n z)]\Big|\\ &\quad \le \sum_{n=k}^{\infty} \frac{1}{6^n}\frac{60\ \mu}{3} = \frac{60\ \mu}{3} \times \frac{1}{6^k} \times \frac{6}{5} \le 24\mu\left(|x|^{\frac{1}{3}}|y|^{\frac{1}{3}}|z|^{\frac{1}{3}} + |x| + |y| + |z|\right).\end{aligned}$$

Thus f_a satisfies (48) for all $x, y, z \in \mathbb{R}$ with $0 < |x|^{\frac{1}{3}}|y|^{\frac{1}{3}}|z|^{\frac{1}{3}} + |x| + |y| + |z| < \frac{1}{6}$.

We claim that the additive functional equation (6) is not stable for $s = \frac{1}{3}$ in condition (iv) of Corollary 3.2. Suppose on the contrary that there exists an additive mapping $A : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ satisfying (49). Since f_a is bounded and continuous for all $x \in \mathbb{R}$, A is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.1, A must have the form $A(x) = cx$ for any x in $\mathbb{R}$. Thus we obtain that

$$|f_a(x)| \leq (\beta + |c|)\, |x|. \tag{51}$$

But we can choose a positive integer m with $m\mu > \beta + |c|$.

If $x \in \left(0, \frac{1}{6^{m-1}}\right)$, then $6^n x \in \left(0, \frac{1}{6}\right)$ for all $n = 0, 1, \ldots, m-1$. For this x, we get

$$f_a(x) = \sum_{n=0}^{\infty} \frac{\alpha(6^n x)}{6^n} \geq \sum_{n=0}^{m-1} \frac{\mu(6^n x)}{6^n} = m\mu x > (\beta + |c|)\, x$$

which contradicts (51). Therefore the additive functional equation (6) is not stable in the sense of Ulam, Hyers and Rassias if $s = \frac{1}{3}$, assumed in the inequality (40). □

Theorem 3.6. *Let $j \in \{-1, 1\}$ and $\alpha, \beta : X^3 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\alpha\left(6^{nj}x, 6^{nj}y, 6^{nj}z\right)}{36^{nj}} = 0 \tag{52}$$

for all $x, y, z \in X$. Let $f_q : X \to Y$ be an even function satisfying the inequality

$$\|Df_q(x, y, z)\| \leq \alpha\,(x, y, z) \tag{53}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ which satisfies (6) and

$$\|f_q(x) - Q(x)\| \leq \frac{1}{72} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\beta(6^{kj}x)}{36^{kj}} \tag{54}$$

where $\beta(6^{kj}x)$ and $Q(x)$ are defined by the two relations:

$$\beta(6^{kj}x) = 2\alpha(6^{kj}x, 6^{kj}x, 6^{kj}x) + \alpha(6^{kj}x, 0, 6^{kj}x) \tag{55}$$

$$Q(x) = \lim_{n\to\infty} \frac{f_q(6^{nj}x)}{36^{nj}} \tag{56}$$

respectively, for all $x \in X$.

Proof. Assume $j = 1$. Replacing (x, y, z) by (x, x, x) in (53) and using evenness of f_q, we get

$$\|f_q(6x) + f_q(2x) + f_q(4x) - 56f_q(x)\| \leq \alpha\,(x, x, x) \tag{57}$$

for all $x \in X$. Again replacing (x, y, z) by $(x, 0, x)$ in (53) and using evenness of f_q, we obtain

$$\|2f_q(4x) + 2f_q(2x) - 40f_q(x)\| \leq \alpha\,(x, 0, x) \tag{58}$$

for all $x \in X$. It follows form (57) and (58) that

$$\begin{aligned} \|2f_q(6x) - 72f_q(x)\| &\leq 2\,\|f_q(6x) + f_q(2x) + f_q(4x) - 56f_q(x)\| \\ &\quad + \|2f_q(4x) + 2f_q(2x) - 40f_q(x)\| \\ &\leq 2\alpha\,(x,x,x) + \alpha\,(x,0,x) \end{aligned} \tag{59}$$

for all $x \in X$. Dividing the above inequality by 72, we arrive

$$\left\| \frac{f_q(6x)}{36} - f_q(x) \right\| \leq \frac{\beta(x)}{72} \tag{60}$$

where

$$\beta(x) = 2\alpha\,(x,x,x) + \alpha\,(x,0,x)$$

for all $x \in X$. The rest of the proof is similar to that one of Theorem 3.1. □

The following Corollary is an immediate consequence of Theorem 3.6 concerning the stability of (6).

Corollary 3.7. *Let λ and s be nonnegative real numbers. Let an even function $f_q : X \to Y$ satisfy the inequality*

$$\|Df_q(x,y,z)\| \leq \begin{cases} \lambda, & \\ \lambda\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s < 2 \quad or \quad s > 2; \\ \lambda||x||^s||y||^s||z||^s, & 3s < 2 \quad or \quad 3s > 2; \\ \lambda\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s < 2 \quad or \quad 3s > 2; \end{cases} \tag{61}$$

for all $x, y, z \in X$. Then there exists a unique quadratic function $Q : X \to Y$ such that

$$\|f_q(x) - Q(x)\| \leq \begin{cases} \dfrac{3\lambda}{70}, \\ \dfrac{4\lambda||x||^s}{|36 - 6^s|}, \\ \dfrac{\lambda||x||^{3s}}{|36 - 6^{3s}|}, \\ \dfrac{5\lambda||x||^{3s}}{|36 - 6^{3s}|} \end{cases} \tag{62}$$

for all $x \in X$.

Now we provide an example to illustrate that the functional equation (6) is not stable for $s = 2$ in Condition (ii) of Corollary 3.7.

Example 3.8. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\alpha(x) = \begin{cases} \mu x^2, & if\ |x| < 2 \\ \mu, & otherwise \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f_q : \mathbb{R} \to \mathbb{R}$ by

$$f_q(x) = \sum_{n=0}^{\infty} \frac{\phi(6^n x)}{36^n} \qquad for\ all \quad x \in \mathbb{R}.$$

Then f_q satisfies the functional inequality

$$|Df_q(x,y,z)| \leq \left(\frac{12\mu \times 36^2}{7}\right)\left(|x|^2+|y|^2+|z|^2\right) \tag{63}$$

for all $x, y, z \in \mathbb{R}$. Then there is no a quadratic mapping $Q : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ such that

$$|f_q(x) - Q(x)| \leq \beta|x|^2 \qquad \text{for all} \quad x \in \mathbb{R}. \tag{64}$$

Proof. Now

$$|f_q(x)| \leq \sum_{n=0}^{\infty} \frac{|\phi(6^n x)|}{|36^n|} = \sum_{n=0}^{\infty} \frac{\mu}{36^n} = \frac{36\mu}{35}.$$

Therefore we see that f_q is bounded. We are going to prove that f_q satisfies (63).

If $x = y = z = 0$ then (63) is trivial. If $|x|^2 + |y|^2 + |z|^2 \geq \frac{1}{36}$ then the left hand side of (63) is less than $\left(\frac{12\mu\times 36}{7}\right)$. Now suppose that $0 < |x|^2 + |y|^2 + |z|^2 < \frac{1}{36}$. Then there exists a positive integer k such that

$$\frac{1}{36^{k+2}} \leq |x|^2 + |y|^2 + |z|^2 < \frac{1}{36^{k+1}}, \tag{65}$$

so that $6^{k-1}|x|^2 < \frac{1}{36}$, $6^{k-1}|y|^2 < \frac{1}{36}, 6^{k-1}|z|^2 < \frac{1}{36}$ and consequently

$$\begin{aligned} &6^{k-1}(x+2y+3z), 6^{k-1}(x-2y+3z), 6^{k-1}(x+2y-3z), 6^{k-1}(x-2y-3z),\\ &\qquad 6^{k-1}(x), 6^{k-1}(-x), 6^{k-1}(y), 6^{k-1}(-y), 6^{k-1}(z), 6^{k-1}(-z) \in \left(-\frac{1}{6}, \frac{1}{6}\right). \end{aligned}$$

Therefore for each $n = 0, 1, \ldots, k-1$, we have

$$\begin{aligned} &6^n(x+2y+3z), 6^n(x-2y+3z), 6^n(x+2y-3z), 6^n(x-2y-3z),\\ &\qquad 6^n(x), 6^n(-x), 6^n(y), 6^n(-y), 6^n(z), 6^n(-z) \in \left(-\frac{1}{6}\frac{1}{6}\right). \end{aligned}$$

and

$$\begin{aligned} &\phi\left(6^n(x+2y+3z)\right) + \phi\left(6^n(x-2y+3z)\right) + \phi\left(6^n(x+2y-3z)\right) + \phi\left(6^n(x-2y-3z)\right)\\ &\qquad - 4\phi(6^n x) - 8[\phi(6^n y) + \phi(-6^n y)] - 18[\phi(6^n z) + \phi(-6^n z)] = 0 \end{aligned}$$

for $n = 0, 1, \ldots, k-1$. From the definition of f_q and (65), we obtain that

$$\begin{aligned} &\Big|Df_q(x,y,z)\Big|\\ &= \sum_{n=0}^{\infty} \frac{1}{36^n}\Big|\phi\left(6^n(x+2y+3z)\right) + \phi\left(6^n(x-2y+3z)\right) + \phi\left(6^n(x+2y-3z)\right) + \phi\left(6^n(x-2y-3z)\right)\\ &\qquad - 4\phi(6^n x) - 8[\phi(6^n y) + \phi(-6^n y)] - 18[\phi(6^n z) + \phi(-6^n z)]\Big|\\ &= \sum_{n=k}^{\infty} \frac{1}{36^n}\Big|\phi\left(6^n(x+2y+3z)\right) + \phi\left(6^n(x-2y+3z)\right) + \phi\left(6^n(x+2y-3z)\right) + \phi\left(6^n(x-2y-3z)\right)\\ &\qquad - 4\phi(6^n x) - 8[\phi(6^n y) + \phi(-6^n y)] - 18[\phi(6^n z) + \phi(-6^n z)]\Big|\\ &\leq \sum_{n=k}^{\infty} \frac{1}{6^n} 60\mu = 60\ \mu \times \frac{1}{36^k} \times \frac{36}{35} \leq \left(\frac{12\mu \times 36^2}{7}\right)\left(|x|^2+|y|^2+|z|^2\right). \end{aligned}$$

Thus f_q satisfies (63) for all $x, y, z \in \mathbb{R}$ with $0 < |x|^2 + |y|^2 + |z|^2 < \frac{1}{6}$.

We claim that the additive functional equation (6) is not stable for $s = 2$ in condition (ii) of Corollary 3.2. Suppose on the contrary that there exists an additive mapping $Q : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ satisfying (64). Since f_q is bounded and continuous for all $x \in \mathbb{R}$, Q is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.1, Q must have the form $Q(x) = cx^2$ for any x in $\mathbb{R}$. Thus we obtain that

$$|f_q(x)| \leq (\beta + |c|)\, |x|^2. \tag{66}$$

But we can choose a positive integer m with $m\mu > \beta + |c|$.

If $x \in \left(0, \frac{1}{6^{m-1}}\right)$, then $6^n x \in (0, \frac{1}{6})$ for all $n = 0, 1, \ldots, m-1$. For this x, we get

$$f_q(x) = \sum_{n=0}^{\infty} \frac{\alpha(6^n x)}{36^n} \geq \sum_{n=0}^{m-1} \frac{\mu(6^n x)^2}{36^n} = m\mu x^2 > (\beta + |c|)\, x^2,$$

which contradicts (66). Therefore the additive functional equation (6) is not stable in the sense of Ulam, Hyers and Rassias if $s = 2$, assumed in the inequality (40). □

A counter example to illustrate the non stability in Condition (iii) of Corollary 3.7:

Example 3.9. *Let s be such that $0 < s < \frac{2}{3}$. Then there is a function $f_q : \mathbb{R} \to \mathbb{R}$ and a constant $\lambda > 0$ satisfying*

$$|Df_q(x, y, z)| \leq \lambda |x|^{\frac{s}{3}}\ |y|^{\frac{s}{3}}\ |z|^{\frac{2-2s}{3}} \tag{67}$$

for all $x, y, z \in \mathbb{R}$ and

$$\sup_{x \neq 0} \frac{|f_q\,(x) - Q\,(x)|}{|x|^2} = +\infty \tag{68}$$

for every quadratic mapping $Q : \mathbb{R} \to \mathbb{R}$.

Proof. If we take

$$f_q\,(x) = \begin{cases} x^2 \ln |x|\,, & if \quad x \neq 0, \\ 0, & if \quad x = 0, \end{cases}$$

then from the relation (68), it follows that

$$\begin{aligned} \sup_{x \neq 0} \frac{|f_q\,(x) - Q\,(x)|}{|x|^2} &\geq \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \frac{|f\,(n) - Q\,(n)|}{|n|^2} \\ &= \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \frac{\left|n^2 (1)^2 \ln |n| - n^2\ Q\,(1)\right|}{|n|^2} \\ &= \sup_{\substack{n \in \mathbb{N} \\ n \neq 0}} \ |\ln |n| - Q\,(1)|\ = \infty. \end{aligned}$$

We have to prove that (67) is true.

Case (i): If $x, y, z > 0$ than $x + 2z + 3z > 0, x - 2z + 3z > 0,$

$x + 2z - 3z > 0, x - 2z - 3z > 0$ and therefore (46) becomes,

$$\begin{aligned} &|f_q(x + 2y + 3z) + f_q(x - 2y + 3z) + f_q(x + 2y - 3z) + f_q(x - 2y - 3z) \\ &\qquad -4f_q(x) - 8[f_q(y) + f_q(-y)] - 18[f_q(z) + f_q(-z)]|\,. \\ &= \left|(x + 2y + 3z)^2 \ln |x + 2y + 3z| + (x - 2y + 3z)^2 \ln |x - 2y + 3z| + (x + 2y - 3z)^2 \ln |x + 2y - 3z|\right. \\ &\qquad \left. +(x - 2y - 3z)^2 \ln |x - 2y - 3z| - 4x^2 \ln |x| - 8[y^2 \ln |y| + y^2 \ln |-y|] - 18[z^2 \ln |z| + z^2 \ln |-z|]\right|. \end{aligned}$$

Set $x = u, y = v, z = w$ it follows that

$$\begin{aligned}
&|f_q(x+2y+3z) + f_q(x-2y+3z) + f_q(x+2y-3z) + f_q(x-2y-3z) \\
&\qquad -4f_q(x) - 8[f_q(y) + f_q(-y)] - 18[f_q(z) + f_q(-z)]| \\
&= \left|(u+2v+3w)^2 \ln|u+2v+3w| + (u-2v+3w)^2 \ln|u-2v+3w| + (u+2v-3w)^2 \ln|u+2v-3w| \right. \\
&\qquad \left. +(u-2v-3w)^2 \ln|u-2v-3w| - 4u^2 \ln|u| - 8[v^2 \ln|v| + v^2 \ln|-v|] - 18[w^2 \ln|w| + w^2 \ln|-w|]\right|. \\
&|f_q(u+2v+3w) + f_q(u-2v+3w) + f_q(u+2v-3w) + f_q(u-2v-3w) \\
&\qquad -4f_q(u) - 8[f_q(v) + f_q(-v)] - 18[f_q(w) + f_q(-w)]|. \\
&\leq \lambda |u|^{\frac{s}{3}} \ |v|^{\frac{s}{3}} \ |w|^{\frac{2-2s}{3}} = \lambda |x|^{\frac{s}{3}} \ |y|^{\frac{s}{3}} \ |z|^{\frac{2-2s}{3}}.
\end{aligned}$$

Case (ii): If $x, y, z < 0$ than $x + 2z + 3z > 0, x - 2z + 3z > 0$, $x + 2z - 3z > 0, x - 2z - 3z > 0$ and therefore (46) becomes,

$$\begin{aligned}
&|f_q(x+2y+3z) + f_q(x-2y+3z) + f_q(x+2y-3z) + f_q(x-2y-3z) \\
&\qquad -4f_q(x) - 8[f_q(y) + f_q(-y)] - 18[f_q(z) + f_q(-z)]|. \\
&= \left|(x+2y+3z)^2 \ln|x+2y+3z| + (x-2y+3z)^2 \ln|x-2y+3z| + (x+2y-3z)^2 \ln|x+2y-3z| \right. \\
&\qquad \left. +(x-2y-3z)^2 \ln|x-2y-3z| - 4x^2 \ln|x| - 8[y^2 \ln|y| + y^2 \ln|-y|] - 18[z^2 \ln|z| + z^2 \ln|-z|]\right|.
\end{aligned}$$

Set $x = -u, y = -v, z = -w$ it follows that

$$\begin{aligned}
&|f_q(x+2y+3z) + f_q(x-2y+3z) + f_q(x+2y-3z) + f_q(x-2y-3z) \\
&\qquad -4f_q(x) - 8[f_q(y) + f_q(-y)] - 18[f_q(z) + f_q(-z)]| \\
&= \left|(-u-2v-3w)^2 \ln|-u-2v-3w| + (-u+2v-3w)^2 \ln|-u+2v-3w| \right. \\
&\qquad +(-u-2v+3w)^2 \ln|-u-2v+3w| + (-u+2v+3w)^2 \ln|-u+2v+3w| \\
&\qquad \left. +4u^2 \ln|-u| - 8[v^2 \ln|-v| + v^2 \ln|v|] - 18[w^2 \ln|-w| + w^2 \ln|w|]\right|. \\
&|f_q(-u-2v-3w) + f_q(-u+2v-3w) + f_q(-u-2v+3w) + f_q(-u+2v+3w) \\
&\qquad -4f_q(-u) - 8[f_q(-v) + f_q(v)] - 18[f_q(-w) + f_q(w)]|. \\
&\leq \lambda |-u|^{\frac{s}{3}} \ |-v|^{\frac{s}{3}} \ |-w|^{\frac{2-2s}{3}} = \lambda |x|^{\frac{s}{3}} \ |y|^{\frac{s}{3}} \ |z|^{\frac{2-2s}{3}}.
\end{aligned}$$

Case (iii): If $x > 0, y < 0, z < 0$ than $x + 2z + 3z < 0, x - 2z + 3z < 0$, $x + 2z - 3z < 0, x - 2z - 3z < 0$ and therefore (46) becomes,

$$\begin{aligned}
&|f_q(x+2y+3z) + f_q(x-2y+3z) + f_q(x+2y-3z) + f_q(x-2y-3z) \\
&\qquad -4f_q(x) - 8[f_q(y) + f_q(-y)] - 18[f_q(z) + f_q(-z)]|. \\
&= \left|(x+2y+3z)^2 \ln|x+2y+3z| + (x-2y+3z)^2 \ln|x-2y+3z| + (x+2y-3z)^2 \ln|x+2y-3z| \right. \\
&\qquad \left. +(x-2y-3z)^2 \ln|x-2y-3z| - 4x^2 \ln|x| - 8[y^2 \ln|y| + y^2 \ln|-y|] - 18[z^2 \ln|z| + z^2 \ln|-z|]\right|.
\end{aligned}$$

Set $x = u, y = -v, z = -w$ it follows that

$$\begin{aligned}
&|f_q(x+2y+3z) + f_q(x-2y+3z) + f_q(x+2y-3z) + f_q(x-2y-3z) \\
&\qquad -4f_q(x) - 8[f_q(y) + f_q(-y)] - 18[f_q(z) + f_q(-z)]| \\
&= \big|(u-2v-3w)^2 \ln|u-2v-3w| + (u+2v-3w)^2 \ln|u+2v-3w| + (u-2v+3w)^2 \ln|u-2v+3w| \\
&\qquad +(u+2v+3w)^2 \ln|u+2v+3w| - 4u^2 \ln|u| - 8[v^2 \ln|-v| + v^2 \ln|v|] - 18[w^2 \ln|-w| + w^2 \ln|w|]\big|. \\
&|f_q(u-2v-3w) + f_q(u+2v-3w) + f_q(u-2v+3w) + f_q(u+2v+3w) \\
&\qquad -4f_q(u) - 8[f_q(-v) + f_q(v)] - 18[f_q(-w) + f_q(w)]|. \\
&\qquad \le \lambda|u|^{\frac{s}{3}}\ |-v|^{\frac{s}{3}}\ |-w|^{\frac{2-2s}{3}} = \lambda|x|^{\frac{s}{3}}\ |y|^{\frac{s}{3}}\ |z|^{\frac{2-2s}{3}}.
\end{aligned}$$

Case (iv): If $x < 0, y > 0, z > 0$ than $x + 2z + 3z < 0, x - 2z + 3z < 0,$
$x + 2z - 3z < 0, x - 2z - 3z < 0$ and therefore (46) becomes,

$$\begin{aligned}
&|f_q(x+2y+3z) + f_q(x-2y+3z) + f_q(x+2y-3z) + f_q(x-2y-3z) \\
&\qquad -4f_q(x) - 8[f_q(y) + f_q(-y)] - 18[f_q(z) + f_q(-z)]|. \\
&= \big|(x+2y+3z)^2 \ln|x+2y+3z| + (x-2y+3z)^2 \ln|x-2y+3z| + (x+2y-3z)^2 \ln|x+2y-3z| \\
&\qquad +(x-2y-3z)^2 \ln|x-2y-3z| - 4x^2 \ln|x| - 8[y^2 \ln|y| + y^2 \ln|-y|] - 18[z^2 \ln|z| + z^2 \ln|-z|]\big|.
\end{aligned}$$

Set $x = -u, y = v, z = w$ it follows that

$$\begin{aligned}
&|f_q(x+2y+3z) + f_q(x-2y+3z) + f_q(x+2y-3z) + f_q(x-2y-3z) \\
&\qquad -4f_q(x) - 8[f_q(y) + f_q(-y)] - 18[f_q(z) + f_q(-z)]| \\
&= \big|(-u+2v+3w)^2 \ln|-u+2v+3w| + (-u-2v+3w)^2 \ln|-u-2v+3w| \\
&\qquad +(-u+2v-3w)^2 \ln|-u+2v-3w| + (-u-2v-3w)^2 \ln|-u-2v-3w| \\
&\qquad +4u^2 \ln|u| - 8[v^2 \ln|v| + v^2 \ln|-v|] - 18[w^2 \ln|w| + w^2 \ln|-w|]\big|. \\
&= |f_q(-u+2v+3w) + f_q(-u-2v+3w) + f_q(-u+2v-3w) + f_q(-u-2v-3w) \\
&\qquad -4f_q(-u) - 8[f_q(v) + f_q(-v)] - 18[f_q(w) + f_q(-w)]|. \\
&\le \lambda|-u|^{\frac{s}{3}}\ |v|^{\frac{s}{3}}\ |w|^{\frac{2-2s}{3}} = \lambda|x|^{\frac{s}{3}}\ |y|^{\frac{s}{3}}\ |z|^{\frac{2-2s}{3}}.
\end{aligned}$$

Case (v): If $x = y = z = 0$ in (46) then it is trivial. □

Now we provide an example to illustrate that the functional equation (6) is not stable for $s = \frac{2}{3}$ in Condition (*iii*) of Corollary 3.7.

Example 3.10. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$\phi(x) = \begin{cases} \mu x^2, & \text{if } |x| < \frac{2}{3}, \\ \frac{2\mu}{3}, & \text{otherwise} \end{cases}$$

where $\mu > 0$ is a constant, and define a function $f_q : \mathbb{R} \to \mathbb{R}$ by

$$f_q(x) = \sum_{n=0}^{\infty} \frac{\phi(6^n x)}{36^n} \qquad \text{for all} \quad x \in \mathbb{R}.$$

Then f_q satisfies the functional inequality

$$|Df_q(x,y,z)| \leq \frac{1440\ \mu}{35}\left(|x|^{\frac{2}{3}}|y|^{\frac{2}{3}}|z|^{\frac{2}{3}}+|x|^2+|y|^2+|z|^2\right), \tag{69}$$

for all $x,y,z \in \mathbb{R}$. Then there doesn't exist a quadratic mapping $Q:\mathbb{R}\to\mathbb{R}$ and a constant $\beta>0$ such that

$$|f_q(x)-Q(x)| \leq \beta|x|^2 \qquad for\ all \quad x\in\mathbb{R}. \tag{70}$$

Proof. Now

$$|f_q(x)| \leq \sum_{n=0}^{\infty}\frac{|\phi(6^n x)|}{|36^n|} = \sum_{n=0}^{\infty}\frac{\mu}{3}\times\frac{1}{36^n} = \frac{2\mu}{5}.$$

Therefore we see that f_q is bounded. We are going to prove that f_q satisfies (69).
If $x=y=z=0$, then (69) is trivial. If $|x|^{\frac{2}{3}}|y|^{\frac{2}{3}}|z|^{\frac{2}{3}}+|x|^2+|y|^2+|z|^2 \geq \frac{1}{36}$ then the left hand side of (48) is less than $\frac{1440\ \mu}{35}$. Now suppose that $0<|x|^{\frac{2}{3}}|y|^{\frac{2}{3}}|z|^{\frac{2}{3}}+|x|^2+|y|^2+|z|^2<\frac{1}{36}$. Then there exists a positive integer k such that

$$\frac{1}{36^{k+2}} \leq |x|^{\frac{2}{3}}|y|^{\frac{2}{3}}|z|^{\frac{2}{3}}+|x|^2+|y|^2+|z|^2 < \frac{1}{36^{k+1}}, \tag{71}$$

so that $6^{k-1}|x|^{\frac{2}{3}}|y|^{\frac{2}{3}}|z|^{\frac{2}{3}}<\frac{1}{6}, 6^{k-1}|x|<\frac{1}{6},\ 6^{k-1}|y|<\frac{1}{6}, 6^{k-1}|z|<\frac{1}{6}$ and consequently

$$\begin{aligned}6^{k-1}(x+2y+3z), 6^{k-1}(x-2y+3z), 6^{k-1}(x+2y-3z), 6^{k-1}(x-2y-3z),\\ 6^{k-1}(x), 6^{k-1}(-x), 6^{k-1}(y), 6^{k-1}(-y), 6^{k-1}(z), 6^{k-1}(-z) \in \left(\frac{-1}{6},\frac{1}{6}\right).\end{aligned}$$

Therefore for each $n=0,1,\ldots,k-1$, we have

$$\begin{aligned}6^n(x+2y+3z), 6^n(x-2y+3z), 6^n(x+2y-3z), 6^n(x-2y-3z),\\ 6^n(x), 6^n(-x), 6^n(y), 6^n(-y), 6^n(z), 6^n(-z) \in \left(\frac{-1}{6},\frac{1}{6}\right)\end{aligned}$$

and

$$\begin{aligned}&\phi\left(6^n(x+2y+3z)\right)+\phi\left(6^n(x-2y+3z)\right)+\phi\left(6^n(x+2y-3z)\right)+\phi\left(6^n(x-2y-3z)\right)\\ &\qquad -4\phi(6^n x)-8[\phi(6^n y)+\phi(-6^n y)]-18[\phi(6^n z)+\phi(-6^n z)]=0\end{aligned}$$

for $n=0,1,\ldots,k-1$. From the definition of f_q and (71), we obtain that

$$\begin{aligned}&\Big|Df_q(x,y,z)\Big|\\ &=\sum_{n=0}^{\infty}\frac{1}{36^n}\Big|\phi\left(6^n(x+2y+3z)\right)+\phi\left(6^n(x-2y+3z)\right)+\phi\left(6^n(x+2y-3z)\right)+\phi\left(6^n(x-2y-3z)\right)\\ &\qquad -4\phi(6^n x)-8[\phi(6^n y)+\phi(-6^n y)]-18[\phi(6^n z)+\phi(-6^n z)]\Big|\\ &=\sum_{n=k}^{\infty}\frac{1}{36^n}\Big|\phi\left(6^n(x+2y+3z)\right)+\phi\left(6^n(x-2y+3z)\right)+\phi\left(6^n(x+2y-3z)\right)+\phi\left(6^n(x-2y-3z)\right)\\ &\qquad -4\phi(6^n x)-8[\phi(6^n y)+\phi(-6^n y)]-18[\phi(6^n z)+\phi(-6^n z)]\Big|\\ &\leq \sum_{n=k}^{\infty}\frac{1}{36^n}\frac{120\ \mu}{3}=\frac{120\ \mu}{3}\times\frac{1}{36^k}\times\frac{36}{35}\leq\frac{1440\ \mu}{35}\left(|x|^{\frac{2}{3}}|y|^{\frac{2}{3}}|z|^{\frac{2}{3}}+|x|^2+|y|^2+|z|^2\right).\end{aligned}$$

Thus f_q satisfies (69) for all $x, y, z \in \mathbb{R}$ with $0 < |x|^{\frac{2}{3}}|y|^{\frac{2}{3}}|z|^{\frac{2}{3}} + |x|^2 + |y|^2 + |z|^2 < \frac{1}{6}$.

We claim that the additive functional equation (6) is not stable for $s = \frac{2}{3}$ in condition (iv) of Corollary 3.2. Suppose on the contrary that there exists an additive mapping $Q : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ satisfying (70). Since f_q is bounded and continuous for all $x \in \mathbb{R}$, Q is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.1, A must have the form $Q(x) = cx^2$ for any x in $\mathbb{R}$. Thus we obtain that

$$|f_q(x)| \leq (\beta + |c|)\,|x|^2. \tag{72}$$

But we can choose a positive integer m with $m\mu > \beta + |c|$.

If $x \in \left(0, \frac{1}{6^{m-1}}\right)$, then $6^n x \in \left(0, \frac{1}{6}\right)$ for all $n = 0, 1, \ldots, m-1$. For this x, we get

$$f_q(x) = \sum_{n=0}^{\infty} \frac{\alpha(6^n x)}{36^n} \geq \sum_{n=0}^{m-1} \frac{\mu(6^n x)^2}{36^n} = m\mu x^2 > (\beta + |c|)\, x^2,$$

which contradicts (72). Therefore the quadratic functional equation (6) is not stable in the sense of Ulam, Hyers and Rassias if $s = \frac{2}{3}$, assumed in the inequality (40). □

Theorem 3.11. *Let $j \in \{-1, 1\}$ and $\alpha, \beta : X^3 \to [0, \infty)$ be a function satisfying (28) and (52) for all $x, y, z \in X$. Let $f : X \to Y$ be a function satisfying the inequality*

$$\|Df(x, y, z)\| \leq \alpha\,(x, y, z) \tag{73}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ satisfying (6) and

$$\|f(x) - A(x) - Q(x)\| \leq \frac{1}{2}\left[\frac{1}{12}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\beta(6^{kj}x) + \beta(-6^{kj}x)}{6^{kj}}\right) + \frac{1}{72}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\beta(6^{kj}x) + \beta(-6^{kj}x)}{36^{kj}}\right)\right], \tag{74}$$

where $\beta(6^{kj}x)$, $A(x)$ and $Q(x)$ are defined in (31), (55),(32) and (56), respectively, for all $x \in X$.

Proof. Let $f_o(x) = \dfrac{f_a(x) - f_a(-x)}{2}$ for all $x \in X$. Then $f_o(0) = 0$ and $f_o(-x) = -f_o(x)$ for all $x \in X$. Hence

$$\|Df_o(x, y, z)\| \leq \frac{\alpha(x, y, z)}{2} + \frac{\alpha(-x, -y, -z)}{2} \tag{75}$$

for all $x, y, z \in X$. By Theorem 3.1, we have

$$\|f_o(x) - A(x)\| \leq \frac{1}{24}\sum_{k=\frac{1-j}{2}}^{\infty}\left(\frac{\beta(6^{kj}x) + \beta(-6^{kj}x)}{6^{kj}}\right) \tag{76}$$

for all $x \in X$. Also, let $f_e(x) = \dfrac{f_q(x) + f_q(-x)}{2}$ for all $x \in X$. Then $f_e(0) = 0$ and $f_e(-x) = f_e(x)$ for all $x \in X$. Hence

$$\|Df_e(x, y, z)\| \leq \frac{\alpha(x, y, z)}{2} + \frac{\alpha(-x, -y, -z)}{2} \tag{77}$$

for all $x, y, z \in X$. By Theorem 3.6, we have

$$\|f_e(x) - Q(x)\| \leq \frac{1}{144} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\beta(6^{kj}x) + \beta(-6^{kj}x)}{36^{kj}} \right) \tag{78}$$

for all $x \in X$. Define

$$f(x) = f_e(x) + f_o(x) \tag{79}$$

for all $x \in X$. From (76),(78) and (79), we arrive

$$\begin{aligned}
\|f(x) - A(x) - Q(x)\| &= \|f_e(x) + f_o(x) - A(x) - Q(x)\| \\
&\leq \|f_o(x) - A(x)\| + \|f_e(x) - Q(x)\| \\
&\leq \frac{1}{2} \left[\frac{1}{12} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\beta(6^{kj}x) + \beta(-6^{kj}x)}{6^{kj}} \right) \right. \\
&\qquad \left. + \frac{1}{72} \sum_{k=\frac{1-j}{2}}^{\infty} \left(\frac{\beta(6^{kj}x) + \beta(-6^{kj}x)}{36^{kj}} \right) \right]
\end{aligned}$$

for all $x \in X$. Hence the theorem is proved. □

Using Corollaries 3.2 and 3.7, we have the following corollary concerning the stability of (6).

Corollary 3.12. *Let λ and s be nonnegative real numbers. Let a function $f : X \to Y$ satisfy the inequality*

$$\|Df(x, y, z)\| \leq \begin{cases} \lambda, & \\ \lambda \left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \neq 1, 2; \\ \lambda ||x||^s ||y||^s ||z||^s, & 3s \neq 1, 2; \\ \lambda \left\{||x||^s ||y||^s ||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \neq 1, 2; \end{cases} \tag{80}$$

for all $x, y, z \in X$. Then there exists a unique additive function $A : X \to Y$ and a unique quadratic function $Q : X \to Y$ such that

$$\|f(x) - A(x) - Q(x)\| \leq \begin{cases} 3\lambda \left(\frac{1}{10} + \frac{1}{70} \right), \\ 4\lambda ||x||^s \left(\frac{1}{|6 - 6^s|} + \frac{1}{|36 - 6^s|} \right), \\ \lambda ||x||^{3s} \left(\frac{1}{|6 - 6^{3s}|} + \frac{1}{|36 - 6^{3s}|} \right), \\ 5\lambda ||x||^{3s} \left(\frac{1}{|6 - 6^{3s}|} + \frac{1}{|36 - 6^{3s}|} \right), \end{cases} \tag{81}$$

for all $x \in X$.

4. Stability Results: Fixed Point Method

In this section, the authors have proved the generalized Ulam - Hyers stability of functional equation (6) in Banach spaces with the help of the fixed point method. Now we will recall the fundamental result in the fixed point theory.

Theorem 4.1. *[24](The alternative of fixed point) Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $T : X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either*

(B_1) $$d(T^n x, T^{n+1}x) = \infty \quad \forall \quad n \geq 0,$$

or

(B_2) *there exists a natural number n_0 such that:*

(i) $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;

(ii) The sequence $(T^n x)$ is convergent to a fixed point y^* of T;

(iii) y^* is the unique fixed point of T in the set $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;

(iv) $d(y^*, y) \leq \frac{1}{1-L}\ d(y, Ty)$ for all $y \in Y$.

Hereafter throughout this section, let us assume V be a vector space and B Banach space respectively. Define a mapping $Df : V \to B$ by

$$\begin{aligned} Df(x, y, z) = f(x + 2y + 3z) + f(x - 2y + 3z) + f(x + 2y - 3z) + f(x - 2y - 3z) \\ - 4f(x) - 8[f(y) + f(-y)] - 18[f(z) + f(-z)] \end{aligned}$$

for all $x, y, z \in V$.

Theorem 4.2. *Let $f_a : V \to B$ be a mapping for which there exists functions $\alpha, \beta, \gamma : V^3 \to [0, \infty)$ with the condition*

$$\lim_{k\to\infty} \frac{\alpha(\mu_i^k x, \mu_i^k y,\ \mu_i^k z)}{\mu_i^k} = 0, \tag{82}$$

where

$$\mu_i = \begin{cases} 6, & i = 0, \\ \frac{1}{6}, & i = 1 \end{cases}$$

satisfying the functional inequality

$$\|Df_a(x, y, z)\| \leq \alpha(x, y, z) \tag{83}$$

for all $x, y, z \in V$. If there exists an $L = L(i) < 1$ such that the function

$$x \to \gamma(x) = \frac{1}{2}\beta\left(\frac{x}{6}\right),$$

one has the property

$$\gamma(x) = L\ \mu_i\ \gamma\left(\frac{x}{\mu_i}\right) \tag{84}$$

for all $x \in V$. Then there exists a unique additive function $A : V \to B$ satisfying the functional equation (6) and

$$\|f_a(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\gamma(x) \tag{85}$$

holds for all $x \in V$.

Proof. Consider the set $X = \{p/p : V \to B,\ p(0) = 0\}$ and introduce the generalized metric on X,

$$d(p,q) = \inf\{K \in (0,\infty) : \| p(x) - q(x) \| \le K\gamma(x), x \in V\}.$$

It is easy to see that (X,d) is complete.

Define $T : X \to X$ by

$$Tp(x) = \frac{1}{\mu_i} p(\mu_i x), \forall \qquad x \in V.$$

Now $p,q \in X$,

$$\begin{aligned} d(p,q) \le K \Rightarrow & \| p(x) - q(x) \| \le K\gamma(x), x \in V. \\ & \Rightarrow \left\| \frac{1}{\mu_i} p(\mu_i x) - \frac{1}{\mu_i} q(\mu_i x) \right\| \le \frac{1}{\mu_i} K\gamma(\mu_i x), x \in V, \\ & \Rightarrow \left\| \frac{1}{\mu_i} p(\mu_i x) - \frac{1}{\mu_i} q(\mu_i x) \right\| \le LK\gamma(x), x \in V, \\ & \Rightarrow \| Tp(x) - Tq(x) \| \le LK\gamma(x), x \in V, \\ & \Rightarrow d(Tp,Tq) \le LK. \end{aligned}$$

This implies

$$d(Tp,Tq) \le Ld(p,q),$$

for all $p,q \in X$. i.e., T is a strictly contractive mapping on X with Lipschitz constant L.

From (36), we have

$$\left\| \frac{f_a(6x)}{6} - f_a(x) \right\| \le \frac{\beta(x)}{12} \tag{86}$$

where

$$\beta(x) = 2\alpha(x,x,x) + \alpha(x,0,x)$$

for all $x \in V$. Using (84) for the case $i = 0$, it reduces to

$$\left\| \frac{1}{6} f_a(6x) - f_a(x) \right\| \le \frac{1}{6}\gamma(x)$$

for all $x \in V$.

$$\text{i.e.,} \qquad d(Tf_a, f_a) \le \frac{1}{6} = L = L^{1-0} = L^{1-i} < \infty.$$

Again replacing $x = \frac{x}{6}$ in (86), we get

$$\left\| f_a(x) - 6f\left(\frac{x}{6}\right) \right\| \le \frac{1}{2}\beta\left(\frac{x}{6}\right).$$

for all $x \in V$. Using (84) for the case $i = 1$, it reduces to

$$\left\| f_a(x) - 6f\left(\frac{x}{6}\right) \right\| \le \gamma(x)$$

for all $x \in V$.

$$\text{i.e.,} \qquad d(f_a, Tf_a) \le 1 = L^0 = L^{1-1} = L^{1-i} < \infty.$$

In the above cases, we arrive

$$d(f_a, Tf_a) \leq L^{1-i}.$$

Therefore $(B_2(i))$ holds.

By $(B_2(ii))$, it follows that there exists a fixed point A of T in X such that

$$A(x) = \lim_{k\to\infty} \frac{f_a(\mu_i^k x)}{\mu_i^k}, \qquad \forall \;\; x \in V. \tag{87}$$

Claim that $A : V \to B$ is additive. Replacing (x, y, z) by $\left(\mu_i^k x, \mu_i^k y, \mu_i^k z\right)$ in (83) and dividing by μ_i^k, it follows from (82) and (87), A satisfies (6) for all $x, y, z \in V$.

By $(B_2(iii))$, A is the unique fixed point of T in the set $Y = \{f_a \in X : d(Tf_a, A) < \infty\}$, using the fixed point alternative result A is the unique function such that

$$\|f_a(x) - A(x)\| \leq K\gamma(x)$$

for all $x \in V$ and $K > 0$. Finally by $(B_2(iv))$, we obtain

$$d(f_a, A) \leq \frac{1}{1-L} d(f_a, Tf_a)$$

implying

$$d(f_a, A) \leq \frac{L^{1-i}}{1-L}.$$

Hence we conclude that

$$\| \; f_a(x) - A(x) \; \| \leq \frac{L^{1-i}}{1-L}\gamma(x).$$

for all $x \in V$. This completes the proof of the theorem. □

From Theorem 4.2, we obtain the following corollary concerning the stability for the functional equation (6).

Corollary 4.3. *Let $f_a : V \to B$ be a mapping and there exist real numbers λ and s such that*

$$\|Df_a(x,y,z)\| \leq \begin{cases} (i) & \lambda, \\ (ii) & \lambda\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s < 1 \quad or \quad s > 1; \\ (iii) & \lambda \; ||x||^s||y||^s||z||^s, & 3s < 1 \quad or \quad 3s > 1; \\ (iv) & \lambda\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s < 1 \quad or \quad 3s > 1; \end{cases} \tag{88}$$

for all $x, y, z \in V$, then there exists a unique additive function $A : V \to B$ such that

$$\|f_a(x) - A(x)\| \leq \begin{cases} (i) & \frac{3\lambda}{10}, \\ (ii) & \frac{4\lambda||x||^s}{|6-6^s|}, \\ (iii) & \frac{\lambda||x||^{3s}}{|6-6^{3s}|} \\ (iv) & \frac{5\lambda||x||^{3s}}{|6-6^{3s}|} \end{cases} \tag{89}$$

for all $x \in V$.

Proof. Let us set

$$\alpha(x,y,z)=\begin{cases}\lambda,\\ \lambda\{||x||^s+||y||^s+||z||^s\},\\ \lambda\ \ ||x||^s\ ||y||^s\ ||z||^s,\\ \lambda\{||x||^s||y||^s||z||^s+(||x||^{3s}+||y||^{3s}+||z||^{3s})\}\end{cases}$$

for all $x,y,z\in V$. Now

$$\frac{\alpha(\mu_i^k x,\mu_i^k y,\mu_i^k z)}{\mu_i^k}=\begin{cases}\frac{\lambda}{\mu_i^k},\\ \frac{\lambda}{\mu_i^k}\left\{||\mu_i^k x||^s+||\mu_i^k y||^s+||\mu_i^k z||^s\right\},\\ \frac{\lambda}{\mu_i^n}||\mu_i^n x||^s\ ||\mu_i^n y||^s\ ||\mu_i^n z||^s,\\ \frac{\lambda}{\mu_i^k}\left\{||\mu_i^k x||^s\ ||\mu_i^k y||^s\ ||\mu_i^k z||^s\left\{||\mu_i^k x||^{3s}+||\mu_i^k y||^{3s}+||\mu_i^k z||^{3s}\right\}\right\}\end{cases}$$

$$=\begin{cases}\to 0 \text{ as } k\to\infty,\\ \to 0 \text{ as } k\to\infty,\\ \to 0 \text{ as } k\to\infty,\\ \to 0 \text{ as } k\to\infty.\end{cases}$$

i.e., (82) is holds. But, we have

$$\gamma(x)=\frac{1}{2}\beta\left(\frac{x}{6}\right)=\frac{1}{2}\left[2\alpha\left(\frac{x}{6},\frac{x}{6},\frac{x}{6}\right)+\alpha\left(\frac{x}{6},0,\frac{x}{6}\right)\right].$$

Hence

$$\gamma(x)=\frac{1}{2}\left[2\alpha\left(\frac{x}{6},\frac{x}{6},\frac{x}{6}\right)+\alpha\left(\frac{x}{6},0,\frac{x}{6}\right)\right]=\begin{cases}\frac{3\lambda}{2},\\ \frac{4\lambda}{6^s}||x||^s,\\ \frac{\lambda}{6^{3s}}||x||^{3s},\\ \frac{5\lambda}{6^{3s}}||x||^{3s}.\end{cases}$$

Also,

$$\frac{1}{\mu_i}\gamma(\mu_i x)=\begin{cases}\frac{3\lambda}{\mu_i\cdot 2},\\ \frac{4\lambda}{\mu_i\cdot 6^s}||\mu_i x||^s,\\ \frac{\lambda}{\mu_i\cdot 6^{3s}}||\mu_i x||^{3s},\\ \frac{5\lambda}{\mu_i\cdot 6^{3s}}||\mu_i x||^{3s}.\end{cases}=\begin{cases}\mu_i^{-1}\frac{3\lambda}{2},\\ \mu_i^{s-1}\frac{4\lambda}{6^s}||x||^s,\\ \mu_i^{3s-1}\frac{\lambda}{6^{3s}}||x||^{3s},\\ \mu_i^{3s-1}\frac{5\lambda}{6^{3s}}||x||^{3s}.\end{cases}=\begin{cases}\mu_i^{-1}\gamma(x),\\ \mu_i^{s-1}\gamma(x),\\ \mu_i^{3s-1}\gamma(x),\\ \mu_i^{3s-1}\gamma(x).\end{cases}$$

Hence the inequality (84) holds either, $L=6^{-1}$ for $s=1$ if $i=0$, or $L=6$ for $s=0$ if $i=1$. Now from (85), we prove the following cases for condition (i).

Case:1 $L=6^{-1}$ for $s=1$ if $i=0$

$$\|f_a(x)-A(x)\|\leq\frac{L^{1-i}}{1-L}\gamma(x)=\frac{\left(6^{-1}\right)^{1-0}}{1-(6)^{-1}}\cdot\frac{3\lambda}{2}=\frac{3\lambda}{10}.$$

Case:2 $L=6$ for $s=0$ if $i=1$

$$\|f_a(x)-A(x)\|\leq\frac{L^{1-i}}{1-L}\gamma(x)=\frac{(6)^{1-1}}{1-6}\cdot\frac{3\lambda}{2}=\frac{-3\lambda}{10}.$$

Again, (84) holds either, $L = 6^{s-1}$ for $s < 1$ if $i = 0$, or $L = \frac{1}{6^{s-1}}$ for $s > 1$ if $i = 1$. Now from (85), we prove the following cases for condition (ii).

Case:1 $L = 6^{s-1}$ for $s < 1$ if $i = 0$

$$\|f_a(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\gamma(x) = \frac{\left(6^{s-1}\right)^{1-0}}{1-6^{s-1}}\frac{4\lambda}{6^s}||x||^s = \frac{6^s}{6-6^s}\frac{4\lambda}{6^s}||x||^s = \frac{4\lambda||x||^s}{6-6^s}.$$

Case:2 $L = \frac{1}{6^{s-1}}$ for $s > 1$ if $i = 1$

$$\|f_a(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\gamma(x) = \frac{\left(\frac{1}{6^{s-1}}\right)^{1-1}}{1-\frac{1}{6^{s-1}}}\frac{4\lambda}{6^s}||x||^s = \frac{6^s}{6^s-6}\frac{4\lambda}{6^s}||x||^s = \frac{4\lambda||x||^s}{6^s-6}.$$

Also, (84) holds either, $L = 6^{3s-1}$ for $3s < 1$ if $i = 0$, or $L = \frac{1}{6^{3s-1}}$ for $3s > 1$ if $i = 1$. Now from (85), we prove the following cases for condition (iii).

Case:1 $L = 6^{3s-1}$ for $3s < 1$ if $i = 0$

$$\|f_a(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\gamma(x) = \frac{\left(6^{3s-1}\right)^{1-0}}{1-6^{3s-1}}\frac{\lambda}{6^{3s}}||x||^{3s} = \frac{6^{3s}}{6-6^{3s}}\frac{\lambda}{6^{3s}}||x||^{3s} = \frac{\lambda||x||^{3s}}{6-6^{3s}}.$$

Case:2 $L = \frac{1}{6^{3s-1}}$ for $3s > 1$ if $i = 1$

$$\|f_a(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\gamma(x) = \frac{\left(\frac{1}{6^{3s-1}}\right)^{1-1}}{1-\frac{1}{6^{3s-1}}}\frac{\lambda}{6^{3s}}||x||^{3s} = \frac{6^{3s}}{6^{3s}-6}\frac{\lambda}{6^{3s}}||x||^{3s} = \frac{\lambda||x||^{3s}}{6^{3s}-6}.$$

Finally, (84) holds either, $L = 6^{3s-1}$ for $3s < 1$ if $i = 0$, or $L = \frac{1}{6^{3s-1}}$ for $3s > 1$ if $i = 1$. Now from (85), we prove the following cases for condition (iv).

Case:1 $L = 6^{3s-1}$ for $3s < 1$ if $i = 0$

$$\|f_a(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\gamma(x) = \frac{\left(6^{3s-1}\right)^{1-0}}{1-6^{3s-1}}\frac{5\lambda}{6^{3s}}||x||^{3s} = \frac{6^{3s}}{6-6^{3s}}\frac{5\lambda}{6^{3s}}||x||^{3s} = \frac{5\lambda||x||^{3s}}{6-6^{3s}}.$$

Case:2 $L = \frac{1}{6^{3s-1}}$ for $3s > 1$ if $i = 1$

$$\|f_a(x) - A(x)\| \leq \frac{L^{1-i}}{1-L}\gamma(x) = \frac{\left(\frac{1}{6^{3s-1}}\right)^{1-1}}{1-\frac{1}{6^{3s-1}}}\frac{5\lambda}{6^{3s}}||x||^{3s} = \frac{6^{3s}}{6^{3s}-6}\frac{5\lambda}{6^{3s}}||x||^{3s} = \frac{5\lambda||x||^{3s}}{6^{3s}-6}.$$

Hence the proof of the corollary. □

The proofs of the following Theorem and Corollary are similar to those proofs of Theorem 4.2 and Corollary 4.3 using (60). Hence we omit the proofs.

Theorem 4.4. *Let $f_q : V \to B$ be a mapping for which there exists functions $\alpha, \beta, \gamma : V^3 \to [0, \infty)$ with the condition*

$$\lim_{k\to\infty} \frac{\alpha(\mu_i^k x, \mu_i^k y, \ \mu_i^k z)}{\mu_i^{2k}} = 0 \tag{90}$$

where

$$\mu_i = \begin{cases} 6, & i = 0, \\ \frac{1}{6}, & i = 1 \end{cases}$$

satisfying the functional inequality

$$\|Df_q(x,y,z)\| \le \alpha(x,y,z) \tag{91}$$

for all $x, y, z \in V$. If there exists $L = L(i) < 1$ such that the function

$$x \to \gamma(x) = \frac{1}{2}\beta\left(\frac{x}{6}\right),$$

has the property

$$\gamma(x) = L\ \mu_i^2\ \gamma\left(\frac{x}{\mu_i}\right) \tag{92}$$

for all $x \in V$. Then there exists a unique quadratic function $Q : V \to B$ satisfying the functional equation (6) and

$$\|f_q(x) - Q(x)\| \le \frac{L^{1-i}}{1-L}\gamma(x) \tag{93}$$

holds for all $x \in V$.

Corollary 4.5. *Let $f_q : V \to B$ be a mapping and there exist real numbers λ and s such that*

$$\|Df_q(x,y,z)\| \le \begin{cases} (i) & \lambda, \\ (ii) & \lambda\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s < 2 \quad or \quad s > 2; \\ (iii) & \lambda\ ||x||^s||y||^s||z||^s, & 3s < 2 \quad or \quad 3s > 2; \\ (iv) & \lambda\left\{||x||^s||y||^s||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s < 2 \quad or \quad 3s > 2; \end{cases} \tag{94}$$

for all $x, y, z \in V$, then there exists a unique quadratic function $Q : V \to B$ such that

$$\|f_q(x) - Q(x)\| \le \begin{cases} (i) & \frac{3\lambda}{70}, \\ (ii) & \frac{4\lambda||x||^s}{|36-6^s|}, \\ (iii) & \frac{\lambda||x||^{3s}}{|36-6^{3s}|} \\ (iv) & \frac{5\lambda||x||^{3s}}{|36-6^{3s}|} \end{cases} \tag{95}$$

for all $x \in V$.

Theorem 4.6. *Let $f : V \to B$ be a mapping for which there exists functions $\alpha, \beta, \gamma : V^3 \to [0, \infty)$ with the condition (82) and (90), where*

$$\mu_i = \begin{cases} 6, & i = 0, \\ \frac{1}{6}, & i = 1 \end{cases}$$

such that the functional inequality

$$\|Df(x,y,z)\| \le \alpha(x,y,z) \tag{96}$$

holds for all $x, y, z \in V$. Assume there exists $L = L(i) < 1$ such that the function

$$x \to \gamma(x) = \frac{1}{2}\beta\left(\frac{x}{6}\right),$$

has the properties (84) and (92) for all $x \in V$. Then there exists a unique additive function $A : V \to B$ and a unique quadratic function $Q : V \to B$ satisfying the functional equation (6) and

$$\|f(x) - A(x) - Q(x)\| \leq \frac{L^{1-i}}{1-L}\left(\gamma(x) + \gamma(-x)\right) \tag{97}$$

holds for all $x \in V$.

Proof. Let $f_o(x) = \dfrac{f_a(x) - f_a(-x)}{2}$ for all $x \in V$. Then $f_o(0) = 0$ and $f_o(-x) = -f_o(x)$ for all $x \in V$. Hence

$$\|Df_o(x, y, z)\| \leq \frac{\alpha(x, y, z)}{2} + \frac{\alpha(-x, -y, -z)}{2}, \tag{98}$$

for all $x, y, z \in V$. By Theorem 4.2, we have

$$\|f_o(x) - A(x)\| \leq \frac{1}{2}\frac{L^{1-i}}{1-L}\left(\gamma(x) + \gamma(-x)\right), \tag{99}$$

for all $x \in V$. Also, let $f_e(x) = \dfrac{f_q(x) + f_q(-x)}{2}$ for all $x \in V$. Then $f_e(0) = 0$ and $f_e(-x) = f_e(x)$ for all $x \in V$. Hence

$$\|Df_e(x, y, z)\| \leq \frac{\alpha(x, y, z)}{2} + \frac{\alpha(-x, -y, -z)}{2}, \tag{100}$$

for all $x, y, z \in V$. By Theorem 4.4, we have

$$\|f_e(x) - Q(x)\| \leq \frac{1}{2}\frac{L^{1-i}}{1-L}\left(\gamma(x) + \gamma(-x)\right), \tag{101}$$

for all $x \in V$. Define

$$f(x) = f_e(x) + f_o(x) \tag{102}$$

for all $x \in V$. From (99),(101) and (102), we arrive

$$\begin{aligned}
\|f(x) - A(x) - Q(x)\| &= \|f_e(x) + f_o(x) - A(x) - Q(x)\| \\
&\leq \|f_o(x) - A(x)\| + \|f_e(x) - Q(x)\| \\
&\leq \frac{1}{2}\frac{L^{1-i}}{1-L}\left(\gamma(x) + \gamma(-x)\right) + \frac{1}{2}\frac{L^{1-i}}{1-L}\left(\gamma(x) + \gamma(-x)\right) \\
&\leq \frac{L^{1-i}}{1-L}\left(\gamma(x) + \gamma(-x)\right)
\end{aligned}$$

for all $x \in V$. Hence the theorem is proved. □

Using Corollaries 4.3 and 4.5, we have the following corollary concerning the stability of (6).

Corollary 4.7. *Let λ and s be nonnegative real numbers. Let a function $f : V \to B$ satisfy the inequality*

$$\|Df(x, y, z)\| \leq \begin{cases} (i) & \lambda, \\ (ii) & \lambda\left\{||x||^s + ||y||^s + ||z||^s\right\}, & s \neq 1, 2; \\ (iii) & \lambda\, ||x||^s ||y||^s ||z||^s, & 3s \neq 1, 2; \\ (iv) & \lambda\left\{||x||^s ||y||^s ||z||^s + \left\{||x||^{3s} + ||y||^{3s} + ||z||^{3s}\right\}\right\}, & 3s \neq 1, 2; \end{cases} \tag{103}$$

for all $x, y, z \in V$. Then there exists a unique additive function $A : V \to B$ and a unique quadratic function $Q : V \to B$ such that

$$\|f(x) - A(x) - Q(x)\| \leq \begin{cases} (i) & 3\lambda\left(\frac{1}{10} + \frac{1}{70}\right), \\ (ii) & 4\lambda||x||^s\left(\frac{1}{|6-6^s|} + \frac{1}{|36-6^s|}\right), \\ (iii) & \lambda||x||^{3s}\left(\frac{1}{|6-6^{3s}|} + \frac{1}{|36-6^{3s}|}\right), \\ (iv) & 5\lambda||x||^{3s}\left(\frac{1}{|6-6^{3s}|} + \frac{1}{|36-6^{3s}|}\right), \end{cases} \tag{104}$$

for all $x \in V$.

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