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Homogenization of a Nonlinear Fibre-Reinforced Structure with Contact Conditions on the Interface Matrix-Fibres

Research Article

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Abstract: We study the homogenization of a nonlinear problem posed in a fibre-reinforced composite with matrix-fibres interfacial condition depending on a parameter $\lambda = \lambda(\varepsilon)$, ε being the size of the basic cell. Using Γ-convergence methods, three homogenized problems are determined according to the limit of the ratio$\gamma = \frac{\lambda(\varepsilon)}{\varepsilon}$. The main result is that the effective constitutive relations reveal non-local terms associated with the microscopic interactions between the matrix and the fibers.

Keywords: Composite material, periodic fibres, interfacial conditions, Γ-convergence, homogenized models.

1. Introduction

The nonlinear differential equation:

$$(-div(|\nabla u|) = f, p > 1)\,, \tag{1}$$

together with appropriate boundary conditions, describes a variety of physical phenomena. This equation appears in some nonlinear diffusion problems [3] as well as in the nonlinear filtration theory of gases and liquids in cracked porous media (see for instance [6]). This equation also occurs in plasticity problems involving a power-hardening stress-strain law given by

$$e_i j(u) = \lambda |\sigma(u)|^{q-2}\sigma_{ij}(u); \quad i,j = 1,2,3,$$

linking the stress tensor σ to the strain tensor $e(u) = (e_{ij}(u))_{i,j=1,2,3}$, with

$$e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

through the Northon-Hoff law, where $q > 1$ is the power-harderning parameter and λ is a non-dimensional positive constant. For a two-dimensional deformation plasticity under longitudinal shear, if $u(x,y)$ represents the component of displacement in the z direction, then the anti-plane stress component σ_{13} and σ_{23} are defined as:

$$\sigma_{13}(u) = |\nabla u|^{p-1}\frac{\partial u}{\partial x},$$
$$\sigma_{13}(u) = |\nabla u|^{p-1}\frac{\partial u}{\partial y}$$

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and equation (1) represents the equilibrium of a material under external loads of density f. The main purpose of this work is to study of the homogenization of a composite material lying in a bounded domain $\Omega \subset \mathbb{R}^3$, which is built with a plastic matrix in contact with plastic circular fibres. We consider a scalar version of a plasticity model obeying a Northon-Hoff's law. Let ωa bounded open of IR^2 with lipschitzian continuous boundary $\partial\omega$.

$$\Omega = \omega \times]0, L[, \omega_0 = \omega \times \{0\}, \omega_L = \omega \times \{L\}, \Gamma = \omega_0 \times \omega_L.$$

The unit cell $Y =]-\frac{1}{2}, -\frac{1}{2}[^2$, $0 < R < \frac{1}{2}$, $D(0,R)$ the disc of radius R centred at the origin, $T = D(0,R) \times]0, L[, S = \partial D(0,R) \times]0, L[, Y^{\#} = Y \backslash \bar{D}$ et$\Sigma = \partial S \times]0, L[$.

We define the following sets:

$Y_\varepsilon^k = (k_1\varepsilon,\ k_2\varepsilon) +]-\frac{\varepsilon}{2}, -\frac{\varepsilon}{2}[\times] - \frac{\varepsilon}{2}, \frac{\varepsilon}{2}[$, $\varepsilon > 0$ and $(k_1, k_2) \in \mathbb{Z}^2$, $D_\varepsilon^k = D(k\varepsilon, \varepsilon R)$, the disc of radius εR centred at $k\varepsilon = (k_1\varepsilon, k_2\varepsilon)$.

$$T_\varepsilon^k = D_\varepsilon^k \times]0, L[, Y_\varepsilon^{\#k} = Y_\varepsilon^k \backslash \overline{D_\varepsilon^k}, \sum_\varepsilon^k = \partial D_\varepsilon^k \times]0, L[.$$

Let us define: $D_\varepsilon = \bigcup\limits_{k \in I_\varepsilon} D_\varepsilon^k$ where $I_\varepsilon = \{k \in \mathbb{Z}^2; \overline{D}_\varepsilon^k \subset \omega\}$.

$$Y_\varepsilon = \bigcup_{k \in I_\varepsilon} Y_\varepsilon^k,$$

$$T_\varepsilon = \bigcup_{k \in I_\varepsilon} T_\varepsilon^k,$$

$$Y_\varepsilon^{\#} = \bigcup_{k \in I_\varepsilon} Y_\varepsilon^{\#k},$$

$$\Sigma_\varepsilon = \bigcup_{k \in I_\varepsilon} \Sigma_\varepsilon^k$$

The fibres do not touch the lateral side of Ω, since $\bar{T}_\varepsilon \cap (\partial\Omega \backslash \Gamma) = \emptyset$. Let $\Omega_\varepsilon = \Omega \backslash \overline{T_\varepsilon}$ and $f \in C_0(\Omega)$. We consider the following problem :

$$(P_\varepsilon^\lambda) \begin{cases} -(div(|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon) = f/_{\Omega_\varepsilon} & \text{in } \Omega_\varepsilon & (1) \\ -(div(|\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon) = f/_{T_\varepsilon} & \text{in } T_\varepsilon & (2) \\ (u_\varepsilon, v_\varepsilon) = (g, g) & \text{on } \partial T_\varepsilon \cap \Gamma & (3) \\ |\nabla u_\varepsilon|^{p-2} \frac{\partial u_\varepsilon}{\partial n} = \lambda |u_\varepsilon - v_\varepsilon|^{p-2}(u_\varepsilon - v_\varepsilon) & \text{on } \Sigma_\varepsilon & (4) \\ |\nabla v_\varepsilon|^{p-2} \frac{\partial v_\varepsilon}{\partial n} = \lambda |u_\varepsilon - v_\varepsilon|^{p-2}(v_\varepsilon - u_\varepsilon) & \text{on } \Sigma_\varepsilon & (5) \end{cases}$$

where $p \in]1, +\infty[$, g is a lipchitzian continuous function and $\lambda > 0$ is a parameter who will tend towards 0 or $+\infty$.

Contact conditions (4) and (5) between the matrix and fibres on the interface Σ_ε mean that the stresses depend on the gap of the displacements on Σ_ε. Contact model, in which the difference between displacements across a linear elastic interface is linearly related to components of tractions, has been formulated in [16]. A analogous model has been used in [2] to describe a contact on interfacial zones between fibres and matrix material. A similar model has been obtained from thermodynamic considerations in [14] and [15], where the proportional coefficient, between the adhesive forces and the gap between the materials, is of the form $k\beta^2$, $k > 0$ being the interface stiffness and β the bonding field which measures the active microscopic bounds with maximal value 1 corresponding to the perfect active bounds.

Our interest in this paper is with the study of the homogenization of the problem (P_ε^λ) when the periodic ε tends to 0 using $\Gamma-$convergence methods (see for example [4, 10], according to the limit of the report $\gamma = \frac{\lambda(\varepsilon)}{\varepsilon}$, $\lambda \to 0$ or $+\infty$.

The homogenization of materials reinforced with fibres have been recently considered by several authors among which [7, 8, 11, 12, 22] . For $p = 2$, the case of a network of tubes for two related mediums was studied by H.Samadi and M.Mabrouk (see [21]) and also in references [19, 20] using the energy method of Tartar [23] with $\lambda = \varepsilon^r, r > 0$. The same problem has been also addressed by Auriault and Ene in references [5], where they exhibit, using the method of matched asymptotic expansion, five models corresponding to $\lambda = \varepsilon^p$, with $p = -1, 0, 1, 2, 3$.

Let us define:

$$W_g^{1,p}(\Omega_\varepsilon) = \{u \in W^{1,p}(\Omega_\varepsilon); u = g \text{ on } \partial\Omega_\varepsilon \cap \Gamma\},$$
$$W_g^{1,p}(T_\varepsilon) = \{u \in W^{1,p}(T_\varepsilon); u = g \text{ on } \partial T_\varepsilon \cap \Gamma\}$$

Definition 1.1. *We say that a sequence* $(u_\varepsilon, v_\varepsilon)_\varepsilon, (u_\varepsilon, v_\varepsilon) \in W_g^{1,p}(\Omega_\varepsilon) \times W_g^{1,p}(T_\varepsilon), \tau-$ *converges to* (u, v) *if*

(i) $u_\varepsilon \to u$ *in* $W^{1,p}(\Omega)-$*weak*

(ii) $\int_\Omega \varphi v_\varepsilon d\mu_\varepsilon \to \int_\Omega \varphi v dx \ \ \forall\, \varphi \in C_0(\Omega)$

The main result obtained in this work is described as follows.
Let $\gamma = \lim \frac{\lambda}{\varepsilon} \in [0, +\infty]$. Then

(1) if $\gamma \in (0, +\infty)$, then the solution $(u_\varepsilon, v_\varepsilon)$ of $(P_\varepsilon^\lambda)\tau-$converge to $(u, v) \in W_g^{1,p}(\Omega) \times L^p\left(\omega, W_g^{1,p}(0, L)\right)$, where (u, v) is the solution of the following problem

$$\begin{cases} -div(\partial j_p^{hom}(\nabla u)) + 2\pi R\gamma|u-v|^{p-2}(u-v) = |Y^\#|f & \text{in } \Omega \\ -\frac{\partial}{\partial x_3}\left(|\frac{\partial v}{\partial x_3}|^{p-2}\frac{\partial v}{\partial x_3}\right) + \frac{2\gamma}{R}|u-v|^{p-2}(v-u) = f & \text{in } \Omega \\ u = v = g & \text{on } \Gamma \\ \partial j_p^{hom}(\nabla u).n = 0 & \text{on } \partial\omega \times]0, L[\end{cases}$$

(2) if $\gamma = 0$, there is no relationship between u and v,

(3) if $\gamma = +\infty$ we obtain that $u = v$ on Ω,

2. Estimates and Compactness Results

We define the functional F_ε^g through:

$$F_\varepsilon^g(u, v) = \begin{cases} \int_{\Omega_\varepsilon} |\nabla u|^p du + \int_{T_\varepsilon} |\nabla v|^p dv + \lambda \int_{\Sigma_\varepsilon} |u - v|^p dv & \text{if } (u, v) \in W_g^{1,p}(\Omega_\varepsilon) \times W_g^{1,p}(T_\varepsilon) \\ +\infty & \text{elsewhere} \end{cases}$$

The problem (P_ε^λ) is equivalent to the following minimization problem:

$$(m_\varepsilon^\lambda) \min\left\{F_\varepsilon^g(u, v) - \int_\Omega \chi(\Omega_\varepsilon) f u \, dx - \int_\Omega f v \chi_{T_\varepsilon} dx; (u, v) \in W_g^{1,p}(\Omega_\varepsilon) \times W_g^{1,p}(T_\varepsilon)\right\},$$

where χ_E is the characteristic function of the set E.

Proposition 2.1. *There exists a unique solution* $(u_\varepsilon, v_\varepsilon) \in W_g^{1,p}(\Omega_\varepsilon) \times W_g^{1,p}(T_\varepsilon)$ *of problem* (m_ε^λ), *such that*

(1) $\sup_\varepsilon \|u_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon)} < +\infty$

(2) $\sup_\varepsilon \|v_\varepsilon\|_{W^{1,p}(\Gamma_\varepsilon)} < +\infty$

(3) $\sup_\varepsilon \lambda \int_{\Sigma_\varepsilon} |u_\varepsilon - v_\varepsilon|^p d\sigma_\varepsilon < +\infty$

Remark 2.2. *Let* $\nu_\varepsilon = \frac{1}{2\pi R}\left(\sum_{k\in I_\varepsilon} \varepsilon\delta_{\Sigma_\varepsilon^k}\right)$, *where* $\delta_{\Sigma_\varepsilon^k}$ *is the Dirac measure of* Σ_ε^k, *then (3) becomes:*

(4) $2\pi R \sup_\varepsilon \frac{\lambda}{\varepsilon} \int_\Omega |u_\varepsilon - v_\varepsilon|^p d\nu_\varepsilon < +\infty.$

Proof. Let $\tilde{g}$ be a lipschitzian continuous extension of g to the whole Ω (Lemma of MacShane [13]). Then, $\forall\ (u,v) \in W_g^{1,p}(\Omega_\varepsilon) \times W_g^{1,p}(T_\varepsilon), (u-\tilde{g}, v-\tilde{g}) \in W_\Gamma^{1,p}(\Omega_\varepsilon) \times W_\Gamma^{1,p}(T_\varepsilon)$, where $W_\Gamma^{1,p} = \{u \in W^{1,p}/u = 0 \text{ in } \Gamma\}$.
Using Poincaré's inequality there exists a positive constant $c_{p,L}$ depending only of p and L such that

$$c_{p,L} \int_{\Omega_\varepsilon} |u-\tilde{g}|^p dx \le \int_{\Omega_\varepsilon} |\nabla(u-\tilde{g})|^p dx.$$

Let c be some positive constant with $c < c_{p,L}$, then by [4] Lemma 2.7, there exists $c_1 > 0$ and $c_2 \ge 0$ depending only of c, $c_{p,L}$ and $\|\tilde{g}\|_{W^{1,p}(\Omega)}$, such that:

$$\int_{\Omega_\varepsilon} |\nabla u|^p dx - c\int_{\Omega_\varepsilon} |u|^p dx \ge c_1\|u\|^p_{W^{1,p}(\Omega_\varepsilon)} - c_2.$$

We similarly obtain

$$\int_{T_\varepsilon} |\nabla v|^p dx - c\int_{T_\varepsilon} |v|^p dx \ge c_1\|v\|^p_{W^{1,p}(T_\varepsilon)} - c_2$$

One deduces that there exists $C > 0$ independent on ε such that (with $q = \frac{p}{p-1}$):

$$\begin{aligned} F_\varepsilon^g(u,v) - \int_\Omega \chi(\Omega_\varepsilon) fu\, dx - \int_\Omega \chi(T_\varepsilon) fv\, dx \\ \ge C\left(\|u\|_{W^{1,p}(\Omega_\varepsilon)} + \|v\|_{W^{1,p}(T_\varepsilon)}\right)\left(\|u\|_{W^{1,p}(\Omega_\varepsilon)} + \|v\|_{W^{1,p}(T_\varepsilon)} - \|f\|_{L^q(\Omega)}\right) - 2c_2, \end{aligned}$$

which implies the coerciveness of $F_\varepsilon^g(u,v) - L_\varepsilon(u,v)$, where $L_\varepsilon(u,v) = \int_\Omega (\chi(\Omega_\varepsilon)u - \chi(T_\varepsilon)v) f dx$.
As F_ε^g- L_ε is strictly convex and $\not\equiv +\infty$ there exists a unique solution $(u_\varepsilon, v_\varepsilon) \in W_g^{1,p}(\Omega_\varepsilon) \times W_g^{1,p}(T_\varepsilon)$ to problem (m_ε^λ). On the other hand,

$$F_\varepsilon^g(u_\varepsilon, v_\varepsilon) - L_\varepsilon(u_\varepsilon, v_\varepsilon) \le F_\varepsilon^g(\tilde{g},\tilde{g}) - L_\varepsilon(\tilde{g},\tilde{g}) \le \int_\Omega |\nabla\tilde{g}|^p dx + \|f\|_{L^p(\Omega)}\|\tilde{g}\|_{L^q(\Omega)} < +\infty,$$

from which we deduce, using the continuity of L_ε, that $\sup_\epsilon F_\epsilon^g(u_\epsilon, v_\epsilon) < +\infty$, $\sup_\epsilon \|u_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon)} < +\infty$ and $\sup_\epsilon \|v_\varepsilon\|_{W^{1,p}(\Gamma_\varepsilon)} < +\infty$.

Now, as Ω_ε is convex, using [21] Theorem 2.1, we have that there exists a extension operator IP_E from $W_g^{1,p}(\Omega_\epsilon)$ into $W_g^{1,p}(\Omega)$, such that $IP_E u_E = u_E$ in Ω_ε and $\|IP_E u_\varepsilon\|_{W^{1,p}(\Omega)} \le C$.

In what follows, we omit the symbol IP_E. Let us define the measure μ_ε by $\mu_\varepsilon = \frac{|\Omega|}{|T_\varepsilon|}\sum_{k\in I_\varepsilon} \delta_{T_\varepsilon^k}$, where $\delta_{T_\varepsilon^k}$ is the Dirac measure supported by T_ε^k. One can easily see that $\mu_\varepsilon \to 1_\Omega dx$ weakly in sense of measure, and as $\sup_\varepsilon \|v_\varepsilon\|_{W^{1,p}(\Gamma_\varepsilon)} < +\infty$ we get, using Holder's inequality :

$$\int_\Omega |v_\varepsilon| d\mu_\varepsilon = \int_{T_\varepsilon} |v_\varepsilon| dx \le C\int_{T_\varepsilon} |v_\varepsilon|^P dx \le C \ \ \forall\ \ \varepsilon > 0,$$

where C is a generic positive constant. □

Lemma 2.3. *Let* $(v_\varepsilon)_\varepsilon \in W_g^{1,p}(T_\varepsilon)$ *such that :* $\sup_\varepsilon \int_{T_\varepsilon} |\nabla v_\varepsilon|^P dx < +\infty$ *and* $\int_\Omega \phi v_\varepsilon d\mu_\varepsilon \to \int_\Omega \varphi v dx \; \forall \, \varphi \in C_0(\Omega)$. *Then* $v, \partial_3 v \in L^p(\Omega)$, *and* $v = g$ *on* Γ.

Proof. We use here Lemma A_3 of [1]. Let us notice that by Hölder's inequality we have :

$$\int_\Omega |\frac{\partial v_\epsilon}{\partial x_3}| d\mu_\epsilon \leq (\mu_\epsilon(\Omega))^{\frac{1}{q}} \left\{ \int_\Omega |\frac{\partial v_\epsilon}{\partial x_3}|^p d\mu_\epsilon \right\}^{\frac{1}{p}}.$$

Since $\mu_\varepsilon(\Omega) = \frac{|\Omega|}{|T_\varepsilon|} |T_\varepsilon| = |\Omega|$, we have $\int_\Omega |\frac{\partial v_\varepsilon}{\partial x_3}| d\mu_\varepsilon \leq |\Omega| \| \frac{\partial v_\varepsilon}{\partial x_3} \|_{L^p(\Omega)} < +\infty$.

The sequence $(\frac{\partial v_\varepsilon}{\partial x_3} \mu_\varepsilon)$ is thus uniformly bounded in variations, hence *-weakly relatively compact. Thus, possibly passing to a subsequence, we can assume that, up to some subsequence, $\frac{\partial v_\varepsilon}{\partial x_3} \mu_\varepsilon \to w \mathbb{L}_\Omega dx$ weakly in the sense of measure. Let $\varphi \in C^1(\bar{\Omega})$. Then with $x' = (x_1, x_2)$, we have

$$\int_\Omega \varphi \frac{\partial v_\varepsilon}{\partial x_3} d\mu_\varepsilon = - \int_\Omega v_\varepsilon \frac{\partial \varphi}{\partial x_3} d\mu_\varepsilon + \frac{|w|}{R\pi^2} \int_{D_\varepsilon} \left\{ \varphi(x', L) v_\varepsilon(x', L) - \varphi(x', 0) v_\varepsilon(x', 0) \right\} dx'. \tag{*}$$

Let us take now $\varphi \in C_0^\infty(\Omega)$ in (*). Then

$$\int_\Omega \varphi w dx = \int_\Omega v \frac{\partial \varphi}{\partial x_3} dx.$$

Thus $w = \frac{\partial v}{\partial x_3}$ in the sense of distribution. By Fenchel's inequality we have, for every $\varphi \in C_0(\Omega)$,

$$\liminf_{\varepsilon \to 0} \left\{ \int_\Omega \frac{\partial v_\varepsilon}{\partial x_3} \varphi d\mu_\varepsilon - \frac{1}{q} \int_\Omega |\varphi|^q d\mu_\varepsilon \right\} \leq \liminf_{\varepsilon \to 0} \frac{1}{p} \int_\Omega |\frac{\partial v_\varepsilon}{\partial x_3}|^p d\mu_\varepsilon < +\infty,$$

From which we deduce that $\sup \left\{ \int_\Omega w \varphi dx; \|\varphi\|_{L^q(\Omega)} \leq 1 \right\} < +\infty$. Thus, according to Riesz representation Theorem, $w \in L^p(\Omega)$. Repeating the same argument, we prove that

$$v_i \in L^p(\Omega), i = 1, 2, 3.$$

Let φ such that $\varphi(x) = \theta(x') \psi(x_3)$, where $\psi \in C^1([0, L])$; $\psi(0) = 1$, $\psi(L) = 0$, and $\theta \in C_0^\infty(G_0)$ with $G_0 = \{x' \in \omega; (x', 0) \in \omega_0\}$. As $v_\varepsilon(x', 0) = g(x', 0)$ in G_0, one has, passing to the limit in (*)

$$\int_\Omega \varphi \frac{\partial v}{\partial x_3} dx + \int_\Omega v \frac{\partial \varphi}{\partial x_3} dx = \lim_{\varepsilon \to 0} \frac{|w|}{R\pi^2} \int_{D_\varepsilon} -(\theta(x') v_\varepsilon(x', 0) dx' = - \int_\omega -(\theta(x') g(x', 0) dx'.$$

On the other hand thanks of the first assertion of Lemma 2.3, we have, using Green's formula, $\int_\Omega \varphi \frac{\partial v}{\partial x_3} dx = - \int_\Omega v \frac{\partial \varphi}{\partial x_3} dx - \int_\omega (\theta(x') v(x', 0) dx'$, from which we deduce that $\int_\omega (\theta(x') v(x', 0) dx' = \int_\omega (\theta(x') g(x', 0) dx' \;\; \forall \, \theta \in C_0^\infty(G_0)$.
This implies that $v = g \; p.p \; \omega_0$. Similarly, with $\psi(L) = 1$, $\psi(0) = 0$ and $\theta \in C_0^\infty(G_L)$ with

$$G_L = \{x' \in \omega \; ; \; (x', L) \in \omega_L\}$$

we get $v = g \; p.p \; \omega_L$. □

Lemma 2.4. *Let* $(u_\varepsilon, v_\varepsilon) \in W_g^{1,p}(\Omega_\varepsilon) \times W_g^{1,p}(T_\varepsilon)$. *Then* $\sup_\varepsilon \int_\Omega |u_\varepsilon - v_\varepsilon|^p d\nu_\varepsilon < +\infty$.

Proof. Let $u \in W^{1,p}(Y^{\#}\times]0,L[)$ such that $u(\cdot,x_3)$ is $Y-$periodic. Let $v \in W^{1,p}(T); v(\cdot,x_3)Y-$periodic. By the trace Theorem

$$\int_{\Sigma}|u|^p ds \leq C\left\{\int_0^L\int_Y |u|^p dy + \int_0^L\int_Y |\nabla u|^p dy\right\}$$

Introducing the change of variables $x' = \varepsilon y'$, $x_3 = y_3$ et $s_3 = \varepsilon s$, we get

$$\int_{\Sigma_\varepsilon^k}|u|^p ds_\varepsilon \leq C\left\{\frac{1}{\varepsilon}\int_0^L\int_{Y_\varepsilon^{\#k}}|u|^p dx + \varepsilon^{p-1}\int_0^L\int_{Y_\varepsilon^{\#k}}\left\{|\frac{\partial u}{\partial x_1}|^p + |\frac{\partial u}{\partial x_2}|^p\right\}dx + \frac{1}{\varepsilon}\int_0^L\int_{Y_\varepsilon^{\#k}}|\frac{\partial u}{\partial x_3}|^p dx\right\}.$$

Then summing over $k \in I_\varepsilon$, we obtain that: $\int_{\Sigma_\varepsilon}|u|^p ds_\varepsilon \leq \frac{C}{\varepsilon}\|u\|^p_{W^{1,p}(\Omega)}$ and, in the same way, $\int_{\Sigma_\varepsilon}|v|^p ds_\varepsilon \leq \frac{C'}{\varepsilon}\|v\|^p_{W^{1,p}(T_\varepsilon)}$. Multiplying these inequalities by $\frac{\varepsilon}{2\pi R}$ we get, $\int_\Omega |v_\varepsilon|^p d\nu_\varepsilon \leq C'\|v\|^p_{W^{1,p}(T_\varepsilon)}$, and $\int_\Omega |u|^p d\nu_\varepsilon \leq C\|v\|^p_{W^{1,p}(\Omega)}$. Thus, using a convexity argument, we obtain that: $\int_\Omega |u_\varepsilon - v_\varepsilon|^p d\nu_\varepsilon < +\infty$. □

3. Convergence

We suppose here that $\gamma = \lim\frac{\lambda}{\varepsilon} \in (0,+\infty)$. We define the functional F^g on $W_g^{1,p}(\Omega)\times L^p(\Omega)$ by

$$F^g(u,v) = \begin{cases} \int_\Omega j_p^{hom}(\nabla u(x))dx + \frac{|T|}{\lambda}\int_\Omega |\frac{\partial v}{\partial x_3}|^p dx + +2\pi R\gamma\int_\Omega |u-v|^p dx & \text{if } (u,v)\in W_g^{1,p}(\Omega)\times L^p(\omega, W_g^{1,p}(0,L)) \\ +\infty & \text{elsewhere} \end{cases}$$

where $j_p^{hom}(Z)$ is defined for $Z \in IR^3$ by :

$$j_p^{hom}(Z) = \min\left\{\int_{Y^{\#}}|Z+\nabla w|^p dy, w \in W^{1,p}(Y^{\#}),\ \text{ w is Y-periodic}\right\},$$

for $p = 2$,

$$j_2^{hom}(Z) = |z_3|^2 + \min\left\{\int_{Y^{\#}}|z+\nabla w|^2 dy, w \in W^{1,p}(Y^{\#}),\ \text{ w is Y-periodic}\right\},$$

where $z = (z_1, z_2)$. In this case we have

$$\int_\Omega j_2^{hom}(\nabla u)dx = \int_\Omega |\frac{\partial u}{\partial x_3}|^2 dx + \int_\Omega j_{0,2}^{hom}(\nabla_{x'}u)dx,$$

with $\nabla_{x'}u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, 0)$. For $z \in IR^2$ we have

$$j_{0,2}^{hom}(z) = \min\left\{\int_{Y^{\#}}|z+\nabla w|^2 dy, w\in H^1(Y^{\#}),\ \text{ w Y-periodic}\right\}.$$

Our main result in this section reads as follows:

Theorem 3.1. *If $\gamma \in (0,+\infty)$ then*

(i) For every $(u,v) \in W_g^{1,p}\times L^p(\omega, W_g^{1,p}(0,L))$, there exists $(u_\varepsilon, v_\varepsilon) \in W_g^{1,p}(\Omega_\varepsilon)\times W_g^{1,p}(T_\varepsilon)$, such that $(u_\varepsilon,v_\varepsilon)_\varepsilon\tau-$converges to (u,v) and $F^g(u,v) \geq \limsup_{\varepsilon\to 0} F_\varepsilon^g(u_\varepsilon,v_\varepsilon)$.

(ii) For every $(u_\varepsilon, v_\varepsilon) \in W_g^{1,p}(\Omega_\varepsilon)\times W_g^{1,p}(T_\varepsilon)$, such that $(u_\varepsilon,v_\varepsilon)_\varepsilon\tau-$converges to (u,v), we have $(u,v) \in W_g^{1,p}\times L^p(\omega, W_g^{1,p}(0,L))$, and $F^g(u,v) \leq \liminf_{\varepsilon\to 0} F_\varepsilon^g(u_\varepsilon,v_\varepsilon)$.

Proof. **1. The limit sup inequality :** Let $z \in IR^3$. We define the following functional

$$F_Z : \begin{cases} W^{1,p}(Y^{\#}) & \to I\bar{R}_+ \\ w & \to \int_{Y^{\#}} |z + \nabla w|^p dy \end{cases}$$

and consider the following minimisation problem: $(P_z)\min\{F_z(w), w \in W^{1,p}(Y^{\#}),$ w is Y-periodic$\}$. It can easily checked that (P_z) have a unique solution $w_z \in W^{1,p}(Y^{\#})$, which can be extended to $W^{1,p}(Y)$ keeping the same notation. We then define the function w_z^ε through : $w_z^\varepsilon(x) = w_z(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon})$. □

We have the following intermediate result.

Lemma 3.2. *$\varepsilon w_z^\varepsilon \to_{\varepsilon\to 0} 0$ in $W^{1,p}(\Omega)-$weak.*

Proof. Observe that

$$\int_\Omega |\varepsilon w_z^\varepsilon(x')|^p dx = L\int_\omega |\varepsilon w_z^\varepsilon(x')|^p dx' \le LC\sum_{k\in I_\varepsilon} \varepsilon^{p+2}\int_{Y^{\#}} |w_z(y)|^p dy LC \le LC|\omega|\varepsilon^p.$$

Thus, $w_z^\varepsilon \to_{\varepsilon\to 0} 0$ in $L^p(\Omega)-$strong. On the other hand

$$\int_\Omega |\varepsilon \nabla w_z^\varepsilon|^p dx \le C' L\varepsilon^p \sum_{k\in I_\varepsilon} \varepsilon^{p+2}\int_{Y_\varepsilon^k} |\nabla w_z(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon})|^p dx' \le LC^{"}|\omega|\int_{Y^{\#}} |\nabla w_z(y)|^p dy.$$

This implies that the sequence $(\nabla(\varepsilon w_z^\varepsilon))$ is bounded in $L^p(\omega, IR^2)$. Combining with the above $L^p(\Omega)-$strong convergence to 0 of the sequence $(\varepsilon w_z^\varepsilon)$, we get $\varepsilon w_z^\varepsilon \to_{\varepsilon\to 0} 0$ in $W^{1,p}(\Omega)-$weak. Let us define $u(x) = z.x + c$, where c is a constant and $z \in IR^3$. We define the test-function u_ε^0 by:

$$u_\varepsilon^0(x) = u(x) + \varepsilon w_z^\varepsilon(x') \quad (*)$$

□

Then, using Lemma 3.2, $u_\varepsilon^0 \to u$ in $W^{1,p}(\Omega)-$weak and

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_\Omega |\nabla u_\varepsilon^0(x)|^p dx &= \lim_{\varepsilon\to 0}\sum_{k\in I_\varepsilon} L\int_{Y_\varepsilon^{\#k}} |z + \varepsilon\nabla w_z^\varepsilon(x')|^p dx' \\
&= L\lim_{\varepsilon\to 0}\sum_{k\in I_\varepsilon}\varepsilon^2\int_{Y^{\#}} |z + \nabla w_z(y)|^p dy \\
&= |\Omega|\int_{Y^{\#}} |z + \nabla w_z(y)|^p dy \\
&= \int_\Omega j_p^{hom}(\nabla u(x))dx,
\end{aligned}$$

where $j_p^{hom}(z) = min\{\int_{Y^{\#}} |z + \nabla w|^p dy, w \in W^{1,p}(Y^{\#}),$ w Y-periodic$\}$. Let us now consider $u \in w^{1,p}(\Omega)$. Then, according to [13], there exists a sequence of piecewise affine functions (u_n), such that $u_n \to_{n\to\infty} u$ in $W^{1,p}(\Omega)-$strong. The sequence (u_n) is define on a partition (Ω_n) of Ω by $u_n(x) = z_n.x + c_n$, where $z_n \in IR^3$ and $c_n \in IR$, $\forall\, n$. We then build the associated test-functions through

$$u_\varepsilon^{0,n}(x) = u_n(x) + \varepsilon w_{z_n}^\varepsilon(x').$$

Then $u_\varepsilon^{0,n} \to_{\varepsilon\to 0} u^n$ in $W^{1,p}(\Omega)-$weak and

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_\Omega |\nabla u_\varepsilon^{0,n}(x)|^p dx &= \lim_{\varepsilon\to 0}\sum_n \int_{\Omega_n} |z_n+\varepsilon\nabla w^\varepsilon_{z_n}(x)|^p dx\\ &= \sum_n \lim_{\varepsilon\to 0}\sum_{k\in I_\varepsilon^n}\int_{Y_\varepsilon^{\#k}} |z_n+\varepsilon\nabla w^\varepsilon_{z_n}(x')|^p dx'\\ &= \sum_n |\Omega_n|\int_{Y^\#}|z_n+\nabla w_{z_n}(y)|^p dy\\ &= \sum_n \int_{\Omega_n} j_p^{hom}(z_n)dx\\ &= \int_\Omega j_p^{hom}(\nabla u_n)dx.\end{aligned}$$

Using the continuity of j_p^{hom} we get $\lim_{n\to\infty}\int_\Omega j_p^{hom}(\nabla u_n)dx = \int_\Omega j_p^{hom}(\nabla u)dx$. Then, using the diagonalization argument of [3], there exists a sequence $n(\varepsilon)\to_{\varepsilon\to 0}+\infty$, such that, with $u_\varepsilon = u_\varepsilon^{0,n(\varepsilon)}$, $u_\varepsilon \to_{\varepsilon\to 0} u$ in $W^{1,p}(\Omega)-$weak, and $\limsup_{\varepsilon\to 0}\int_\Omega |\nabla u_\varepsilon|^p dx \le \int_\Omega j_p^{hom}(\nabla u)dx$.

Let v be a Lipschitzian continuous function on Ω, such that $v=g$ on Γ. We define $v_\varepsilon^\# = \sum_{k\in I_\varepsilon} v(k_1\varepsilon,k_2\varepsilon,x_3) l_{Y_\varepsilon^k}(x')$. Then $v_\varepsilon^\#$ is a piecewise affine function with respect to x'. Let $E_\varepsilon = \{x\in\Omega; d(x,\Gamma)<\varepsilon\}$ and φ_ε a smooth function such that: $\varphi_\varepsilon = 1$ on Γ, $\varphi_\varepsilon=0$ in $\Omega\backslash\bar{E}_\varepsilon$ and $|\nabla\varphi_\varepsilon|\le \frac{C}{\varepsilon}$. We then define the following test-function in the fibres $v_\varepsilon^0 = (1-\varphi_\varepsilon)v_\varepsilon^\# + \varphi_\varepsilon v$. One can see that $v_\varepsilon^0 \in W_g^{1,p}(T_\varepsilon)$ and, after some computations,

$$\int_\Omega \varphi v_\varepsilon^0 d\mu_\varepsilon \to_{\varepsilon\to 0}\int_\Omega \varphi v dx,\ \ \forall\,\varphi\in C_0(\Omega).$$

Besides $\int_{T_\varepsilon}|\nabla v_\varepsilon^0|^p dx = \int_{T_\varepsilon\backslash E_\varepsilon}|\nabla v_\varepsilon^\#|^p dx + \int_{T_\varepsilon\cap E_\varepsilon}|\nabla v_\varepsilon^\# + \varphi_\varepsilon\nabla(v_\varepsilon^\#-v)+(v_\varepsilon^\#-v)\nabla\varphi_\varepsilon|^p dx$. We have the following estimate for the second right term:

$$\int_{T_\varepsilon\cap E_\varepsilon}|\nabla v_\varepsilon^\# + \varphi_\varepsilon\nabla(v_\varepsilon^\#-v)+(v_\varepsilon^\#-v)\nabla\varphi_\varepsilon|^p dx \le C\left\{\int_{T_\varepsilon\cap E_\varepsilon}|\nabla v_\varepsilon^\#|^p dx + \int_{T_\varepsilon\cap E_\varepsilon}|\nabla(v_\varepsilon^\#-v)|^p dx + \frac{1}{\varepsilon^p}\int_{T_\varepsilon\cap E_\varepsilon}|(v_\varepsilon^\#-v)|^p dx\right\}$$

As $|v_\varepsilon^\# - v|\le C\varepsilon$ in $T_\varepsilon\cap E_\varepsilon$, we get $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^p}\int_{T_\varepsilon\cap E_\varepsilon}|(v_\varepsilon^\#-v)|)^p dx = 0$. Observing that

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{T_\varepsilon\backslash\bar{E}_\varepsilon}|\nabla v_\varepsilon^\#|^p dx &= \lim_{\varepsilon\to 0}\sum_{k\in I_\varepsilon}\int_{T_\varepsilon\backslash\bar{E}_\varepsilon}|\frac{\partial v}{\partial x_3}(k_1\varepsilon,k_2\varepsilon,x_3)|^p dx_3\\ &= \frac{|T|}{L}\lim_{\varepsilon\to 0}\sum_{k\in I_\varepsilon}\varepsilon^2\int_\varepsilon^{L-\varepsilon}|\frac{\partial v}{\partial x_3}(k_1\varepsilon,k_2\varepsilon,x_3)|^p dx_3\\ &= \frac{|T|}{L}\int_\Omega|\frac{\partial v}{\partial x_3}(x)|^p dx\end{aligned}$$

and

$$\lim_{\varepsilon\to 0}\int_{T_\varepsilon\cap E_\varepsilon}|\nabla v_\varepsilon^\#|^p dx = \lim_{\varepsilon\to 0}\int_{T_\varepsilon\cap E_\varepsilon}|\nabla(v_\varepsilon^\#-v)|^p dx = 0.$$

We thus obtain that

$$\lim_{\varepsilon\to 0}\int_{T_\varepsilon}|\nabla v_\varepsilon^0|^p dx = \frac{|T|}{L}\int_\Omega|\frac{\partial v}{\partial x_3}|^p dx.$$

Now, taking a sequence of the Lipschitzian continuous functions (v_n) such that $v_n = g$ on Γ and $v_n\to v$ in $L^p(\omega, W^{1,p}(0,L))-$strong, we build, as before a sequence of functions $(v_\varepsilon^{\#n})$:

$$v_\varepsilon^{\#n} = \sum_{k\in I_\varepsilon} v_n(k_1\varepsilon,k_2\varepsilon,x_3) l_{Y_\varepsilon^k}$$

and define the sequence of test-functions $(v_\varepsilon^{0,n})$; $v_\varepsilon^{0,n} = (1-\varphi_\varepsilon)v_\varepsilon^{\#} + \varphi_\varepsilon v_n$. Then, for every $\varphi \in C_0(\Omega)$,

$$\lim_{\varepsilon\to 0}\int_\Omega \varphi v_\varepsilon^{0,n} d\mu_\varepsilon = \int_\Omega \varphi v_n dx,$$
$$\lim_{\varepsilon\to 0}\int_{T_\varepsilon} |\nabla v_\varepsilon^{0,n}|^p dx = \int_\Omega |\frac{\partial v_n}{\partial x_3}|^p dx \text{ and}$$
$$\lim_{n\to+\infty}\int_\Omega |\frac{\partial v_n}{\partial x_3}|^p dx = \int_\Omega |\frac{\partial v}{\partial x_3}|^p dx.$$

Thus, using the diagonalization argument of [3], there exists a sequence (v_ε), such that $\int_\Omega \varphi v_\varepsilon d\mu_\varepsilon \to_{\varepsilon\to 0} \int_\Omega \varphi v dx \quad \forall \quad \varphi \in C_0(\Omega)$, and

$$\limsup_{\varepsilon\to 0}\int_{T_\varepsilon} |\nabla v_\varepsilon|^p dx \le \int_\Omega |\frac{\partial v}{\partial x_3}|^p dx.$$

Let $(u_\varepsilon^{0,n})$ and $(v_\varepsilon^{0,n})$ be the sequences previously built. Let us compute the limit

$$\lim_{\varepsilon\to 0}\int_\Omega |u_\varepsilon^{0,n} - v_\varepsilon^{0,n}|^p d\nu_\varepsilon.$$

We first have

$$\sum_n \int_{\Omega_n} |\varepsilon\nabla w_{z_n}^\varepsilon|^p d\nu_\varepsilon = \sum_n L_n \sum_{k\in I_\varepsilon^n} \varepsilon^{p+1}\int_{\partial D_\varepsilon^k} |w_{z_n}(\frac{s_\varepsilon}{\varepsilon})|^p \frac{ds_\varepsilon}{2\pi R}$$
$$\le C\sum_n |\omega_n| L_n \|w_{z_n}\|_{W^{1,p}(Y\#)}\varepsilon^p,$$

where L_n is the length of the set $\{(0,x_3) \in \Omega_n\}$. Observing that $\sum_n L_n\|w_{z_n}\|_{W^{1,p}(Y\#)} \le C|\Omega|$, we get $\lim_{\varepsilon\to 0}\sum_n\int_{\Omega_n} |\varepsilon\nabla w_{z_n}^\varepsilon|^p d\nu_\varepsilon = 0$. Since $v_\varepsilon \to_{\varepsilon\to 0} dx\mathbb{L}_\Omega$, we have

$$\int_\Omega |u_n|^p d\nu_\varepsilon \to_{\varepsilon\to 0} \int_\Omega |u_n|^p dx,$$

Thus $\lim_{\varepsilon\to 0}\int_\Omega |u_\varepsilon^{0,n}|^p d\nu_\varepsilon = \int_\Omega |u_n(x)|^p dx$. On the other hand

$$\lim_{\varepsilon\to 0}\int_\Omega |v_\varepsilon^{0,n}|^p d\nu_\varepsilon = \lim_{\varepsilon\to 0}\frac{1}{2\pi R}\sum_{k\in I_\varepsilon} \varepsilon\int_{\partial T_\varepsilon^k} |v_n(k_1\varepsilon, k_2\varepsilon, x_3)|^p dx_3$$
$$= \lim_{\varepsilon\to 0}\int_0^L \sum_{k?I_\varepsilon} \varepsilon^2 |v_n(k_1\varepsilon, k_2\varepsilon, x_3)|^p dx_3$$
$$= \int_\Omega |v_n(x)|^p dx.$$

Thus, using the uniform convexity property of $L^p(\Omega)p > 1)$, we have

$$\lim_{\varepsilon\to 0}\int_\Omega |u_\varepsilon^{0,n} - v_\varepsilon^{0,n}|^p d\nu_\varepsilon = \int_\Omega |u_n - v_n|^p dx$$

and, since

$$\lim_{n\to+\infty}\int_\Omega |u_n - v_n|^p dx = \int_\Omega |u - v|^p dx,$$

we get $\lim_{\varepsilon\to 0}\sup\int_\Omega |u_n - v_n|^p dv_\varepsilon \le \int_\Omega |u-v|^p dx$, where (u_ε) and (v_ε) are the sequences obtained previously by diagonalisation. We thus proved that, for every $(u,v) \in W_g^{1,p} \times L^p(\omega, W_g^{1,p}(0,L))$, there exists $(u_\varepsilon, v_\varepsilon) \in W_g^{1,p}(\Omega_\varepsilon) \times W_g^{1,p}(T_\varepsilon)$, such that $(u_\varepsilon, v_\varepsilon)_\varepsilon \tau-$converges to (u,v) and

$$F^g(u,v) \ge \limsup_{\varepsilon\to 0} F_\varepsilon^g(u_\varepsilon, v_\varepsilon).$$

2. The limit inf inequality:

Let $(u_\varepsilon, v_\varepsilon) \in W^{1,p}(\Omega_\varepsilon) \times W_g^{1,p}(T_\varepsilon)$ such that $(u_\varepsilon, v_\varepsilon)\tau-$converge to (u, v). We may suppose that $\sup F_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty$, otherwise the result is trivial.

Let $(u_\varepsilon^{0,n})$; $u_\varepsilon^{0,n} = u_n + \varepsilon w_{z_n}^\varepsilon$, be the sequence previously built in **(*)**, with here $u_n \to_{n\to\infty} uW^{1,p}(\Omega)-$strong and $z_n = \nabla u_n$ in Ω_n. Let $\varphi_n \in C_0^\infty(\Omega_n); 0 \le \varphi_n \le 1$. Let us introduce the following sub differential inequality

$$\int_\Omega |\nabla u_\varepsilon|^p dx \ge \sum_n \int_{\Omega_n} |\nabla u_\varepsilon|^p \varphi_n dx$$
$$\ge \sum_n \int_{\Omega_n} |\nabla u_\varepsilon^{0,n}|^p \varphi_n dx + p \sum_n \int_{\Omega_n} |\nabla u_\varepsilon^{0,n}|^{p-2} \nabla u_\varepsilon^{0,n} (\nabla u_\varepsilon - \nabla u_\varepsilon^{0,n}) \varphi_n dx.$$

We have

$$\int_{\Omega_n} |\nabla u_\varepsilon^{0,n}|^{p-2} \nabla u_\varepsilon^{0,n} (\nabla u_\varepsilon - \nabla u_\varepsilon^{0,n}) \varphi_n dx = -\int_{\Omega_n} div(|\nabla u_\varepsilon^{0,n}|^{p-2} \nabla u_\varepsilon^{0,n})(u_\varepsilon - u_\varepsilon^{0,n}) \varphi_n dx$$
$$+ \int_{\Omega_n} |\nabla u_\varepsilon^{0,n}|^{p-2} \nabla u_\varepsilon^{0,n} \nabla \varphi_n (u_\varepsilon - u_\varepsilon^{0,n}) dx.$$

Observe that, for every $\psi_n \in C_0^\infty(\Omega_n)$,

$$-\int_{\Omega_n} div(|\nabla u_\varepsilon^{0,n}|^{p-2} \nabla u_\varepsilon^{0,n}) \psi_n dx = -\sum_{k\in I_\varepsilon^n} \int_{Y_\varepsilon^{\# k}} div(|z_n + \varepsilon \nabla w_{z_n}(\frac{x'}{\varepsilon})|^{p-2}(z_n + \varepsilon \nabla w_{z_n}(\frac{x'}{\varepsilon}))\psi_n dx'$$
$$= -\int_{\Omega_n(x_3)} \sum_{k\in I_\varepsilon^n} \varepsilon^2 \psi_n(k_1\varepsilon, k_2\varepsilon, x_3) \int_Y div(|z_n + \nabla w_{z_n}(y)|^{p-2})(z_n + \nabla w_{z_n}(y)) dx' + O_n(\varepsilon),$$

where $\Omega_n(x_3) = \Omega_n \cap (0,0)\times]0, L[$. As $div(|z_n + \nabla w_{z_n}(y)|^{p-2})(z_n + \nabla w_{z_n}(y)) = 0$, we obtain that $\lim_{\varepsilon\to 0} \int_{\Omega_n} div(|\nabla u_\varepsilon^{0,n}|^{p-2} \nabla u_\varepsilon^{0,n}) \psi_n dx = 0$, hence, recalling that $(u_\varepsilon - u_\varepsilon^{0,n})\varphi_n \to (u - u^n)\varphi_n$ dans $L^p(\Omega_n)-$strong, we get

$$\lim_{\varepsilon\to 0} \sum_n \int_{\Omega_n} div(|\nabla u_\varepsilon^{0,n}|^{p-2} \nabla u_\varepsilon^{0,n})(u_\varepsilon - u_\varepsilon^{0,n}) \varphi_n dx = 0.$$

On the other hand

$$\lim_{\varepsilon\to 0} \sum_n \int_{\Omega_n} |\nabla u_\varepsilon^{0,n}|^{p-2} \nabla u_\varepsilon^{0,n} \nabla \varphi_n (u_\varepsilon - u_\varepsilon^{0,n}) dx = \sum_n \int_{\Omega_n} |\nabla u^n|^{p-2} \nabla u^n \nabla \varphi_n (u - u^n) dx.$$

Then letting n tend to $+\infty$ in the above sub differential inequality we get

$$\liminf_{\varepsilon\to 0} \int_\Omega |\nabla u_\varepsilon|^p dx \ge \int_\Omega j_p^{hom}(\nabla u) dx.$$

Observe that $\int_{T_\varepsilon} |\nabla v_\varepsilon|^p dx \ge \int_{T_\varepsilon} |\frac{\partial v_\varepsilon}{\partial x_3}|^p dx$. Then, using the proof of Lemma 2.3, we obtain that, for every $\varphi \in L^q(\Omega)$; $q = \frac{p}{p-1}$,

$$\liminf_{\varepsilon\to 0} \int_\Omega |\frac{\partial v_\varepsilon}{\partial x_3}|^p d\mu_\varepsilon \ge p \int_\Omega \frac{\partial v}{\partial x_3} \varphi dx - \frac{p}{q} \int_\Omega |\varphi|^q dx, \text{ and } v \in L^p(\omega, W_g^{1,p}(0, L)).$$

This implies that, with $\varphi = |\frac{\partial v}{\partial x_3}|^{p-2} \frac{\partial v}{\partial x_3}$,

$$\liminf_{\varepsilon\to 0} \int_{T_\varepsilon} |\nabla v_\varepsilon|^p dx \ge \frac{|T|}{L} \int_\Omega |\frac{\partial v}{\partial x_3}|^p dx.$$

Now according to Lemma 2.4, we have

$$\sup_\varepsilon \int_\Omega |u_\varepsilon - v_\varepsilon|^p d\nu_\varepsilon < +\infty,$$

From which we deduce, using Hölder's inequality, that the sequence $((u_\varepsilon - v_\varepsilon)v_\varepsilon)$ is uniformly bounded in variation and thus, weakly converges, up to some subsequence, to some $\chi l_\Omega dx$ in the sense of measure. Then, using Fenchel's inequality, we have, for every $\varphi \in C_0(\Omega)$,

$$\liminf_{\varepsilon\to 0} \int_\Omega |u_\varepsilon - v_\varepsilon|^p dv_\varepsilon \geq p\int_\Omega \chi\varphi dx - \frac{p}{q}\int_\Omega |\varphi|^q dx,$$

from which we deduce, using the proof of lemma1, that $\chi = u - v \in L^p(\Omega)$. Then taking $\varphi = |u-v|^{p-2}(u-v)$, we get

$$\liminf_{\varepsilon\to 0} \int_\Omega |u_\varepsilon - v_\varepsilon|^p dv_\varepsilon \geq \int_\Omega |u-v|^p dx.$$

We thus have proved that for every $(u_\varepsilon, v_\varepsilon) \in W_g^{1,p}(\Omega_\varepsilon) \times W_g^{1,p}(T_\varepsilon)$, such that $(u_\varepsilon, v_\varepsilon)_\varepsilon \tau-$converges to (u,v), we have $(u,v) \in W_g^{1,p} \times L^p(\omega, W_g^{1,p}(0,L))$, and $F^g(u,v) \leq \liminf_{\varepsilon\to 0} F_\varepsilon^g(u_\varepsilon, v_\varepsilon)$. One can easily see that

$$\begin{aligned}\chi(\Omega_\varepsilon) \to& |Y^{\#}| \text{ in } L^p(\Omega) - \text{weak},\\ \chi(T_\varepsilon) \to& |\Omega| R^2\pi \text{ in } L^1(\Omega) - \text{strong},\end{aligned}$$

And, as $f \in C_0(\Omega)$,

$$\begin{aligned}\int_\Omega \chi(T_\varepsilon) f v_\varepsilon dx &= \frac{|T_\varepsilon|}{|\Omega|}\int_\Omega f v_\varepsilon d\mu_\varepsilon,\\ &\to R^2\pi \int_\Omega f v dx.\end{aligned}$$

Then, using the properties of Γ-convergence [4] we obtain the following

Corollary 3.3.

(1) If $\gamma \in (0,+\infty)$ then : the solution $(u_\varepsilon, v_\varepsilon)$ of $(m_\varepsilon^\lambda)\tau-$converges to $(u,v) \in W_g^{1,p}(\Omega) \times L^p(\omega, W_g^{1,p}(0,L))$ where (u,v) is the solution of the following problem

$$\begin{cases} -div(\partial j_p^{hom}(\nabla u)) + 2\pi R\gamma |u-v|^{p-2}(u-v) = |Y^{\#}| f & \text{in } \Omega,\\ -\frac{\partial}{\partial x_3}(|\frac{\partial v}{\partial x_3}|^{p-2}\frac{\partial v}{\partial x_3}) + \frac{2\gamma}{R}|u-v|^{p-2}(v-u) = f & \text{in } \Omega,\\ u = v = g & \text{on } \Gamma,\\ \partial j_p^{hom}(\nabla u).n = 0 & \text{on } \partial\omega\times]0,L[. \end{cases}$$

(2) if $\gamma = 0$, there is no relation between u and v,

(3) if $\gamma = +\infty$, then $u = v$ in Ω.

Representation of Deny-Beurling (case of $p = 2$ and $g = 0$) : In this case for p.p $x' \in w, v(x',\cdot)$ is the solution of the differential equation in $]0,L[$:

$$(P_w)\begin{cases} -w'' + \frac{2\gamma}{R} w = \frac{2\gamma}{R} u(x',\cdot) + f(x',\cdot)\\ w(0) = w(L) = 0 \end{cases}$$

The solution of (P_w) is given by:

$$w(s) = \int_0^L G(s,t)u(x',t)dt + \frac{R}{2\gamma}\int_0^L G(s,t)f(x',t)dt$$

for $t \in]0, L[$, the kernel Poisson $G(\cdot, t)$ is the solution of the equation:

$$(P_y) \begin{cases} -y'' + c^2\phi = c^2\delta_t \\ y(0) = y(L) = 0 \end{cases}$$

$$c = \sqrt{\frac{2\gamma}{R}}$$
$$G(s,t) = \frac{csh(c(L - s \vee t)sh(c(s \wedge t)}{sh(cL)}$$

by an integration by parts :

$$\begin{aligned} \int_\Omega |\frac{\partial v}{\partial x_3}|^2 dx &= \int_w \left(\int_0^L |\frac{\partial v}{\partial x_3}|^2 dx_3 \right) dx' \\ &= \int_w \left([v \frac{\partial v}{\partial x_3}]_0^L - \int_0^L \frac{\partial^2 v}{\partial x_3^2} v dx_3 \right) dx' \\ &= \frac{2\gamma}{R} \int_\Omega [-v(x', s)^2 + u(x', s)v(x', s)] ds dx' + \int_\Omega fv dx \\ &= \frac{2\gamma}{R} \int_\Omega (uv - v^2) dx + \int_\Omega fv dx \end{aligned}$$

then

$$\pi R^2 \int_\Omega |\frac{\partial v}{\partial x_3}|^2 dx = 2\gamma\pi R \int_\Omega (uv - v^2) dx + \pi^2 R \int_\Omega fv dx$$

deferring in the total energy :

$$\begin{aligned} \varphi(u, v) &= F(u, v) - |Y^\#| \int_\Omega fu dx - \pi^2 R \int_\Omega fv dx \\ &= \int_\Omega j_2^{hom}(\nabla u(x)) dx - |Y^\#| \int_\Omega fu dx + 2\gamma\pi R \int_\Omega u^2 dx - 2\gamma\pi R \int_\Omega uv dx \\ &= \int_\Omega j_2^{hom}(\nabla u(x)) dx - |Y^\#| \int_\Omega fu dx + 2\gamma\pi R \int_\Omega u^2 dx \\ &\quad - 2\gamma\pi R \int_w \left(\int_{(0,L)^2} u(x', s) u(x', t) G(s,t) ds dt \right) dx' - \pi^2 R \int_w \left(\int_{(0,L)^2} u(x, s) G(s,t) f(x', t) ds dt \right) dx' \end{aligned}$$

however

$$\begin{aligned} 2\gamma\pi R \int_\Omega u^2 dx - 2\gamma\pi R \int_w \left(\int_{(0,L)^2} u(x', s) u(x', t) G(s,t) ds dt \right) dx' &= \gamma\pi R \int_w \left(\int_{(0,L)^2} (u(x', s) - u(x', t))^2 G(s,t) ds dt \right) dx' \\ &\quad + 2\gamma\pi R \int_w \left(\int_0^L (u(x', s))^2 (1 - \int_0^L G(s,t) ds) ds \right) \\ &= 2\gamma\pi R \int_\Omega u^2 p(x_3) dx \\ &\quad + \gamma\pi R \int_w \left(\int_{(0,L)^2} (u(x', s) - u(x', t))^2 G(s,t) ds dt \right) dx' \end{aligned}$$

where $p(s) = \frac{\cosh\left(\sqrt{\frac{2\gamma}{R}}(s - \frac{L}{2})\right)}{\cosh\left(\sqrt{\frac{2\gamma}{R}}(\frac{L}{2})\right)}$. Let

$$\begin{cases} p_{\gamma,R}(s) = 2\pi\gamma R p(s) \\ k_{\gamma,R}(s,t) = \frac{\pi R \gamma \sqrt{\frac{2\gamma}{R}}}{sh\left(\sqrt{\frac{2\gamma}{R}} L\right)} sh\left(\sqrt{\frac{2\gamma}{R}}(L - s \vee t)\right) sh\left(\sqrt{\frac{2\gamma}{R}}(s \wedge t)\right) \end{cases}$$

We obtain

$$\begin{aligned}\phi(u,v) &= \int_\Omega j_2^{hom}(\nabla u(x))dx + \int_\Omega u^2 p_{\gamma,R}(x_3)dx \\ &+ \int_w \left(\int_{(0,L)^2} (u(x',s)-u(x',t))^2 G(s,t)k_{\gamma,R}(s,t)dsdt \right) dx' \\ &- \int_w \left(\int_{(0,l)^2} u(x,s)f(x',t)k_{\gamma,R}(s,t)dsdt \right) dx' - |Y^\#| \int_\Gamma fudx\end{aligned}$$

let $\mu(dx) = p_{\gamma,R}(x_3)dx, J(dxdy) = \frac{1}{2}\Delta(dx'dy') \otimes k_{\gamma,R}(x_3,y_3)d_3dy_3$, where $\Delta(dx'dy')$ is the measure in w^2 defined by:

$$\iint_{w^2} \phi(x',y')\Delta(dx'dy') = \int_w \phi(x',x')dx'$$

and $v = (v_{ij})\ \ i,j = 1,2,3$ the measure defined by:

$$\begin{aligned} v_{ij}(dx) &= a_{ij}^{hom} dx \text{ for } i,j = 1,2\text{où} \\ a_{ij}^{hom} &= \int_{Y\#} \{\delta_{ij} - \sum_{k=1,2} \delta_{ik} \frac{\partial \chi^j}{\partial y_k}\} dy \end{aligned}$$

with χ^j is the solution of the problem min $\left\{\int_{Y\#} |\nabla w + e_j|^2 dx, w \in H^1(Y^\#), Y - \text{periodique}\right\}$, where $(e_j)_{j=1,2}$ the canonical base of IR^2, $v_{3,3}(dx) = dx$. Then

$$\begin{aligned}\phi(u,v) = &\int_\Omega \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_j} v_{ij}(dx) + \int_\Omega (u(x))^2 \mu(dx) + \int_{\Omega\times\Omega} (u(x)-u(y))^2 J(dxdy) \\ &- \iint_{\Omega\times\Omega} u(x)f(y)J(dxdy) - |Y^\#| \int_\Omega f(x)u(x)dx\end{aligned}$$

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