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On the Boundedness of Some Nonlinear Differential Equation of Second Order

Research Article

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Abstract: In this paper we study the boundedness of the solutions of some nonlinear differential equation using as a key tool the Second Lyapunov method, i.e. find sufficient conditions under which the solutions of this equation are bounded. Various particular cases and methodological remarks are included at the end of paper.

Keywords: Lyapunov's Second Method, boundedness, second order nonlinear equation.

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1. Preliminaries

1892 was an annus mirabalis in the theory of differential equations and nonlinear mechanics. That was the year of the publication of the famous Lyapunov memory General Problem of the Stability of Motion and the first volume of the famous New Methods of Celestial Mechanicsof Poincaré. While the origin of both works is different, we note that both have much in common and many differences, each of these have their methods and styles. Poincaré method is more geometric, while the more analytical Lyapunov. It is not necessary to add that the successors have taken advantages of both styles. The fundamental work of Lyapunov in Stability Theory, initially received little attention and for a long time, was almost forgotten, just 25 years later, these investigations were taken over by the soviet mathematicians, noting that the Second Method of Lyapunov was applicable to concrete problems of physics and engineering. Hereafter was greater the number of publications and researchers devoted to the subject, not only Europe but all over the world who have taken the extensions and refinements of this topic to many areas of theoretical and practical. Thus, the Second Method of Lyapunov also called Direct Method is now not only used to prove theorems of stability in Theoretical Mechanics, but practical problems of mechanical and electrical oscillations, particularly in the engineering control. The name Second Method is of historical origin, Lyapunov also used a First Method, comprising all procedures in which the explicit form of the solution is used.

The Second Method is a powerful tool to known the behavior of a solution of a differential equation (or system) without knowing the solution, i.e., without using the explicit expression of the perturbed motion, which in the vast majority of problems applied is impossible, for dealing with nonlinear equations not solvable in quadratures.

The conclusions about the behavior of the trajectory of a given differential equation (or system) is done using, in addition to the equation o system, a function defined in the Plan Phase, usually called Functions of Lyapunov and the sign of its

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first derivative along equation or system is used together the sign of the function. This idea supports a simple geometric interpretation, which is probably due to Chetaev in 1965 and is particularly useful in applications.

Since the fifties we have witnessed a remarkable development of a field of mathematical physics designated by the name of Nonlinear Mechanics. This term is probably not entirely correct, because the changes have not occurred in Mechanic itself but in the techniques of solving their problems, especially those using instead linear differential equations, nonlinear equations. The aim of this work is to study the boundedness of the solutions of the nonlinear differential equation:

$$(a(t)x')' + \varphi(t,x,x')f(x) + b(t)g(x)h(x') = (t,x,x'),$$

using as a key tool the Second Lyapunov method, i.e. find sufficient conditions under which the solutions of this equation are bounded and assume, that the equations, satisfy a certain condition of existence and uniqueness. The boundedness of the solutions is basic to all Qualitative Theory, since the study of behavior to the future depends heavily on it.

2. Results

The precise meaning of boundedness of the solutions of a differential equation, and we use throughout this paper, is as follows. Consider a system of differential equations

$$\frac{dx}{dt} = F(t,x), \tag{i}$$

where x is a n-vector. Suppose that $F(t,x)$ is continuous in its arguments on $I \times D$, where D is a connected open set in $\mathbf{R}^n$ and $I = [t_0, +\infty)$ with $t_0 \geq 0$. The solution $x(t;x_0,t_0)$ of (i) is said bounded, if there exists a $\beta > 0$ such that $|x(t;x_0,t_0)| < \beta$ for all $t \geq t_0$ where β may depend on each solution. In our work we first consider a particular case of (i). Thus, the differential equation is

$$x'' + \theta(t,x,x')f(x) + a(t)g(x)h(x') = 0. \tag{1}$$

By making the change of variables $x' = y$, equation (1) becomes the equivalent system:

$$\begin{aligned} x' &= y, \\ y' &= -\theta(t,x,y)f(x) - a(t)g(x)h(y). \end{aligned} \tag{2}$$

And consider the following Lyapunov Function:

$$V(t,x,y) = G(x) + \frac{H(y)}{a(t)}, \tag{3}$$

with $G(x) = \int\limits_0^x g(s)ds$ and $H(y) = \int\limits_0^y \frac{s}{h(s)}ds$. Thus, we can state our first result.

Theorem 2.1. *Under conditions:*

a) $f(x) > 0,\ \forall x$.

b) $y\theta(t,x,y) > 0,\ \forall\ y \neq 0$ *and any* (t,x).

c) $xg(x) > 0, x \neq 0$.

d) $0 < a(t) \leq a; a \in C^1[0,+\infty, 0 \leq \delta < a'(t), \delta \in R$.

e) $0 < m \leq h(y) \leq M < +\infty$.

Then, all the solutions $(x(t), y(t))$ *of the system (2) are stable and bounded.*

Proof. From the conditions (c), (d) and (e), we easily see that the function (3) is positive definite: $V(t,0,0) = G(0) + \frac{H(0)}{a(0)} = 0$ and $V(t,x,y) = G(x) + \frac{H(y)}{a(t)} > 0$ for all $x, y \neq 0$, since $G(x) > 0$ for all $x \neq 0$, $a(t) > 0$ and

$$H(y) = \int_0^y \frac{s}{h(s)} ds \geq \int_0^y \frac{s}{M} ds = \frac{1}{M} \int_0^Y s ds = \frac{y^2}{2M} > 0, \ \forall \, y \neq 0.$$

The derivative of the function (3) along the system (2) is negative; in fact:

$$\begin{aligned}
V'_{(2)}(t,x,y) &= G'(x) + (\frac{1}{a(t)} H(y))', \\
V'_{(2)}(t,x,y) &= -\left[\frac{a'(t)}{a^2(t)} H(y) + \frac{1}{a(t)} \frac{y\theta(t,x,y)}{h(y)} f(x)\right] \leq -\left[\frac{a'(t)}{a^2(t)} H(y) + \frac{1}{a(t)} \frac{y\theta(t,x,y)}{M} f(x)\right], \\
V'_{(2)}(t,x,y) &\leq -\frac{1}{M}\left[\frac{a'(t)}{a^2(t)} \frac{y^2}{2} + \frac{1}{a(t)} y\theta(t,x,y) f(x)\right] \leq 0,
\end{aligned}$$

from here we have

$$V'_{(2)}(t,x,y) \leq 0 \tag{4}$$

The only way a solution of (2) is not bounded, is that there is $T > 0$ such that:

$$\lim_{t \to T^-} [|x(t)| + |y(t)|] = +\infty. \tag{5}$$

Integrating the inequality (4), we have:

$$V(t,x,y) - V_0(t_0, x(t_0), y(t_0)) = -\frac{1}{M} \int_0^t (\frac{a'(t)}{a^2(t)} \frac{y^2}{2} + \frac{1}{a(t)} y\theta(s,x,y) f(x)) ds \leq -\frac{1}{M} \int_0^t \frac{a'(t)}{a^2(t)} \frac{y^2}{2} ds \leq 0.$$

If the integral

$$\int_0^t \frac{a'(s)}{a^2(s)} \frac{y^2}{2} ds \tag{6}$$

was divergent, then $\lim_{t \to +\infty} A(t) = -\infty$, with $A(t) = -\frac{1}{M} \int_0^t \frac{a'(s)}{a^2(s)} \frac{y^2}{2} ds$. Hence $\lim_{t \to +\infty} A(t) + V_0(t_0, x(t_0), y(t_0)) = -\infty$, which contradicts the fact that the function (3) is positive definite, i.e. that $V(t,x,y) \geq 0$ for all $x, y \neq 0$.

Then, the integral (6) is convergent for all $t > 0$, which implies that the integrand is bounded, in particular, $y^2(t)$ is bounded, from which it is obtained that is $y(t)$, i.e. $\exists \, k > 0$ such that

$$|y| \leq k \tag{7}$$

Integrating now the first equation (2) we obtain

$$|x| - |x_0| \leq |x - x_0| \leq \int_{t_0}^t |x'(s)| ds \leq k(t - t_0) \leq k(T - t_0),$$

where $T \geq t > t_0$. From here we have

$$|x| \leq |x_0| + k(T - t_0), \tag{8}$$

that is, the boundedness of $|x|$. (7) and (8) contradicts the assumption (5); this and the positivity of V and (4), prove the Theorem. □

Theorem 2.2. *Under conditions:*

a) $f(x) > 0,\ \forall\, x.$

b) $y\theta(t,x,y) > 0,\ \forall\, y \neq 0$ *and any* (t,x).

c) $G(x) \geq -G_0, (G_0 > 0)$.

d) $0 < a(t) \leq a < +\infty, a \in C^1[0, +\infty.$

e) $\lim_{y\to\pm\infty} H(y) = +\infty.$

all the solutions $(x(t), y(t))$ *of the system (2) are bounded.*

Proof. As we cannot guarantee that the function $V(t,x,y)$ is positive definite, we will use another way to show the boundedness of the solutions of the system (2). We obtain the derivative of $V(t,x,y)$ along the system (2):

$$V'_{(2)}(t,x,y) = g(x)y - \frac{a'(t)}{a^2(t)}H(y) + \frac{1}{a(t)}\frac{y}{h(y)}(-\theta(t,x,y)f(x) - a(t)g(x)h(y)),$$
$$V'_{(2)}(t,x,y) = -\left[\frac{a'(t)}{a^2(t)}H(y) + \frac{1}{a(t)}\frac{\theta(t,x,y)f(x)}{h(y)}\right] \leq 0.$$

As before, assume that there is a solution of the system (2) which is not bounded, i.e. that (5) is fulfilled. Using the fact that $V'_{(2)}(t,x,y) \leq 0$, we obtain

$$V_0(t_0, x(t_0), y(t_0)) \geq V(t,x,y) = G(x) + \frac{H(y)}{a(t)} \geq -G_0 + \frac{H(y)}{a(t)} \geq -G_0 + \frac{H(y)}{a}.$$

Of the last inequality, and considering the condition (e), we deduce that y(t) is bounded, i.e. $\exists\, k > 0$ such that

$$|y| \leq k. \tag{9}$$

So

$$|x| - |x_0| \leq |x(t) - x_0| \leq \int_{t_0}^{t} |x'(s)|ds \leq k(t - t_0) \leq k(T - t_0)$$

with $T \geq t > t_0$. From here we have

$$|x| \leq |x_0| + k(T - t_0),$$

that is, the boundedness of $|x|$. But this last inequality and (9) contradict (5). From this we obtain the desired boundedness of solutions of (2), i.e. the Theorem. □

Consider now a generalization of the differential equation (1), that is, consider the equation

$$(a(t)x')' + \phi(t,x,x')f(x) + b(t)g(x)h(x') = \gamma(t,x,x'). \tag{10}$$

Clearly, if a $(t) = 1$, and $\gamma \equiv 0$, (11) becomes (1). So any result for (11) is also a result of (1). Consider the Lyapunov Function:

$$V(t,x,y) = G(x) + \frac{a(t)}{b(t)}H(y). \tag{11}$$

With $G(x)$ and $H(y)$ as before. By making $x' = y$, equation (10) becomes the equivalent:

$$\begin{aligned} x' &= y, \\ y' &= \frac{1}{a(t)}[-a'(t)y - \phi(t,x,y)f(x) - b(t)g(x)h(y) + \gamma(t,x,y)]. \end{aligned} \tag{12}$$

So we have our final result.

Theorem 2.3. *Under conditions:*

a) $a(t)$ and $b(t)$ are continuous and positive in their arguments.

b) $\varphi(t,x,y), f(x), \gamma(t,x,y)$ are continuous in their arguments, $y\varphi(t,x,y) \geq 0$, $\forall\, y \neq 0$ and any (t,x).

c) $g(x)$ continuous, $xg(x) > 0$ for $x \neq 0$ and $G(\pm\infty) = +\infty$.

d) $h \in C^1$ with $h(y) > 0$.

If, in addition the following assumptions are fullfiled:

i) $\int\limits_0^\infty \frac{|a'(t)|}{a(t)} dt < \infty$ and $\int_0^\infty \frac{b'(t)_-}{b(t)} dt < \infty$

ii) $\exists\, k \geq 0$, $\exists\, M : 0 < M < 1$, such that if $|y| \geq k$ then $\frac{y^2}{h(y)} \leq MH(y)$.

iii) $|\gamma(t,x,y)| \leq \frac{a(t)|b'(t)|}{Mb(t)}$, $\forall\, t \geq 0$.

Then the solutions of (10) are bounded. If in addition we assume:

iv) $H(y) \to \infty, |y| \to \infty y b(t) \leq b_2$.

Then, every solution $(x(t), y(t))$ of (12) are bounded.

Remark 2.4. *Condition (i) implies that there are constants a_1, a_2, b_1 such that $a_1 \leq a(t) \leq a_2$ and $b_1 \leq b(t)$. Moreover (c) together with (ii) is equivalent to $\exists M' > 0$, $\frac{y^2}{h(y)} \leq M'H(y)$ in $\boldsymbol{R}$; from (ii) it follows that:*

$*$) $\frac{|y|}{h(y)} \leq m + MH(y)$,

$**$) $\frac{y^2}{h(y)} \leq m' + MH(y)$,

for some positive constants m and m'.

Proof. As before, it is easy to see that the function $V(t,x(t),y(t))$ defined by (11) is positive definite. Calculating the derivative along the system (12) we obtain:

$$V'_{(12)}(t,x,y) = \left[G(x) + \frac{a(t)}{b(t)}H(y)\right]',$$
$$V'_{(12)}(t,x,y) \leq V(t,x,y)(\frac{M+1}{a(t)}|a'(t)| + c^2) + \frac{|\gamma(t,x,y)|y}{b(t)h(y)} \leq V(t,x,y)p(t) + \frac{y}{h(y)}\frac{a(t)|b'(t)|}{Mb^2(t)},$$

where $p(t) = (\frac{M+1}{a(t)}|a'(t)| + c^2)$. From here, and using (ii)

$$V'_{(12)}(t,x,y) \leq V(t,x,y)p(t) + \frac{|y|}{h(y)}\frac{a(t)|b'(t)|}{Mb^2(t)} \leq V(t,x,y)p(t) + \frac{a(t)}{b(t)}H(y)\frac{1}{k}\frac{|b'(t)|}{b(t)},$$
$$V'_{(12)}(t,x,y) \leq V(t,x,y)(p(t) + \frac{1}{k}\frac{|b'(t)|}{b(t)}) = V(t,x,y)q(t),$$

with $q(t) = (p(t) + \frac{1}{k}\frac{|b'(t)|}{b(t)})$. By integrating the last inequality we have

$$V(t,x,y) \leq V_0(t_0,x(t_0),y(t_0)) \exp\left(\int_0^t q(s)ds\right).$$

Taking into account that $p(t)$ and $q(t)$ are integrable, by (i) we obtain the boundedness of $V(t,x,y)$, i.e. there is certain positive constant C such that $V(t,x,y) \leq C$. Using (c) we obtain the boundedness of $x(t)$ of (10). Using this and (iv) the boundedness of the solutions of (12) is obtained. This completes the proof of Theorem. □

3. Conclusion and Methodological Remarks

We will always reference the equation (10) as it is the most general studied by us.

Remark 3.1. *We consider the differential equation*

$$x'' + a(t)f(x)g(x') = 0, \tag{13}$$

studied in [22] and obtained from (11) making $a \equiv 1$*,* $f \equiv 0$ *and* $\gamma \equiv 0$*. Suppose that all assumptions of Theorem 2.3 are fulfilled and* $a(t) > 0$ *for* $t \geq T$ *and there exists a non-negative function* $\alpha(t)$ *such that* $-a(t) \leq (t)a(t)$ *with* $\int\limits_{t_0}^{\infty} \alpha(s)ds < \infty$ *then all solutions of (13) are bounded. So our Theorem 2.3 become in the Theorem 2.1 of that paper, which make use of the following assumptions:*

A_1) $g(x')$ is a positive and continuous function,

A_2) $f(x)$ is a continuous function satisfying $xf(x) > 0$, for $x \neq 0$,

A_3) $a(t)$ is a continuous function,

A_4) $\lim\limits_{|x|\to\pm\infty} \int_0^x f(s)ds = \infty$.

Proof. Making $x' = y$ in (13) we obtain the equivalent system

$$\begin{aligned} x' &= y, \\ y' &= -a(t)f(x)g(y). \end{aligned} \tag{14}$$

And we consider the following Lyapunov Function:

$$V(t,x,y) = \int_0^x f(s)ds + \frac{1}{a(t)} \int_0^y \frac{s}{g(s)}ds. \tag{15}$$

Is clear that $V(t,x,y) > 0$ if $x^2 + y^2 \neq 0$, and taking into account A_4, $V \to 0$ as $x \to 0$. The derivative of V along the system (14) is $V'(t,x,y) \leq \frac{a'(t)}{a^2(t)} \int\limits_0^y \frac{s}{g(s)}ds \leq \alpha(t)V'(t,x,y)$, and we have

$$V(t,x,y) \leq V'(T,x(T),y(T)) \left\{ \exp \int_T^t \alpha(s)ds \right\} < \infty. \tag{16}$$

For all t; from this all solutions of (14) are bounded. Also if we suppose that

A_5) $\lim\limits_{|u|\to\infty} \int_0^u \frac{s}{g(s)}ds = \infty$,

and $a'(t)$ is a bounded function (so $V \to \infty$ as $y \to \infty$) then all solutions of (14) are bounded. As an example of the consistency of our results, we see that the example discussed in [22] is still valid. Consider the equation $x'' + (1 + e^{-t}sint)x^{\frac{3}{2}}[2 + cosx'] = 0$, and the Liapunov Function

$$V(t,x,y) = F(x) + \frac{1}{a(t)}G(y), \tag{17}$$

With $F(x) = \int_0^x f(s)ds, G(y) = \int_0^y \frac{s}{g(s)}ds$. Derivating (17) we obtain:

$$V'(t,x,y) \leq -\frac{a'(t)}{a^2(t)} \int_0^y \frac{s}{g(s)}ds \leq \frac{|a'(t)|}{a(t)} \int_0^y \frac{s}{g(s)}ds \leq \frac{|a'(t)|}{a(t)} V(t,x,y).$$

From this we have $\frac{|a'(t)|}{a(t)} = \frac{|e^{-t}(sint - cost)|}{1+e^{-t}sint} \leq \frac{\sqrt{2}e^{-t}}{1-e^{-\frac{3}{2}\pi}}$ and $\frac{\sqrt{2}}{1-e^{-\frac{3}{2}\pi}} \int\limits_0^{\infty} e^{-s}ds < \infty$. Then $\int\limits_0^{\infty} \frac{|e^{-t}(sint-cost)|}{1+e^{-t}sint} < \infty$. Therefore, we can choose $\alpha(t) = \frac{|e^{-t}(sint-cost)|}{1+e^{-t}sint}$ from here the boundedness of the solutions is followed. □

Remark 3.2. *Returning again to equation (13) studied in [22]*

$$x'' + a(t)f(x)g(x^{'}) = 0. \tag{18}$$

The author doesn't prove the Theorem 6 exposed in that article since the integral $\int_{t_0}^{t} a'(s)F(x)ds$ *used in* $a(t)F(x) \leq c_0 + c_1 + \int_{t_0}^{t} a'(s)F(x)ds$ *(with* $c_0 = G(x'(t_0)) + a(t_0)F(u(t_0))$ *and* $c_1 = \int_{t_0}^{\infty} \gamma(s)ds$ *nonnegative constants), as part of an upper bound, it not guaranteed to be convergent.*

Instead, as the equation (18) is a particular case of equation (11) studied in Theorem 2.3, the boundedness of solutions is obtained without any doubt.

Remark 3.3. *One might think that the condition* $f(x) > 0$ *is very restrictive, but nevertheless, this condition has some "overtones"of necessity because in the equation* $x'' - 6x' + 5x = 0$ *the positivity of* f *is violates and has a solution unbounded* $x = e^t$.

Remark 3.4. *If in (11) we make* $a \equiv 1$, $\varphi(t, x, x') = x'$, $h \equiv 1$, $b \equiv 1$ *and* $\gamma \equiv 0$ *we obtain the classical Liénard equation, focus of many research in recent decades for its variety of applications. We invite readers to check that our results are consistent with those reported in the literature, in particular [5, 6] and [8].*

Remark 3.5. *Similarly if we let* $a \equiv 1$, $\varphi(t, x, x') = x'$, $h \equiv 1$, *and* $\gamma(t, x, x') \equiv \gamma(t)$ *in (11), we obtain the non autonomous Liénard equation, subjected to an external force* $x'' + f(x)x' + b(t)g(x) = \gamma(t)$, *we refer the reader to the papers [2] and [10] to check the validity of our results.*

Remark 3.6. *In his doctoral thesis (see [13]), Sakata get different results on boundedness and attractivity for equations* $(a(t)x')' + b(t)f_1(x)g_1(x')x' + c(t)f_2(x)g_2(x') = e(t, x, x')$ *and* $(a(t)x')' + h(t, x, x') + c(t)f(x)g(x') = e(t, x, x')$.

We invite the reader to see that our Theorem 2.3 is consistent with Theorem 2.1 proved in this work. In particular, the examples presented there are still valid under Theorem 2.3. Taking the Example 2.4, which it is considered the equation

$$x'' + \frac{1}{4(t+1)}x' + \frac{1}{8(t+1)^2}x = 0.$$

You can easily check that all conditions of our Theorem 2.3 are fullfiled, except the first bound of (i) and the above equation has unbounded solution $x(t) = \sqrt{t+1}$.

The Example 2.6 consider the equation $(a(t)x')' + c(t)x = e(t)$, *with* $a(t) = (t+7)^2 \log(t+7), c(t) = \frac{1}{4}(t+7)$ *and* $e(t) = \frac{1}{4}(t+11)\log(t+7) + 1$ *that does not fulfill the aforementioned condition and has unbounded solution* $x(t) = \log(t+7)$.

The Example 2.7 consider the equation $x'' + x' + x = (2+t)x'$ *which satisfies all the conditions of Theorem 3, except (iii) and has the unbounded solution* $x(t) = t + 1$.

Remark 3.7. *Other particular cases of Equation (11) and have been studied in different papers are:*

(1). [18] studied Tunc (11) with $f \equiv 1$, $\theta(t, x, x') = \varphi(t, x, x')$ *con*ϕ *>0 and* $h \equiv 1$.

(2). In [3] is considered (11) making $\theta(t, x, x')f(x) = h(t, x, x')$ *and our Theorem 2.3 coincides with Theorem 2.1 proved in this work.*

(3). [1] considered the equation (11) with $\theta(t, x, x')f(x) = \varphi(t, x, x')x'$, $h \equiv 1$, *and* $\gamma \equiv 0$, *we invite the reader to check the consistency of Theorem 2.1 of that paper with our Theorem 2.3.*

(4). Tunc consider in [19] the equation

$$x'' + f(t, x, x')g(t, x, x') + b(t)h(x) = e(t, x, x'). \tag{19}$$

and Theorem 2.2 of this work is requested that in addition to the initial considerations (continuous in their arguments and b of class C^1) the following conditions are fulfilled:

i). $f(t, x, x')g(t, x, x') \geq 0$ for all $t \geq 0$ and all $x, y \in R$, $b(t) \geq 1$, $b'(t) \leq 0$ for all $t > 0$, $\frac{h(x)}{x} \geq \alpha$ for all $x \neq 0$,

ii). $|e(t, x, x')| \leq |p(t)|$, with $\int_0^t |p(s)|ds < \infty$.

Then all solutions of equation (19) are bounded. Is easy to obtain that assumption (ii) is equivalent to our assumptions (i), (iii) and that $\frac{h(x)}{x} \geq \alpha$ for all $x \neq 0$ is similar to our assumption c), while we do not make use the rest of the conditions imposed in that Theorem 2.2.

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