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On Certain Subclasses of Multivalent Functions Involving A Differintegral Operator

Research Article

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Abstract: In this paper, we introduce the class $S_p^{m,l,\lambda}(\alpha,\beta,\gamma,\sigma;A,B)$ of p-valent functions in the unit disc $\mathbb{U}=\{z:|z|<1\}$. We obtain coefficient estimate, distortion and closure thoerems and radii of close-to-convexity, starlikeness and convexity for this class.

MSC: 30C45.

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1. Introduction

Let $\mathcal{A}_p$ denote the class of functions normalized by

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{n+p}z^{n+p}\quad (p\in\mathbb{N}:=1,2,3,...),\tag{1}$$

which are analytic in the open unit disk $\mathbb{U}$.

Let f and F be members of $H(\mathbb{U})$, the function $f(z)$ is said to be subordinate to $F(z)$, or $F(z)$ is said to be superordinate to $f(z)$, if there exists a function $w(z)$ analytic in $\mathbb{U}$ with $w(0)=0$ and $w(z)<1(z\in\mathbb{U})$, such that $f(z)=F(w(z))$. In such a case we write $f(z)\prec F(z)$. In particular, if F is univalent, then $f(z)\prec F(z)$ if and only if $f(0)=F(0)$ and $f(\mathbb{U})\subset F(\mathbb{U})$ (see [4]).

For two functions f (z) given by (1) and

$$g(z)=z^p+\sum_{n=1}^{\infty}b_{n+p}z^{n+p}\quad (p\in\mathbb{N}:=1,2,3,...),\tag{2}$$

The Hadamard product (or convolution) of f and g is defined by

$$(f*g)(z)=z^p+\sum_{n=1}^{\infty}a_{n+p}b_{n+p}z^{n+p}=(g*f)(z).\tag{3}$$

We recall the definitions of the fractional derivative and integral operators introduced and studied by Saigo (cf. [14, 15]).

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Definition 1.1. *let $\alpha > 0$ and $\beta, \gamma \in R$, then the generalized fractional integral operator, $I_{0,z}^{\alpha,\beta,\gamma}$ of order α of a function $f(z)$ is defined by*

$$I_{0,z}^{\alpha,\beta,\gamma} = \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; 1-\frac{t}{z}\right) f(t)\, dt, \tag{4}$$

where the function $f(z)$ is analytic in a simply-connected region of the z - plane containing the origin and the multiplicity of $(z-t)^{\alpha-1}$ is removed by requiring $log(z-t)$ to be real when $(zt) > 0$ provided further that

$$f(z) = O\left(|z|^{\epsilon}\right), z \to 0 \quad for \quad \epsilon > max\left(0, \beta-\gamma\right) - 1. \tag{5}$$

Definition 1.2. *Let $0 \le \alpha < 1$ and $\beta, \gamma \in R$, then the generalized fractional derivative operator $J_{0,z}^{\alpha,\beta,\gamma}$ of order α of a function $f(z)$ defined by*

$$\begin{aligned} J_{0,z}^{\alpha,\beta,\gamma} f(z) =& \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \left[z^{\alpha-\beta} \int_0^z (z-t)^{-\alpha} {}_2F_1\left(\beta-\alpha, 1-\gamma; 1-\alpha; 1-\frac{t}{z}\right) f(t)\, dt \right], \\ =& \frac{d^n}{dz^n} J_{0,z}^{\alpha-n,\beta,\gamma} f(z). \quad (n \le \alpha < n+1; n \in \mathbb{N}) \end{aligned} \tag{6}$$

where the function $f(z)$ is analytic in a simply-connected region of the z - plane containing the origin, with the order as given in (5) and multiplicity of and multiplicity of $(z-t)^{\alpha}$ is removed by requiring $log\,(z-t)$ to be real when $(z-t) > 0$.

Definition 1.3. *For real number $\alpha\,(-\infty < \alpha < 1)$ and $\beta\,(-\infty < \beta < 1)$ and a positive real number γ, the fractional operator $U_{0,z}^{\alpha,\beta,\gamma} : \mathcal{A}_p \to \mathcal{A}_p$ is defined in terms of $J_{0,z}^{\alpha,\beta,\gamma}$ and $I_{0,z}^{\alpha,\beta,\gamma}$ by (see [8] and [11])*

$$U_{0,z}^{\alpha,\beta,\gamma} = z^p + \sum_{n=1}^{\infty} \frac{(1+p)_n (1+p+\gamma-\beta)_n}{(1+p-\beta)_n (1+p+\gamma-\alpha)_n} a_{n+p} z^{n+p}, \tag{7}$$

which for $f(z) = 0$ may be written as

$$U_{0,z}^{\alpha,\beta,\gamma} f(z) = \begin{cases} \frac{\Gamma(1+p-\beta)\Gamma(1+p+\gamma-\alpha)}{\Gamma(1+p)\Gamma(1+p+\gamma-\alpha)} z^{\beta} J_{0,z}^{\alpha,\beta,\gamma} f(z) & (0 \le \alpha \le 1), \\ \\ \frac{\Gamma(1+p-\beta)\Gamma(1+p+\gamma-\alpha)}{\Gamma(1+p)\Gamma(1+p+\gamma-\alpha)} z^{\beta} I_{0,z}^{-\alpha,\beta,\gamma} f(z) & (-\infty \le \alpha < 0). \end{cases} \tag{8}$$

Definition 1.4. *Using the operator $U_{0,z}^{\alpha,\beta,\gamma}$, Ahmed S. Galiz [3] introduce the following linear operator $\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f : \mathcal{A}_p \to \mathcal{A}_p$. If $f \in \mathcal{A}_p$, then from (1) and (7), we can easily see that*

$$\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z) = z^p + \sum_{n=1}^{\infty} \left[\frac{p+l+\lambda n}{p+l}\right]^m \frac{(1+p)_n (1+p+\gamma-\beta)_n}{(1+p-\beta)_n (1+p+\gamma-\alpha)_n} a_{n+p} z^{n+p}, \tag{9}$$

where $m \in N_0 = N \cup 0, l \ge 0, \lambda \ge 0$, and $p \in N$. The above operator generates several operators studied by many authors such as El-Ashwah and Aouf [6], Selvaraj and Karthikeyan [15], DziokSrivastava operator [5], Kamali and Orhan [7], Kumar et al. [9], Salagean [12], Al-Oboudi [1] and others.

It is easily verified from (8) that

$$\lambda z \left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)\right)' = (p+l)\, \phi_{\alpha,\beta,\gamma}^{m+1,l,\lambda} f(z) - \left[p\,(1-\lambda)+l\right] \phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z) \tag{10}$$

By making use of the differintegral operator, $\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f$, and the above mentioned principle of subordination between analyticfunctions, we introduce and investigate the following subclass of the class of p -valent analytic functions.

For fixed parameters A and B $(-1 \leq B < A \leq 1)$ and $0 \leq \sigma < p$, we say that a function $f \in \mathcal{A}_p$ is in the class $\mathcal{S}_p^{m,l,\lambda}(\alpha, \beta, \gamma, \sigma; A, B)$ if it satisfies the following condition:

$$\frac{1}{p-\sigma}\left(\frac{z\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)\right)'}{\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)} - \sigma\right) \prec \frac{1+Az}{1+Bz} \tag{11}$$

or, equivalently, if

$$\left|\frac{z\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)\right)' - p\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)}{(A-B)(p-\sigma)\,\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z) - B\left\{z\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)\right)' - p\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)\right\}}\right| < 1 \quad (z \in \mathbb{U}). \tag{12}$$

Furthermore, we say that a function $f \in \mathcal{S}_p^{m,l,\lambda}(\alpha, \beta, \gamma, \sigma; A, B)$ is in the analogous class $\mathcal{T}_p^{m,l,\lambda}(\alpha, \beta, \gamma, \sigma; A, B)$ if f is of the following form:

$$f(z) = z^p - \sum_{n=1}^{\infty} |a_{n+p}|\, z^{n+p} \quad (p \in \mathbb{N} := 1, 2, 3, ...), \tag{13}$$

To prove our results, we need the following definitions and lemmas.

2. Preliminary Lemmas

In proving our main results, we need each of the following lemmas.

Lemma 2.1. *(Miller and Mocanu [10]). Let $-1 \leq B < A \leq 1$, and $\beta > 0$. Also let the complex number γ be constrained by*

$$\Re(\gamma) \geq -\frac{\beta(1-A)}{1-B}.$$

Then the following differential equation:

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U})$$

has a univalent solution in $\mathbb{U}$ given by

$$q(z) = \begin{cases} \dfrac{z^{\beta+\gamma}(1+Bz)^{\beta(A-B)/B}}{\beta\int_0^z t^{\beta+\gamma-1}(1+Bt)^{\beta(A-B)/B}dt} - \dfrac{\gamma}{\beta} & (B \neq 0), \\[2ex] \dfrac{z^{\beta+\gamma}exp(\beta Az)}{\beta\int_0^z t^{\beta+\gamma-1}exp(\beta At)dt} - \dfrac{\gamma}{\beta} & (B = 0). \end{cases} \tag{14}$$

Furthermore, if

$$\phi(z) = 1 + c_1 z + c_2 z^2 + \cdots$$

is analytic in U and satisfies the following differential subordination:

$$\phi(z) + \frac{z\phi'(z)}{\beta\phi(z) + \gamma} = \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U}) \tag{15}$$

then

$$\phi(z) \prec q(z) \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathbb{U})$$

and $q(z)$ is the best dominant of (15).

Lemma 2.2. *(Wilken and Feng [18]). Let ν be a positive measure on [0, 1]. Also let $h(z,t)$ be a complex-valued function defined on $\mathbb{U}\times[0,1]$ such that $h(z,t)$ is analytic in $\mathbb{U}$ for each $t\in[0,1]$, and $h(z,t)$ is ν-integrable on $t\in[0,1]$ for all $z\in\mathbb{U}$. In addition, suppose that $\Re(h(z,t))>0$, $h(-r,t)$ is real and*

$$\Re\left(\frac{1}{h(z,t)}\right)\geqq\frac{1}{h(-r,t)}$$

If

$$\eta(z):=\int_0^1 h(z,t)\,d\nu(t),$$

then

$$\Re\left(\frac{1}{\eta(z)}\right)\geqq\frac{1}{\eta(-r)}\qquad(|z|\leqq r<1).$$

Each of the identities (asserted by Lemma 3 below) is well known (see, for example, [[14], Chapter 14]).

Lemma 2.3. *For real or complex parameters $\alpha_1,\alpha_2,\beta_1\ (\beta_1\notin\mathbb{Z}_0^-)$.*

$$\int_0^1 t^{\alpha_2-1}(1-t)^{\beta_1-\alpha_2-1}(1-zt)^{-\alpha_1}\,dt=\frac{\Gamma(\alpha_2)\,\Gamma(\beta_1-\alpha_2)}{\Gamma\beta_1}\,{}_2F_1(\alpha_1,\alpha_2;\beta_1;z)$$

$$(\Re(\beta_1)>\Re(\alpha_2)>0);\tag{16}$$

$${}_2F_1(\alpha_1,\alpha_2;\beta_1;z)={}_2F_1(\alpha_2,\alpha_1;\beta_1;z);\tag{17}$$

$${}_2F_1(\alpha_1,\alpha_2;\beta_1;z)=(1-z)^{-\alpha_1}\,{}_2F_1\left(\alpha_1,\beta_1-\alpha_2;\beta_1;\frac{z}{z-1}\right).\tag{18}$$

3. Inclusion Relationships for the Class $\mathcal{S}_p^{m,l,\lambda}(\alpha,\beta,\gamma,\sigma;A,B)$

Unless otherwise mentioned, we assume throughout this section that $0\leq\sigma<p$, and $1\leq B<A\leq 1$.

Theorem 3.1. *If $f\in\mathcal{S}_p^{m+1,l,\lambda}(\alpha,\beta,\gamma,\sigma;A,B)$, and*

$$(1-A)(p-\sigma)+(1-B)\left(\frac{p+l}{\lambda}-(p-\sigma)\right)\geq 0\tag{19}$$

then

$$\frac{1}{p-\sigma}\left(\frac{z\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)\right)'}{\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)}-\sigma\right)\prec\frac{1}{p-\sigma}\left(\frac{1}{Q(z)}-\left(\frac{(p+l)-(p-\sigma)\lambda}{\lambda}\right)\right)=q(z)$$
$$\prec\frac{1+Az}{1+Bz}\qquad(z\in\mathbb{U})\tag{20}$$

where

$$Q(z)=\begin{cases}\displaystyle\int_0^1 t^{\left(\frac{p+l}{\lambda}-1\right)}\left(\frac{1+Btz}{1+Bz}\right)^{(A-B)(p-\sigma)/B}dt, & (B\neq 0),\\[2ex] \displaystyle\int_0^1 t^{\left(\frac{p+l}{\lambda}-1\right)}exp\left(A(p-\sigma)(t-1)z\right)dt, & (B=0).\end{cases}\tag{21}$$

and q(z) is the best dominant of (20). If, in addition to (19),

$$A \leq \frac{-B\left(\frac{p+l}{\lambda} - p + \sigma + 1\right)}{p - \sigma} \quad (-1 \leq B < 0) \tag{22}$$

then

$$\mathcal{S}_p^{m+1,l,\lambda}(\alpha, \beta, \gamma, \sigma; A, B) \subset \mathcal{S}_p^{m,l,\lambda}(\alpha, \beta, \gamma, \sigma; 1 - 2\rho, -1), \tag{23}$$

where

$$\rho = \frac{1}{p-\sigma}\left[\frac{p+l}{\lambda}\left\{{}_2F_1\left(1, \frac{(B-A)(p-\sigma)}{B}; \frac{p+l}{\lambda}+1; \frac{B}{B-1}\right)\right\}^{-1} - \frac{p+l}{\lambda} + p\right].$$

The result is the best possible.

Proof. By setting

$$\phi(z) = \frac{1}{p-\sigma}\left(\frac{z\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)\right)'}{\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)} - \sigma\right) \quad (z \in \mathbb{U}) \tag{24}$$

we note that $\phi(z)$ is analytic in $\mathbb{U}$ with $\phi(0) = 1$. Using the identity (6) in (24), and then differentiating the resulting equation logarithmically with respect to z, we obtain

$$\begin{aligned} &\frac{1}{p-\sigma}\left(\frac{z\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)\right)'}{\phi_{\alpha,\beta,\gamma}^{m,l,\lambda} f(z)} - \sigma\right) \\ &= \phi(z) + \frac{\lambda z \phi'(z)}{\lambda(p-\sigma)\phi(z) + [(p+l) - (p-\sigma)\lambda]} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}). \end{aligned} \tag{25}$$

which shows that $\phi(z)$ satisfies the differential subordination (15). Hence, by applying Lemma 1, we get

$$\phi(z) \prec q(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}),$$

where $q(z)$ is given by (14) with $\beta = p\sigma$ and $\gamma = \frac{[(p+l)-(p-\sigma)\lambda]}{\lambda}$ and this $q(z)$ is the best dominant of (20). This proves the assertion (19) of Theorem 1. Next we show that

$$\inf_{|z|<1} \{\Re(q(z))\} = q(-1). \tag{26}$$

If, in Lemma 3, we set

$$\alpha_1 = \frac{(p-\sigma)(B-A)}{B}, \quad \alpha_2 = \frac{p+l}{\lambda} \quad \textit{and} \quad \beta_1 = \frac{p+l}{\lambda} + 1$$

then $\beta_1 > \alpha_2 > 0$. By using (16) to (18), we see from (21) that, for $B \neq 0$,

$$\begin{aligned} Q(z) &= (1+Bz)^{\alpha_1} \int_0^1 t^{\alpha_2 - 1}(1+Btz)^{-\alpha_1}\, dt \\ &= \frac{\Gamma(\alpha_2)}{\Gamma\beta_1}\,{}_2F_1\left(1, \alpha_1; \beta_1; \frac{Bz}{Bz+1}\right). \end{aligned} \tag{27}$$

In order to prove (26), we need to show that

$$\Re\left(\frac{1}{Q(z)}\right) \geq \frac{1}{Q(-1)} \qquad (z \in \mathbb{U}). \tag{28}$$

Since the hypothesis (22) implies that $\beta_1 > \alpha_1 > 0$, by using (16), (27) yields

$$Q(z) = \int_0^1 h(z,t)\, dv(t)$$
$$h(z,t) = \frac{1+Bz}{1+(1-t)Bz} \qquad (0 \le t \le 1)$$

and

$$dv(t) = \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_1)\Gamma(\beta_1-\alpha_1)} t^{\alpha_1-1}(1-t)^{\beta_1-\alpha_1-1}\, dt,$$

where $d(t)$ is a positive measure on $t \in [0,1]$. For $1 \le B < 0$, it may be noted that $\Re\{h(z,t)\} > 0$ and $h(-r,t)$ is real for $0 \le r < 1$ and $t \in [0,1]$. Therefore, by Lemma 2, we have

$$\Re\left(\frac{1}{Q(z)}\right) \ge \frac{1}{Q(-r)} \qquad (|z| \le r < 1).$$

Thus, by letting $r \to 1-$, we obtain (28). Moreover, by letting

$$A \to \left(-\frac{B\left(\frac{p+l}{\lambda} - p + \sigma + 1\right)}{p-\sigma}\right) +$$

for the case when

$$A = -\frac{B\left(\frac{p+l}{\lambda} - p + \sigma + 1\right)}{p-\sigma},$$

and using (20), we get the inclusion relationship (23) asserted by Theorem 1.

The result is the best possible as the function $q(z)$ is the best dominant of (20). This completes the proof of Theorem 1. □

Setting $A = 1$ and $B = -1$ in Theorem 1, we get the following consequence.

Corollary 3.2. *For* $max\{\frac{p+l}{\lambda} - p, \frac{1}{2}\left(2p - \frac{p+l}{\lambda} - 1\right)\} \le \sigma < p$*,*

$$\mathcal{S}_p^{m+1,l,\lambda}(\alpha,\beta,\gamma,\sigma;1,-1) \subset \mathcal{S}_p^{m,l,\lambda}(\alpha,\beta,\gamma,\sigma;1,-1) \tag{29}$$

where

$$\rho = \frac{p+l}{\lambda}\left\{{}_2F_1\left(1, 2(p-\sigma); \frac{p+l}{\lambda}+1; \frac{1}{2}\right)\right\}^{-1} - \frac{p+l}{\lambda} + p.$$

The result is the best possible.

For a function $f \in \mathcal{A}_p$*, the generalized BernardiLiberaLivingston integral operator* $\mathcal{F}_{\lambda,p}(f)$ *is defined by*

$$\mathcal{F}_{\lambda,p}(f) = \mathcal{F}_{\lambda,p}(f)(z) := \frac{\mu+p}{z^\mu}\int_0^1 t^{\mu-1} f(t)\, dt$$
$$= \left(z^p + \sum_{n=1}^{\infty} \frac{\mu+p}{\mu+p+n} z^{n+p}\right) * f(z) \qquad (\mu > -p) \tag{30}$$

Theorem 3.3. *let* μ *be a real number satisfying the following inequality:*

$$(1-A)(p-\sigma) + (1-B)(\mu+\sigma) \ge 0. \tag{31}$$

1. *If* $f \in \mathcal{S}_p^{m,l,\lambda}(\alpha,\beta,\gamma,\sigma;A,B)$*, then the function* $\mathcal{F}_{\lambda,p}$ *defined by (3.12) belongs to the class* $\mathcal{S}_p^{m,l,\lambda}(\alpha,\beta,\gamma,\sigma;A,B)$*. Furthermore,*

$$\frac{1}{p-\sigma}\left(\frac{z\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)\right)'}{\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)}-\sigma\right)\prec\frac{1}{p-\sigma}\left(\frac{1}{Q(z)}-(\mu-\sigma)\right)=\widetilde{q}(z)$$
$$\prec\frac{1+Az}{1+Bz}\qquad(z\in\mathbb{U}),\tag{32}$$

where

$$Q(z)=\begin{cases}\int\limits_0^1 t^{\mu+p-1}\left(\frac{1+Btz}{1+Bz}\right)^{(A-B)(p-\sigma)/B}dt, & (B\neq 0),\\ \\ \int\limits_0^1 t^{\mu+p-1}exp\left(A(p-\sigma)(t-1)z\right)dt, & (B=0).\end{cases}\tag{33}$$

and $\widetilde{q}(z)$ *is the best dominant of (32).*

2. *If* $-1\leq B<0$ *and*

$$\mu\geq max\left\{\frac{(B-A)(p-\sigma)}{B}-p-1,-\frac{(1-A)(p-\sigma)}{1-B}-\sigma\right\},\tag{34}$$

then for $f\in\mathcal{S}_p^{m,l,\lambda}(\alpha,\beta,\gamma,\sigma;A,B)$*,*

$$\mathcal{F}_{\lambda,p}(f)\in\mathcal{S}_p^{m,l,\lambda}(\alpha,\beta,\gamma,1-2\chi,-1),$$

where

$$\chi:=\frac{1}{p-\sigma}\left[(\mu+p)\left\{{}_2F_1\left(1,\frac{(B-A)(p-\sigma)}{B};\mu+p+1;\frac{B}{B-1}\right)\right\}^{-1}-(\mu+\sigma)\right].$$

The result is the best possible.

Proof. From (5) and (30), it follows that

$$z\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}\mathcal{F}_{\lambda,p}f(z)\right)'=(\mu+p)\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)\right)-\mu\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}\mathcal{F}_{\lambda,p}f(z)\right).\tag{35}$$

By setting

$$\phi(z)=\frac{1}{p-\sigma}\left(\frac{\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}\mathcal{F}_{\lambda,p}f(z)\right)'}{\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}\mathcal{F}_{\lambda,p}f(z)\right)}-\sigma\right).\qquad(z\in\mathbb{U}),\tag{36}$$

we see that $\varphi(z)$ is analytic in $\mathbb{U}$ and $\phi(0)=1$. Using the identity (35) in (36), and then differentiating the resulting equation logarithmically with respect to z, we obtain

$$\frac{1}{p-\sigma}\left(\frac{\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}\mathcal{F}_{\lambda,p}f(z)\right)'}{\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}\mathcal{F}_{\lambda,p}f(z)\right)}-\sigma\right)=\phi(z)+\frac{z\phi'(z)}{(p-\sigma)\phi(z)+\mu+\sigma}\qquad(z\in\mathbb{U}),$$

Thus, by applying Lemma 1, we get

$$\phi(z)\prec\widetilde{q}(z)=\frac{1}{p-\sigma}\left(\frac{1}{Q(z)}-(\mu+\sigma)\right)\prec\frac{1+Az}{1+Bz}\qquad(z\in\mathbb{U}),$$

where $Q(z)$ is given by (33). This proves the first part of Theorem 2.

Following the same lines as in our demonstration of Theorem 1, we can prove the second part of Theorem 2. The result is the best possible as $\widetilde{q}(z)$ is the best dominant. □

By putting $A = 1$ and $B = -1$ in Theorem 2, we deduce the following consequence.

Corollary 3.4. *If μ is a real number satisfying*

$$\mu \geq max\{-\sigma, p - 2\sigma - 1\} \text{ and } \quad f \in \mathcal{S}_p^{m,l,\lambda}(\alpha, \beta, \gamma, \sigma),$$

then

$$\mathcal{F}_{\lambda,p}(f) \in \mathcal{S}_p^{m,l,\lambda}(\tau, \beta, \gamma, \sigma),$$

where

$$\tau := \frac{1}{p-\sigma}\left[(\mu+p)\left\{{}_2F_1\left(1, 2(p-\sigma); \mu+p+1; \frac{1}{2}\right)\right\}^{-1} - (\mu+\sigma)\right].$$

The result is the best possible.

4. Basic Properties of the Class $\mathcal{T}_p^{m,l,\lambda}(\alpha, \beta, \gamma, \sigma; A, B)$ and the Generalized Neighbourhoods.

Theorem 4.1. *Let $m > 0, l > 0, \lambda > 0,\ \alpha > 0, \beta > 0, \gamma > 0$ and $-1 \leq B < 0$. Also let $f \in A_p$ be given by (1.9). Then $f \in \mathcal{T}_p^{m,l,\lambda}(\alpha, \beta, \gamma, \sigma; A, B)$ if and only if*

$$\sum_{n=1}^{\infty} \frac{\{(1-B)n + (A-B)(p-\sigma)\}}{(A-B)(p-\sigma)} \Omega(n, \alpha, \beta, \gamma, \sigma)|a_{n+p}| \leq 1.$$

$$\text{where} \quad \Omega(n, \alpha, \beta, \gamma, \sigma) = \left[\frac{p+l+\mu n}{p+l}\right]^m \frac{(1+p)_n(1+p+\gamma-\beta)_n}{(1+p-\beta)_n(1+p+\gamma-\alpha)_n} \tag{37}$$

The result is sharp.

Proof. First of all, suppose that $f \in \mathcal{T}_p^{m,l,\lambda}(\alpha, \beta, \gamma, \sigma; A, B)$ is given by (1.9). Then the inequality (1.8) readily yields

$$\left|\frac{z\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)\right)' - p\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)}{(A-B)(p-\sigma)\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z) - B\left\{z\left(\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)\right)' - p\phi_{\alpha,\beta,\gamma}^{m,l,\lambda}f(z)\right\}}\right|$$
$$= \left|\frac{\sum\limits_{n=1}^{\infty} n\Omega(n, \alpha, \beta, \gamma, \sigma)|a_{n+p}|z^n}{(A-B)(p-\sigma) + \sum\limits_{n=1}^{\infty}\{(A-B)(p-\sigma) - Bn\}\Omega(n, \alpha, \beta, \gamma, \sigma)|a_{n+p}|z^n}\right| < 1, \tag{38}$$

Since $||Re(z)| \leq |z|$, for any z, choosing z to be real and letting $z \to 1-$ through real values, we find from (4.2) that

$$\sum_{n=1}^{\infty} n\Omega(n, \alpha, \beta, \gamma, \sigma)|a_{n+p}| \tag{39}$$
$$\leq (A-B)(p-\sigma) - \sum_{n=1}^{\infty}\{(A-B)(p-\sigma) - Bn\}\Omega(n, \alpha, \beta, \gamma, \sigma)|a_{n+p}|,$$

which gives the desired inequality (4.1).

To prove the converse part of Theorem 3, we assume that the inequality (4.1) holds true. Letting $|z| = 1$, we find from (1.9) and (4.1) that

$$\begin{aligned} &\left| z\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} f(z)\right)' - p\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} f(z)\right| \\ &\qquad - \left|(A-B)(p-\sigma)\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} f(z) - B\left\{z\left(\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} f(z)\right)' - p\phi^{m,l,\lambda}_{\alpha,\beta,\gamma} f(z)\right\}\right| \\ &= \sum_{n=1}^{\infty} n\Omega(n,\alpha,\beta,\gamma,\sigma)|a_{n+p}| \\ &\qquad - \left((A-B)(p-\sigma) - \sum_{n=1}^{\infty}\{(A-B)(p-\sigma) - Bn\}\Omega(n,\alpha,\beta,\gamma,\sigma)|a_{n+p}|\right) \\ &\leq 0, \end{aligned}$$

by the hypothesis of Theorem 3. Hence, by the *Maximum Modulus Theorem*, the function $f \in \mathcal{A}_p$ defined by (1.9) belongs to the class $\mathcal{T}^{m,l,\lambda}_p(\alpha,\beta,\gamma,\sigma;A,B)$.

Finally, we note that the inequality (4.1) is sharp for the function f defined in $\mathbb{U}$ by

$$f(z) = z^p - \frac{(A-B)(p-\sigma)}{\{(1-B)n + (A-B)(p-\sigma)\}\Omega(n,\alpha,\beta,\gamma,\sigma)} z^{n+p} \quad (n \in \mathbb{N}). \tag{40}$$

□

Corollary 4.2. *If the function $f(z)$ defined by (1.4) belongs to the class $\mathcal{T}^{m,l,\lambda}_p(\alpha,\beta,\gamma,\sigma;A,B)$, then*

$$|a_{n+p}| \leq \frac{(A-B)(p-\sigma)}{\{(1-B)n + (A-B)(p-\sigma)\}\Omega(n,\alpha,\beta,\gamma,\sigma)} z^{n+p} \quad (n \in \mathbb{N}).$$

The estimate is sharp for the function f given by (4.4).

5. Distortion Bounds

Theorem 5.1. *A function $f(z)$ defined by (1.4) is in $\mathcal{S}^{m,l,\lambda}_p(\alpha,\beta,\gamma,\sigma;A,B)$, then for $|z| = r$, we have*

$$\begin{aligned} r^p &- \frac{(p+l)^m(1+p)(1+p+\gamma-\beta)(A-B)(p-\sigma)}{(p+l+\lambda)^m(1+p-\beta)(1+p+\gamma-\alpha)\{(1-B)n + (A-B)(p-\sigma)\}} r^{p+1} \leq |f(z)| \\ &\leq r^p + \frac{(p+l)^m(1+p)(1+p+\gamma-\beta)(A-B)(p-\sigma)}{(p+l+\lambda)^m(1+p-\beta)(1+p+\gamma-\alpha)\{(1-B)n + (A-B)(p-\sigma)\}} r^{p+1} \end{aligned} \tag{41}$$

for $z \in \mathbb{U}$. The result is sharp.

Proof. Since $f(z)$ belongs to the class $\mathcal{S}^{m,l,\lambda}_p(\alpha,\beta,\gamma,\sigma;A,B)$ in view of Theorem 2.1, we obtain

$$\begin{aligned} &\frac{(p+l+\lambda)^m(1+p-\beta)(1+p+\gamma-\alpha)\{(1-B)n + (A-B)(p-\sigma)\}}{(p+l)^m(1+p)(1+p+\gamma-\beta)} \sum_{n=1}^{\infty} a_{n+p} \\ &\leq \sum_{n=1}^{\infty}\{(1-B)n + (A-B)(p-\sigma)\}\Omega(n,\alpha,\beta,\gamma,\sigma)a_{n+p} \leq (A-B)(p-\sigma) \end{aligned}$$

which is equivalent to

$$\sum_{n=1}^{\infty} a_{n+p} \leq \frac{(p+l)^m(1+p)(1+p+\gamma-\beta)(A-B)(p-\sigma)}{(p+l+\lambda)^m(1+p-\beta)(1+p+\gamma-\alpha)\{(1-B)n + (A-B)(p-\sigma)\}} \tag{42}$$

Using (1.4) (5.2), we obtain

$$\begin{aligned}|f(z)| &\leq |z|^p+|z|^{p+1}\sum_{n=1}^{\infty}a_{n+p}\\ &\leq r^p+r^{p+1}\sum_{n=1}^{\infty}a_{n+p}\\ &\leq r^p+\frac{(p+l)^m(1+p)(1+p+\gamma-\beta)(A-B)(p-\sigma)}{(p+l+\lambda)^m(1+p-\beta)(1+p+\gamma-\alpha)\{(1-B)n+(A-B)(p-\sigma)\}}r^{p+1}.\end{aligned}$$

Similarly,

$$\begin{aligned}&|f(z)|\\ &\geq r^p-\frac{(p+l)^m(1+p)(1+p+\gamma-\beta)(A-B)(p-\sigma)}{(p+l+\lambda)^m(1+p-\beta)(1+p+\gamma-\alpha)\{(1-B)n+(A-B)(p-\sigma)\}}r^{p+1}.\end{aligned}$$

This completes the proof of Theorem 5.1. □

Theorem 5.2. *A function $f(z)$ defined by (1.4) is in $\mathcal{S}_p^{m,l,\lambda}(\alpha,\beta,\gamma,\sigma;A,B)$, then for $|z|=r$, we have*

$$\begin{aligned}&pr^{p-1}-\frac{(p+l)^m(1+p)(1+p+\gamma-\beta)(A-B)(p-\sigma)(p+1)}{(p+l+\lambda)^m(1+p-\beta)(1+p+\gamma-\alpha)\{(1-B)n+(A-B)(p-\sigma)\}}r^p\\ &\leq |f\prime(z)|\leq\\ &pr^{p-1}+\frac{(p+l)^m(1+p)(1+p+\gamma-\beta)(A-B)(p-\sigma)(p+1)}{(p+l+\lambda)^m(1+p-\beta)(1+p+\gamma-\alpha)\{(1-B)n+(A-B)(p-\sigma)\}}r^p\end{aligned} \tag{43}$$

for $z\in\mathbb{U}$. The result is sharp.

Proof. Since $f(z)$ belongs to the class $\mathcal{S}_p^{m,l,\lambda}(\alpha,\beta,\gamma,\sigma;A,B)$ in view of Theorem 2.1, we obtain

$$\begin{aligned}&\frac{(p+l+\lambda)^m(1+p-\beta)(1+p+\gamma-\alpha)\{(1-B)n+(A-B)(p-\sigma)\}}{(p+l)^m(1+p)(1+p+\gamma-\beta)}\sum_{n=1}^{\infty}(n+p)a_{n+p}\\ &\leq\sum_{n=1}^{\infty}\{(1-B)n+(A-B)(p-\sigma)\}\Omega(n,\alpha,\beta,\gamma,\sigma)a_{n+p}\leq(A-B)(p-\sigma)\end{aligned}$$

which is equivalent to

$$\sum_{n=1}^{\infty}(n+p)a_{n+p}\leq\frac{(p+l)^m(1+p)(1+p+\gamma-\beta)(A-B)(p-\sigma)(p+1)}{(p+l+\lambda)^m(1+p-\beta)(1+p+\gamma-\alpha)\{(1-B)n+(A-B)(p-\sigma)\}} \tag{44}$$

Using (1.4) (5.4), we obtain

$$\begin{aligned}|f'(z)| &\leq p|z|^{p-1}+|z|^p\sum_{n=1}^{\infty}(n+p)a_{n+p}\\ &\leq pr^{p-1}+r^p\sum_{n=1}^{\infty}(n+p)a_{n+p}\\ &\leq pr^{p-1}+\frac{(p+l)^m(1+p)(1+p+\gamma-\beta)(A-B)(p-\sigma)(p+1)}{(p+l+\lambda)^m(1+p-\beta)(1+p+\gamma-\alpha)\{(1-B)n+(A-B)(p-\sigma)\}}r^p.\end{aligned}$$

Similarly,

$$\begin{aligned}&|f'(z)|\\ &\geq pr^{p-1}-\frac{(p+l)^m(1+p)(1+p+\gamma-\beta)(A-B)(p-\sigma)(p+1)}{(p+l+\lambda)^m(1+p-\beta)(1+p+\gamma-\alpha)\{(1-B)n+(A-B)(p-\sigma)\}}r^p.\end{aligned}$$

This completes the proof of Theorem 5.2. □

6. Closure Theorems

Theorem 6.1. *Let the functions*

$$f_j(z) = z^p - \sum_{n=1}^{\infty} a_{n+p,j} z^{n+p} \qquad (a_{n+p,j} \geq 0) \tag{45}$$

be in the class $\mathcal{S}_p^{m,l,\lambda}(\alpha, \beta, \gamma, \sigma; A, B)$ *for every* $j = 1, 2, 3, \ldots, m$. *Then the function* $h(z)$ *defined by*

$$h(z) = \sum_{j=1}^{\infty} c_j f_j(z) \qquad (c_j \geq 0) \tag{46}$$

is also in the same class $\mathcal{S}_p^{m,l,\lambda}(\alpha, \beta, \gamma, \sigma; A, B)$, *where*

$$\sum_{j=1}^{m} c_j = 1. \tag{47}$$

Proof. By means of the definition of $h(z)$, we can write

$$h(z) = z^p - \sum_{n=1}^{\infty} \left(\sum_{j=1}^{m} c_j a_{n+p,j} \right) z^{n+p}. \tag{48}$$

Now, since $f_j(z) \in \mathcal{S}_p^{m,l,\lambda}(\alpha, \beta, \gamma, \sigma; A, B)$ for every $j = 1, 2, 3, \ldots, m$. We obtain

$$\sum_{n=1}^{\infty} \Omega\left(n, \alpha, \beta, \gamma, \sigma\right) \left[(1 - B) n + (A - B) (p - \sigma)\right] a_{n+p,j} \leq (A - B) (p - \sigma), \tag{49}$$

for every $j = 1, 2, 3, \ldots, m$, by virtue of theorem (4.1). Consequently, with the aid of (6.5) we can see that

$$\begin{aligned} &\sum_{n=1}^{\infty} \Omega\left(n, \alpha, \beta, \gamma, \sigma\right) \left[(1 - B) n + (A - B) (p - \sigma)\right] \left(\sum_{j=1}^{m} c_j a_{n+p,j} \right) \\ &= \sum_{j=1}^{m} c_j \left\{ \sum_{n=1}^{\infty} \Omega\left(n, \alpha, \beta, \gamma, \sigma\right) \left[(1 - B) n + (A - B) (p - \sigma)\right] a_{n+p,j} \right\} \\ &\leq \left(\sum_{j=1}^{m} c_j \right) (A - B) (p - \sigma) = (A - B) (p - \sigma) \end{aligned}$$

This proves that the function $h(z)$ belongs to the class $\mathcal{S}_p^{m,l,\lambda}(\alpha, \beta, \gamma, \sigma; A, B)$. □

Theorem 6.1. *Let the functions*

$$f_p(z) = z^p \qquad \text{and} \tag{50}$$

$$f_{n+p}\left(z\right) = z^p - \frac{(A - B) (p - \sigma)}{\left\{(1 - B) n + (A - B) (p - \sigma)\right\} \Omega\left(n, \alpha, \beta, \gamma, \sigma\right)} z^{n+p} \tag{51}$$

for $-1 \leq B < A \leq 1, -1 \leq B < 0, 0 \leq \sigma < p$ *and* $\Omega\left(n, \alpha, \beta, \gamma, \sigma\right)$ *is defined by (4.1). Then* $f(z)$ *is in the class* $\mathcal{S}_p^{m,l,\lambda}(\alpha, \beta, \gamma, \sigma; A, B)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=0}^{\infty} \xi_{n+p} f_{n+p}\left(z\right) \qquad (\xi_{n+p} \geq 0) \tag{52}$$

and

$$\sum_{n=0}^{\infty} \xi_{n+p} = 1 \tag{53}$$

Proof. Assume that

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \xi_{n+p} f_{n+p}(z) \\ &= z^p - \sum_{n=1}^{\infty} \frac{(A-B)(p-\sigma)}{\{(1-B)n + (A-B)(p-\sigma)\}\,\Omega(n,\alpha,\beta,\gamma,\sigma)} \xi_{n+p} z^{n+p} \end{aligned} \tag{54}$$

Then, we get

$$\begin{aligned} &\sum_{n=1}^{\infty} \Omega(n,\alpha,\beta,\gamma,\sigma)\{(1-B)n + (A-B)(p-\sigma)\} \\ &\qquad \times \frac{(A-B)(p-\sigma)}{\{(1-B)n + (A-B)(p-\sigma)\}\,\Omega(n,\alpha,\beta,\gamma,\sigma)} \xi_{n+p} \\ &\le (A-B)(p-\sigma). \end{aligned}$$

By virtue of theorem (4.1). This proves that the function $f(z)$ belongs to the class $\mathcal{S}_p^{m,l,\lambda}(\alpha,\beta,\gamma,\sigma;A,B)$.

Conversely, assume that $f(z)$ belongs to the class $\mathcal{S}_p^{m,l,\lambda}(\alpha,\beta,\gamma,\sigma;A,B)$. Again, by virtue of theorem (4.1). We have

$$a_{n+p} \le \frac{(A-B)(p-\sigma)}{\{(1-B)n + (A-B)(p-\sigma)\}\,\Omega(n,\alpha,\beta,\gamma,\sigma)}$$

Next, setting

$$\xi_{n+p} \le \frac{\{(1-B)n + (A-B)(p-\sigma)\}\,\Omega(n,\alpha,\beta,\gamma,\sigma)}{(A-B)(p-\sigma)} a_{n+p} \qquad \textit{and}$$

$$\xi_p = 1 - \sum_{n=1}^{\infty} \xi_{n+p},$$

We have the representation (6.8). This complete the proof of the theorem. □

7. Radii of Close-to-Convexity, Starlikeness and Convexity

Theorem 7.1. *Let $f \in \mathcal{S}_p^{m,l,\lambda}(\alpha,\beta,\gamma,\sigma;A,B)$. Then f is p-valently close-to-convex of order $\eta\,(0 \le \eta < p)$ in $|z| < R_1$, where*

$$R_1 = \inf_n \left\{ \left[\frac{\{(1-B)n + (A-B)(p-\sigma)\}\,\Omega(n,\alpha,\beta,\gamma,\sigma)}{(A-B)(p-\sigma)} \left(\frac{p-\eta}{n+p} \right) \right]^{\frac{1}{n}} \right\} \tag{55}$$

and $\Omega(n,\alpha,\beta,\gamma,\sigma)$ is defined by (37).

Theorem 7.2. *Let $f \in \mathcal{S}_p^{m,l,\lambda}(\alpha,\beta,\gamma,\sigma;A,B)$. Then f is p-valently starlike of order $\eta\,(0 \le \eta < p)$ in $|z| < R_2$, where*

$$R_2 = \inf_n \left\{ \left[\frac{\{(1-B)n + (A-B)(p-\sigma)\}\,\Omega(n,\alpha,\beta,\gamma,\sigma)}{(A-B)(p-\sigma)} \left(\frac{p-\eta}{n+p-\eta} \right) \right]^{\frac{1}{n}} \right\} \tag{56}$$

and $\Omega(n,\alpha,\beta,\gamma,\sigma)$ is defined by (37).

Theorem 7.3. *Let $f \in \mathcal{S}_p^{m,l,\lambda}(\alpha,\beta,\gamma,\sigma;A,B)$. Then f is p-valently convex of order*

$\eta\,(0 \le \eta < p)$ in $|z| < R_3$, where

$$R_3 = \inf_n \left\{ \left[\frac{\{(1-B)n + (A-B)(p-\sigma)\}\,\Omega(n,\alpha,\beta,\gamma,\sigma)}{(A-B)(p-\sigma)} \left(\frac{p(p-\eta)}{(n+p)(n+p-\eta)} \right) \right]^{\frac{1}{n}} \right\} \tag{57}$$

and $\Omega(n,\alpha,\beta,\gamma,\sigma)$ *is defined by (37).*

In order to establish the required results in Theorems 7.1, 7.2 and 7.3, it is sufficient to show that

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\le p-\eta \qquad for \qquad |z|<R_1,$$
$$\left|\frac{zf'(z)}{f(z)}-p\right|\le p-\eta \qquad for \qquad |z|<R_2, \quad and$$
$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right)-p\right|\le p-\eta \qquad for \qquad |z|<R_3,$$

respectively.

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