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The Global Maximal Domination Number of a Graph

Research Article

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Abstract: A maximal dominating set D of a graph G is a global maximal dominating set if D is also a maximal dominating set of $\overline{G}$. The global maximal domination number $\gamma_{gm}(G)$ of G is the minimum cardinality of a global maximal dominating set. In this paper, bounds for $\gamma_{gm}(G)$ and exact values of $\gamma_{gm}(G)$ for some standard graphs are obtained. We characterize maximal dominating sets of G which are global maximal dominating sets. Also Nordhaus-Gaddum type results are obtained.

MSC: 05C.

Keywords: Dominating set, maximal dominating set, global maximal dominating set, global maximal domination number.

1. Introduction

In this paper, a graph is a finite, undirected graph without loops or multiple edges. Any undefined term in this paper may be found in [1, 2]. Let $G = (V, E)$ be a graph with $|V| = p$ vertices and $|E| = q$ edges. A set D of vertices in G is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. Recently many new domination parameters are given in the books by Kulli [2–4]. A dominating set D of a graph G is a maximal dominating set if $V - D$ is not a dominating set of G. The maximal domination number $\gamma_m(G)$ of G is the minimum cardinality of a maximal dominating set of G. This concept was introduced by Kulli and Janakiram in [5] and was studied, for example, in [6]. (the term "nil domination number"was used instead of maximal domination number in [7]). Different types of maximal domination parameters have been studied, for example, in [8–12]. A dominating set D of G is a global dominating set if D is also dominating set of $\overline{G}$. The global domination number $\gamma_g(G)$ of G is the minimum cardinality of a global dominating set. This concept was studied, for example, in [13, 14]. Let $\lceil x \rceil$ denote the least integer greater than or equal to x. Let $\overline{G}$ be the complement of a graph G. In this paper, we introduce the concept of the global maximal domination number in graphs and establish some results on this new parameter.

2. Global Maximal Domination in Graphs

Definition 2.1. *A maximal dominating set D of a graph G is a global maximal dominating set, if D is also a maximal dominating set of $\overline{G}$. The global maximal domination number $\gamma_{gm}(G)$ of G is the minimum cardinality of a global maximal dominating set of G.*

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A γ_{gm}-set is a minimum global maximal dominating set.

Theorem 2.2. *Let G be a graph such that neither G nor $\overline{G}$ have an isolated vertex. Then*

a) $\gamma_{gm}(G) = \gamma_{gm}(\overline{G})$;

b) $\gamma_m(G) \leq \gamma_{gm}(G)$;

c) $\gamma|g(G) \leq \gamma_{gm}(G)$;

d) $\frac{(\gamma_m(G)+\gamma_m(\overline{G}))}{2} \leq \gamma_{gm}(G) \leq \gamma_m(G) + \gamma_m(\overline{G})$.

Exact values of $\gamma_{gm}(G)$ for some standard graphs are given below.

Proposition 2.3. *For a complete graph K_p with $p \geq 2$ vertices $\gamma_{gm}(Kp) = p$.*

Proposition 2.4. *For a cycle C_p with p vertices,*

$$\begin{aligned} \gamma_{gm}(Cp) &= p-1, \text{ if } p \geq 6, \\ &= 3, \text{ if } p = 3, 4 \text{ or } 5. \end{aligned}$$

Proposition 2.5. *For a path P_p with p vertices,*

$$\begin{aligned} \gamma_{gm}(P_p) &= \left\lceil \frac{p}{3} \right\rceil + 1, \text{ if } p \geq 4, \\ &= 2, \text{ if } p = 2 \text{ or } 3. \end{aligned}$$

Proposition 2.6. *For a complete bipartite graph $K_{r,s}$ $1 \leq r \leq s$, $\gamma_{gm}(K_{r,s}) = r + 1$.*

Proposition 2.7. *For a wheel W_p with $p \geq 4$ vertices, $\gamma_{gm}(W_p) = 4$.*

We characterize maximal dominating sets of G which are global maximal dominating sets.

Theorem 2.8. *A maximal dominating set D of G is a global maximal dominating set if and only if at least one of the following conditions is satisfied:*

a) for each vertex $v \in V - D$, there exists a vertex $u \in D$ such that u is not adjacent to v.

b) there exists a vertex $w \in D$ such that w is adjacent to all vertices in $V - D$.

Proof. Suppose D is a global maximal dominating set of G. On the contrary, there exists a vertex $v \in D$ such that v does not satisfy any of the given conditions. Then by (a) and (b), it follows that $D - \{v\}$ is a dominating set of G, a contradiction. Hence every vertex v of D satisfies at least one of the given conditions.
Converse is obvious. □

We characterize graphs G which have global maximal domination number equal to the order p of G.

Theorem 2.9. *Let G be a graph. Then $\gamma_{gm}(G) = p$ if and only if $G = K_p$ or $\overline{K_p}$.*

Proof. Suppose $\gamma_{gm}(G) = p$. We now prove that $G = K_p$ or $\overline{K_p}$. On the contrary, assume $G \neq K_p$ or $\overline{K_p}$. Let D be a γ_{gm}- set of G. Then there exist two nonadjacent vertices $u, v \in D$. It implies that $D - \{u\}$ is a global maximal dominating set of G, which is a contradiction, by Theorem 2.8 Hence $G = K_p$ or $\overline{K_p}$.
Converse is obvious. □

Theorem 2.10. *Let D be a minimum dominating set of G. If there exists a vertex v in $V - D$ which is adjacent only to the vertices of D. Then $\gamma_{gm}(G) \leq \gamma(G) + 1$.*

Proof. Suppose D is a minimum dominating set of G. If there exists a vertex v in $V - D$ which is adjacent only to the vertices of D, then $D \cup \{v\}$ is a global maximal dominating set of G. Then

$$\begin{aligned}\gamma_{gm}(G) &\leq |D \cup \{v\}| \\ &\leq |D| + 1 \\ &\leq \gamma(G) + 1. \qquad (1)\end{aligned}$$

□

Corollary 2.11. *If a graph G has a pendant vertex, then $\gamma_{gm}(G) \leq (G) + 1$. In particular, this inequality holds for a tree.*

Corollary 2.12. *If D is a minimum dominating set of G and if $V - D$ is independent, then $\gamma_{gm}(G) \leq \gamma(G) + 1$.*

Corollary 2.13. *Let $G = (V_1, V_2, E)$ be a bipartite graph without isolated vertices, where $|V_1| = r$, $|V_2| = s$ and $r \leq s$. Then*

$$\gamma_{gm}(G) \leq r + 1. \qquad (2)$$

Proof. (2) follows from (1), since $\gamma(G) \leq r$.

In a graph G, a vertex and an edge incident with it are said to cover each other. A set of vertices that covers all the edges of G is a cover of G. The vertex covering number $\alpha_0(G)$ of G is the minimum number of vertices in a vertex cover. A set S of vertices in G is independent if no two vertices in S are adjacent. The independence number $\beta_0(G)$ of G is the maximum cardinality of an independent set of vertices. The clique number $\omega(G)$ of G is the maximum order among the complete subgraphs of G. □

We establish another upper bound for $\gamma_{gm}(G)$.

Theorem 2.14. *For any graph G without isolated vertices,*

$$\gamma_{gm}(G) \leq p - \beta_0(G) + 1. \qquad (3)$$

Proof. Let S be an independent set of G with $\beta_0(G)$ vertices. Then $|S| = \beta_0(G)$. Since G has no isolated vertices, it implies that $V - S$ is a dominating set of G. Clearly for any vertex $v \in S$, $(V - S) \cup \{v\}$ is a global maximal dominating set of G. Therefore

$$\begin{aligned}\gamma_{gm}(G) &\leq |(V - S) \cup \{v\}| \\ &\leq p - \beta_0(G) + 1.\end{aligned}$$

□

Corollary 2.15. *For any graph G without isolated vertices,*

$$\gamma_{gm}(G) \leq \alpha_0(G) + 1. \qquad (4)$$

Proof. It is known that for a graph G, $\alpha_0(G) + \beta_0(G) = p$. By Theorem 2.14, $\gamma_{gm}(G) \leq p - \beta_0(G) + 1$. Therefore (4) holds. □

Nordhaus-Gaddum type results were obtained for several parameters, for example, in [15–28]. We now establish Nordhaus-Gaddum type results.

Theorem 2.16. *Let G and $\overline{G}$ have no isolated vertices. Then*

$$\gamma_{gm}(G) + \gamma_{gm}\left(\overline{G}\right) \leq p + \alpha_o(G) - \omega(G) + 2.$$

Proof. By Corollary 2.15, we have

$$\begin{aligned} \gamma_{gm}(G) &\leq \alpha_o(G) + 1 \\ \gamma_{gm}\left(\overline{G}\right) &\leq \alpha_o\left(\overline{G}\right) + 1 \\ &\leq p - \beta_o\left(\overline{G}\right) + 1 \\ &\leq p - \omega(G) + 1. \end{aligned}$$

Hence $\gamma_{gm}(G) + \gamma_{gm}\left(\overline{G}\right) \leq p + \alpha_o(G) - \omega(G) + 2$. □

Theorem 2.17. *Let G and $\overline{G}$ be connected graphs. Then*

$$\gamma_{gm}(G) + \gamma_{gm}\left(\overline{G}\right) \leq 2(p-1). \tag{5}$$

Furthermore, the equality holds if and only if $G = P_4$.

Proof. By Theorem 2.14, $\gamma_{gm}(G) \leq p\beta_0(G) + 1$. Since both G and $\overline{G}$ are connected, it implies that $\Delta(G)$, $\Delta\left(\overline{G}\right) \leq p - 1$. Thus $\beta_0(G)$, $\beta_0\left(\overline{G}\right) \geq 2$. Hence $\gamma_{gm}(G) \leq p1$. Similarly $\gamma_{gm}\left(\overline{G}\right) \leq p - 1$. Hence $\gamma_{gm}(G) + \gamma_{gm}\left(\overline{G}\right) \leq 2(p-1)$.
We now prove the second part. Suppose the equality holds. Then $\gamma_{gm}(G) + \gamma_{gm}\left(\overline{G}\right) = 2(p-1)$. This shows that both G and $\overline{G}$ are trees. Suppose G is a tree with $p \geq 5$ vertices. Then $\overline{G}$ contains a cycle, which is a contradiction. Suppose p = 2 or 3. Then both G and $\overline{G}$ are not connected. Thus we conclude that $G = P_4$. □

Similarly we prove that following

Theorem 2.18. *Let G and $\overline{G}$ be connected graphs. Then $\gamma_{gm}(G) . \gamma_{gm}\left(\overline{G}\right) \leq (p-1)^2$. Furthermore, the equality holds if and only if $G = P_4$.*

References

[1] V.R.Kulli, *College Graph Theory*, Vishwa International Publications, Gulbarga, India (2012).

[2] V.R.Kulli, *Theory of Domination in Graphs*, Vishwa International Publications, Gulbarga, India (2010).

[3] V.R.Kulli, *Advances in Domination Theory I*, Vishwa International Publications, Gulbarga, India (2012).

[4] V.R.Kulli, *Advances in Domination Theory II*, Vishwa International Publications, Gulbarga, India (2013).

[5] V.R.Kulli and B.Janakiram, *The maximal domination number of a graph*, Graph Theory Notes of New York, New York Academy of Sciences, 33(1997), 11-13.

[6] V.R.Kulli and M.B.Kattimani, *A note on the maximal domination number of a graph*, Graph Theory Notes of New York, New York Academy of Sciences, 39(2000), 35-36.

[7] T.T.Chelvam and S.R.Chellathurai, *Complementary nil domination number of a graph*, Tamkang J. of Mathematics, 40(2)(2009), 165-172.

[8] V.R.Kulli, *Maximal total edge domination in graphs*. In Advances in Domination Theory I, V.R.Kulli, ed., Vishwa International Publications, Gulbarga, India (2012), 25-33.

[9] V.R.Kulli, *Maximal domination and maximal total domination in digraphs*, Journal of Computer and Mathematical Sciences, 5(1)(2014), 59-63.

[10] V.R.Kulli and M.B.Kattimani, *Connected maximal domination in graphs*, In Advances in Domination Theory I, V.R.Kulli, ed., Vishwa International Publications, Gulbarga, India (2012), 79-85.

[11] N.D.Soner, B.Chaluvaraju and B.Janakiram, *The maximal neighbourhood number of a graph*, Far East J. Appl. Math., 5(3)(2001), 301-307.

[12] N.D.Soner and Puttaswamy, *Maximal total domination in graphs*, Aligarh Bull. Math., 22(2003), 29-33.

[13] V.R.Kulli and B.Janakiram, *The total global domination number of a graph*, Indian J. Pure Appl. Math., 27(1996), 537-542.

[14] V.R.Kulli and B.Janakiram, *Global nonsplit domination in graphs*, Nat. Acad. Sci. Lett., 28(2005), 389-392.

[15] V.R.Kulli, *Set independence number of a graph*, Journal of Computer and Mathematical Sciences, 4(5)(2013), 322-324.

[16] V.R.Kulli, *The block point tree of a graph*, Indian J. Pure Appl. Math., 7(1976), 620-624.

[17] V.R.Kulli, *On nonbondage numbers of a graph*, International J. of Advanced Research in Computer Science and Technology, 1(1)(2013), 42-45.

[18] V.R.Kulli and Janakiram, *The cobondage number of a graph*, Discussiones Mathmaticae, 16(1996), 111-117.

[19] V.R.Kulli and B.Janakiram, *The split domination number of a graph*, Graph Theory Notes of New York, New York Academy of Sciences, 32(1997), 16-19.

[20] V.R.Kulli and B.Janakiram, *The nonsplit domination number of a graph*, Indian J. Pure Appl. Math., 31(2000), 545-550.

[21] V.R.Kulli and B.Janakiram, *The block nonsplit domination number of a graph*, Inter. J. Management Systems, 20(2004), 219-228.

[22] V.R.Kulli and B.Janakiram, *The strong nonsplit domination number of a graph*, Inter. J. Management Systems, 19(2003), 145-156.

[23] V.R.Kulli and B.Janakiram and R.R.Iyer, *Regular number of a graph*, J. Discrete Mathematical Sciences and Cryptography, 4(1)(2001), 57-64.

[24] V.R.Kulli and M.B.Kattimani, *The inverse neighbourhood number of a graph*, South East Asain J. Math. and Math. Sci., 6(3)(2008), 23-28.

[25] V.R.Kulli and S.C.Sigarkanti, *Further results on the neighbourhood number of a graph*, Indian J. Pure Appl. Math., 23(8)(1992), 575-577.

[26] V.R.Kulli and S.C.Sigarkanti, *On the tree and star numbers of a graph*, Journal of Computer and Mathematical Sciences, 6(1)(2015), 25-32.

[27] V.R.Kulli, S.C.Sigarkanti and N.D.Soner, *Entire domination in graphs*, In V.R. Kulli, ed., *Advances in Graph Theory*, Vishwa International Publications, Gulbarga, India, (1991), 237-243.

[28] V.R.Kulli and N.D.Soner, *Complementary edge domination in graphs*, Indian J. Pure Appl. Math., 28(1997), 917-920.