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Employing weak $\psi - \varphi$ Contraction in Common Coupled Fixed Point Results for Hybrid Pairs of Mappings Satisfying (EA) Property

Research Article

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Abstract: We establish some common coupled fixed point theorems for two hybrid pairs of mappings under weak $\psi - \varphi$ contraction on a non complete metric space, which is not partially ordered. It is to be noted that to find coupled coincidence point, we do not employ the condition of continuity of any mapping involved therein. An example is also given to validate our results.

MSC: 47H10, 54H25.

Keywords: Coupled fixed point, coupled coincidence point, weak $\psi - \varphi$ contraction, $w-$compatibility, $F-$weakly commutativity.

1. Introduction and Preliminaries

Let (X, d) be a metric space. We denote by 2^X the class of all nonempty subsets of X, by $CL(X)$ the class of all nonempty closed subsets of X, by $CB(X)$ the class of all nonempty closed bounded subsets of X. A functional $H : CL(X) \times CL(X) \to \mathbb{R}_+ \cup \{+\infty\}$ is said to be the Pompeiu-Hausdorff generalized metric induced by d is given by

$$H(A,\ B) = \begin{cases} \max\left\{\sup_{a\in A} D(a,\ B),\ \ \sup_{b\in B} D(b,\ A)\right\}, & \text{if maximum exists,} \\ +\infty, & \text{otherwise,} \end{cases}$$

for all $A,\ B \in CB(X)$, where $D(x,\ A) = \inf_{a\in A} d(x,\ a)$ denote the distance from x to $A \subset X$. For simplicity, if $x \in X$, we shall denote $g(x)$ by gx.

The existence of fixed points for multivalued contractions and non-expansive mappings using the Hausdorff metric studied by many authors under different contractive conditions. The theory of multivalued mappings has found application in control theory, convex optimization, differential inclusions and economics.

The idea of the coupled fixed point was initiated by Guo and Lakshmikantham [22] in 1987, which was well followed by Bhaskar and Lakshmikantham [5] where the authors introduced the notion of mixed monotone property for a linear contraction (mapping) $F : X^2 \to X$ (wherein X is an ordered metric space) and utilized the same to study the existence

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and uniqueness of solution for periodic boundary value problems. In 2009, Lakshmikantham and Ciric [26] generalized these results for nonlinear contraction mappings by introducing the notions of coupled coincidence point and mixed g-monotone property. Very recently, Samet et al. [36] have shown that the coupled fixed results can be more easily obtained using well-known fixed point theorems on ordered metric spaces. In recent years, the existence results on coupled fixed point were generalized and improved by various authors [4, 6, 10, 11, 19, 20, 23, 30, 31, 35, 39].

Abbas et al. [2] obtained coupled fixed/coincidence point theorems involving hybrid pair of mappings satisfying generalized contractive conditions in complete metric spaces by extending the coupled fixed point theory to multivalued mappings. For more details on coupled fixed point theory for hybrid pair of mappings, one can refer [2, 12, 14–18, 28, 38] and the references involved therein.

In [2], Abbas et al. introduced the following for multivalued mappings:

Definition 1.1. *Let X be a nonempty set, $F : X \times X \to 2^X$ and g be a self-mapping on X. An element $(x, y) \in X \times X$ is called*

(1) a coupled fixed point of F if $x \in F(x, y)$ and $y \in F(y, x)$.

(2) a coupled coincidence point of hybrid pair $\{F, g\}$ if $gx \in F(x, y)$ and $gy \in F(y, x)$.

(3) a common coupled fixed point of hybrid pair $\{F, g\}$ if $x = gx \in F(x, y)$ and $y = gy \in F(y, x)$.

We denote the set of coupled coincidence points of mappings F and g by $C(F, g)$. Note that if $(x, y) \in C(F, g)$, then (y, x) is also in $C(F, g)$.

Definition 1.2. *Let $F : X \times X \to 2^X$ be a multivalued mapping and g be a self-mapping on X. The hybrid pair $\{F, g\}$ is called $w-$compatible if $gF(x, y) \subseteq F(gx, gy)$ whenever $(x, y) \in C(F, g)$.*

Definition 1.3. *Let $F : X \times X \to 2^X$ be a multivalued mapping and g be a self-mapping on X. The mapping g is called $F-$weakly commuting at some point $(x, y) \in X \times X$ if $g^2x \in F(gx, gy)$ and $g^2y \in F(gy, gx)$.*

Aamri and ElMoutawakil [1] defined (EA) property for self-mappings which contained the class of noncompatible mappings. Kamran [25] extended the property (EA) for hybrid pair $g : X \to X$ and $T : X \to 2^X$. Liu et al. [27] introduced common (EA) property for a hybrid pair of single and multivalued mappings and gave some new common fixed point theorems under hybrid contractive conditions. In [24], Jungck and Rhoades introduced the notion of occasionally weakly compatibility for self mappings. Abbas and Rhoades [3] extended the concept of occasionally weakly compatible mappings for hybrid pair $g : X \to X$ and $T : X \to 2^X$. Deshpande and Handa [13] introduced the concept of (EA) property and occasionally $w-$compatibility for hybrid pair $g : X \to X$ and $F : X \times X \to 2^X$. They also introduced the concept of common (EA) property for hybrid pairs $f, g : X \to X$ and $F, G : X \times X \to 2^X$.

Definition 1.4 ([13])**.** *Mappings $g : X \to X$ and $F : X \times X \to CB(X)$ are said to satisfy the (EA) property if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X, some s, t in X and A, B in $CB(X)$ such that*

$$\begin{aligned} \lim_{n\to\infty} gx_n &= s \in A = \lim_{n\to\infty} F(x_n,\ y_n), \\ \lim_{n\to\infty} gy_n &= t \in B = \lim_{n\to\infty} F(y_n,\ x_n). \end{aligned}$$

Definition 1.5 ([13])**.** *Let $f, g : X \to X$ and $F, G : X \times X \to CB(X)$. The pairs $\{F, f\}$ and $\{G, g\}$ are said to satisfy the common (EA) property if there exist sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ in X, some u, v in X and A, B, C, D in $CB(X)$*

such that

$$\begin{aligned}\lim_{n\to\infty} F(x_n,\, y_n) &= A,\ \ \lim_{n\to\infty} G(u_n,\, v_n) = B,\\ \text{then } \lim_{n\to\infty} fx_n &= \lim_{n\to\infty} gu_n = u \in A\cap B,\\ \text{and } \lim_{n\to\infty} F(y_n,\, x_n) &= C,\ \ \lim_{n\to\infty} G(v_n,\, u_n) = D,\\ \text{then } \lim_{n\to\infty} fy_n &= \lim_{n\to\infty} gv_n = v \in C\cap D.\end{aligned}$$

Definition 1.6 ([13]). *Mappings $F : X\times X \to 2^X$ and $g : X \to X$ are said to be occasionally $w-$compatible if and only if there exists some point $(x,\, y) \in X\times X$ such that $gx \in F(x,\, y)$, $gy \in F(y,\, x)$ and $gF(x,\, y) \subseteq F(gx,\, gy)$.*

There exists considerable literature about fixed point properties for two hybrid pairs of mappings, including [3, 7–9, 13, 27, 29, 32, 33, 37, 40]. In [21], Gordji et al. established some fixed point theorems for $(\psi,\, \varphi)$-weak contractive mappings in a complete metric space endowed with a partial order. Ciric et al. [6] proved coupled fixed point theorems for mixed monotone mappings satisfying a generalized Mizoguchi-Takahashi's condition in the setting of ordered metric spaces. In [19], Ding et al. proved coupled coincidence and common coupled fixed point theorems for generalized nonlinear contraction on partially ordered metric spaces which generalize the results of Lakshmikantham and Ciric [26].
The main objective of this article is to establish some common coupled fixed point theorems for two hybrid pairs of mappings under weak $\psi-\varphi$ contraction satisfying some weaker conditions on a noncomplete metric space, which is not partially ordered. It is to be noted that to find coupled coincidence point, we do not employ the condition of continuity of any mapping involved therein. We improve, extend and generalize the results of Bhaskar and Lakshmikantham [5], Ciric et al. [6], Ding et al. [19], Gordji et al. [21] and Lakshmikantham and Ciric [26]. The effectiveness of our generalization is demonstrated with the help of an example.

2. Main Results

Let Ψ denote the set of all functions $\psi : [0,\, +\infty) \to [0,\, +\infty)$ satisfying
(i_ψ) ψ is continuous and non-decreasing,
(ii_ψ) $\psi(t) = 0 \Leftrightarrow t = 0$,
(iii_ψ) $\limsup_{s\to 0+} \frac{s}{\psi(s)} < \infty$.
Let Φ denote the set of all functions $\varphi : [0,\, +\infty) \to [0,\, +\infty)$ satisfying
(i_φ) φ is lower semi-continuous,
(ii_φ) $\varphi(t) = 0 \Leftrightarrow t = 0$,
(iii_φ) for any sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = 0$, there exists $k \in (0,\, 1)$ and $n_0 \in \mathbb{N}$, such that $\varphi(t_n) \geq kt_n$ for each $n \geq n_0$.
Let Θ denote the set of all functions $\theta : [0,\, +\infty) \to [0,\, +\infty)$ satisfying
(i_θ) θ is continuous,
(ii_θ) $\theta(t) = 0 \Leftrightarrow t = 0$.

For example, if $\psi(t) = \ln(t + 1)$, $\varphi(t) = t - \ln\left(\frac{t}{2} + 1\right)$ and $\theta(t) = \frac{t}{4}$. Obviously, then $\psi \in \Psi$, $\varphi \in \Phi$ and $\theta \in \Theta$, because ψ is continuous, non-decreasing ($\psi'(t) = \frac{1}{t+1} > 0$), positive in $(0,\, +\infty)$, $\psi(0) = 0$ and $\limsup_{s\to 0+} \frac{s}{\psi(s)} = 1 < \infty$.

Also, φ is continuous, positive in $(0,\, +\infty)$ and $\varphi(0) = 0$, now let (t_n) be a sequence such that $t_n \to 0$ as $n \to \infty$, since $\lim_{n\to\infty} \frac{\varphi(t_n)}{t_n} = \varphi'(0) = \frac{1}{2}$, then for any $\varepsilon > 0$, there exists n_0 such that $\left|\frac{\varphi(t_n)}{t_n} - \frac{1}{2}\right| < \varepsilon$ for all $n \geq n_0$, hence $\varphi(t_n) \geq \frac{1}{2}t_n$

for all $n \geq n_0$. Furthermore θ is continuous and $\theta(0)=0$. For convenience, we denote

$$M(x,\ y,\ u,\ v)=\max\left\{\begin{array}{c} d(fx,\ gu),\ D(fx,\ F(x,\ y)),\ D(gu,\ G(u,\ v)),\\ \frac{D(fx,\ G(u,\ v))+D(gu,\ F(x,\ y))}{2},\\ d(fy,\ gv),\ D(fy,\ F(y,\ x)),\ D(gv,\ G(v,\ u)),\\ \frac{D(fy,\ G(v,\ u))+D(gv,\ F(y,\ x))}{2} \end{array}\right\},$$

and

$$N(x,\ y,\ u,\ v)=\min\left\{\begin{array}{c} D(fx,\ F(x,\ y)),\ D(gu,\ G(u,\ v)),\\ D(fy,\ F(y,\ x)),\ D(gv,\ G(v,\ u)),\\ D(fx,\ G(u,\ v)),\ D(gu,\ F(x,\ y)),\\ D(fy,\ G(v,\ u)),\ D(gv,\ F(y,\ x)) \end{array}\right\}.$$

Theorem 2.1. *Let $(X,\ d)$ be a metric space. Assume $F,\ G: X\times X \to CB(X)$ and $f,\ g: X\to X$ be mappings satisfying*

(i) $\{F,\ f\}$ and $\{G,\ g\}$ satisfy the common (EA) property.

(ii) For all $x,\ y,\ u,\ v \in X$, there exist some $\psi\in\Psi$, $\varphi\in\Phi$ and $\theta\in\Theta$ such that

$$\psi\left(H(F(x,\ y),\ G(u,\ v))\right) \leq \psi\left(M(x,\ y,\ u,\ v)\right)-\varphi\left(\psi\left(M(x,\ y,\ u,\ v)\right)\right)+\theta\left(N(x,\ y,\ u,\ v)\right).$$

(iii) $f(X)$ and $g(X)$ are closed subsets of X. Then

(a) F and f have a coupled coincidence point,

(b) G and g have a coupled coincidence point,

(c) F and f have a common coupled fixed point, if f is $F-$weakly commuting at $(x,\ y)$ and $f^2x=fx$ and $f^2y=fy$ for $(x,\ y)\in C(F,\ f)$,

(d) G and g have a common coupled fixed point, if g is $G-$weakly commuting at $(\widetilde{x},\ \widetilde{y})$ and $g^2\widetilde{x}=g\widetilde{x}$ and $g^2\widetilde{y}=g\widetilde{y}$ for $(\widetilde{x},\ \widetilde{y})\in C(G,\ g)$,

(e) F, G, f and g have common coupled fixed point provided that both (c) and (d) are true.

Proof. Since $\{F,\ f\}$ and $\{G,\ g\}$ satisfy the common (EA) property, there exist sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ in X, some u, v in X and A, B, C, D in $CB(X)$ such that

$$\begin{aligned} \lim_{n\to\infty} F(x_n,\ y_n) &= A,\ \lim_{n\to\infty} G(u_n,\ v_n)=B,\\ \lim_{n\to\infty} fx_n &= \lim_{n\to\infty} gu_n = u\in A\cap B,\\ \lim_{n\to\infty} F(y_n,\ x_n) &= C,\ \lim_{n\to\infty} G(v_n,\ u_n)=D,\\ \lim_{n\to\infty} fy_n &= \lim_{n\to\infty} gv_n = v\in C\cap D. \end{aligned} \tag{1}$$

Since $f(X)$ and $g(X)$ are closed subsets of X, then there exist $x,\ y,\ \widetilde{x},\ \widetilde{y}\in X$, we have

$$u=fx=g\widetilde{x} \text{ and } v=fy=g\widetilde{y}. \tag{2}$$

Now, by using condition (ii) and (i_ψ), we get

$$\psi\left(H(F(x,\ y),\ G(u_n,\ v_n))\right)\leq \psi\left(M(x,\ y,\ u_n,\ v_n)\right)-\varphi\left(\psi\left(M(x,\ y,\ u_n, v_n)\right)\right)+\theta\left(N(x,\ y,\ u_n, v_n)\right),$$

where

$$M(x,\ y,\ u_n,\ v_n) = \max \left\{ \begin{array}{c} d(fx,\ gu_n),\ D(fx,\ F(x,\ y)),\ D(gu_n,\ G(u_n,\ v_n)), \\ \frac{D(fx,\ G(u_n,\ v_n))+D(gu_n,\ F(x,\ y))}{2}, \\ d(fy,\ gv_n),\ D(fy,\ F(y,\ x)),\ D(gv_n,\ G(v_n,\ u_n)), \\ \frac{D(fy,\ G(v_n,\ u_n))+D(gv_n,\ F(y,\ x))}{2} \end{array} \right\},$$

and

$$N(x,\ y,\ u_n,\ v_n) = \min \left\{ \begin{array}{c} D(fx,\ F(x,\ y)),\ D(gu_n,\ G(u_n,\ v_n)), \\ D(fx,\ G(u_n,\ v_n)),\ D(gu_n,\ F(x,\ y)), \\ D(fy,\ F(y,\ x)),\ D(gv_n,\ G(v_n,\ u_n)), \\ D(fy,\ G(v_n,\ u_n)),\ D(gv_n,\ F(y,\ x)) \end{array} \right\}.$$

Letting $n \to \infty$ in the above inequality, by using (i_ψ), (i_φ), (i_θ), (ii_θ), (1), (2), $fx \in A$ and $fy \in C$, we get

$$\psi\left(D(F(x,\ y),\ fx)\right) \leq \psi\left(\max\left\{D(fx,\ F(x,\ y)),\ D(fy,\ F(y,\ x))\right\}\right) - \varphi\left(\psi\left(\max\left\{D(fx,\ F(x,\ y)),\ D(fy,\ F(y,\ x))\right\}\right)\right).$$

Similarly, we can obtain that

$$\psi\left(D(F(y,\ x),\ fy)\right) \leq \psi\left(\max\left\{D(fx,\ F(x,\ y)),\ D(fy,\ F(y,\ x))\right\}\right) - \varphi\left(\psi\left(\max\left\{D(fx,\ F(x,\ y)),\ D(fy,\ F(y,\ x))\right\}\right)\right).$$

Combining them, we get

$$\begin{aligned} \max\{\psi\left(D(F(x,\ y),\ fx)\right),\ \psi\left(D(F(y,\ x),\ fy)\right)\} &\leq \psi\left(\max\left\{D(fx,\ F(x,\ y)),\ D(fy,\ F(y,\ x))\right\}\right) \\ &\quad - \varphi\left(\psi\left(\max\left\{D(fx,\ F(x,\ y)),\ D(fy,\ F(y,\ x))\right\}\right)\right). \end{aligned}$$

Since ψ is non-decreasing, therefore

$$\begin{aligned} \psi\left(\max\left\{D(fx,\ F(x,\ y)),\ D(fy,\ F(y,\ x))\right\}\right) &\leq \psi\left(\max\left\{D(fx,\ F(x,\ y)),\ D(fy,\ F(y,\ x))\right\}\right) \\ &\quad - \varphi\left(\psi\left(\max\left\{D(fx,\ F(x,\ y)),\ D(fy,\ F(y,\ x))\right\}\right)\right), \end{aligned}$$

which, by (ii_φ) and (ii_ψ), implies that $\max\{D(fx,\ F(x,\ y)),\ D(fy, F(y,x))\} = 0$, it follows that $fx \in F(x,y)$ and $fy \in F(y,x)$, that is, $(x,\ y)$ is a coupled coincidence point of F and f. This proves (a). Again, by using condition (ii) and (i_ψ), we get

$$\psi\left(H(F(x_n,y_n),G(\widetilde{x},\widetilde{y}))\right) \leq \psi\left(M(x_n,y_n,\widetilde{x},\widetilde{y})\right) - \varphi\left(\psi\left(M(x_n,y_n,\widetilde{x},\widetilde{y})\right)\right) + \theta\left(N(x_n,y_n,\widetilde{x},\widetilde{y})\right),$$

where

$$M(x_n,\ y_n,\widetilde{x},\ \widetilde{y}) = \max \left\{ \begin{array}{c} d(fx_n,\ g\widetilde{x}),\ D(fx_n,\ F(x_n,\ y_n)),\ D(g\widetilde{x},\ G(\widetilde{x},\ \widetilde{y})), \\ \frac{D(fx_n,\ G(\widetilde{x},\ \widetilde{y}))+D(g\widetilde{x},\ F(x_n,\ y_n))}{2}, \\ d(fy_n,\ g\widetilde{y}),\ D(fy_n,\ F(y_n,\ x_n)),\ D(g\widetilde{y},\ G(\widetilde{y},\ \widetilde{x})), \\ \frac{D(fy_n,\ G(\widetilde{y},\ \widetilde{x}))+D(g\widetilde{y},\ F(y_n,\ x_n))}{2} \end{array} \right\},$$

and

$$N(x_n,\ y_n,\ \widetilde{x},\ \widetilde{y}) = \min \left\{ \begin{array}{c} D(fx_n,\ F(x_n,\ y_n)),\ D(g\widetilde{x},\ G(\widetilde{x},\ \widetilde{y})), \\ D(fx_n,\ G(\widetilde{x},\ \widetilde{y})),\ D(g\widetilde{x},\ F(x_n,\ y_n)), \\ D(fy_n,\ F(y_n,\ x_n)),\ D(g\widetilde{y},\ G(\widetilde{y},\ \widetilde{x})), \\ D(fy_n,\ G(\widetilde{y},\ \widetilde{x})),\ D(g\widetilde{y},\ F(y_n,\ x_n)) \end{array} \right\}.$$

Letting $n \to \infty$ in the above inequality, by using (i_ψ), (i_φ), (i_θ), (ii_θ), (1), (2), $g\widetilde{x} \in B$ and $g\widetilde{y} \in D$, we get

$$\psi\left(D(g\widetilde{x},\ G(\widetilde{x},\ \widetilde{y}))\right) \leq \psi\left(\max\left\{D(g\widetilde{x},\ G(\widetilde{x},\ \widetilde{y})),\ D(g\widetilde{y},\ G(\widetilde{y},\ \widetilde{x}))\right\}\right) - \varphi\left(\psi\left(\max\left\{D(g\widetilde{x},\ G(\widetilde{x},\ \widetilde{y})),\ D(g\widetilde{y},\ G(\widetilde{y},\ \widetilde{x}))\right\}\right)\right).$$

Similarly, we can obtain that

$$\psi\left(D(g\widetilde{y},\ G(\widetilde{y},\ \widetilde{x}))\right) \leq \psi\left(\max\left\{D(g\widetilde{x},\ G(\widetilde{x},\ \widetilde{y})),\ D(g\widetilde{y},\ G(\widetilde{y},\ \widetilde{x}))\right\}\right) - \varphi\left(\psi\left(\max\left\{D(g\widetilde{x},\ G(\widetilde{x},\ \widetilde{y})),\ D(g\widetilde{y},\ G(\widetilde{y},\ \widetilde{x}))\right\}\right)\right).$$

Combining them, we get

$$\max\{\psi\left(D(g\widetilde{x},\ G(\widetilde{x},\ \widetilde{y}))\right),\ \psi\left(D(g\widetilde{y},\ G(\widetilde{y},\ \widetilde{x}))\right)\} \leq \psi\left(\max\left\{D(g\widetilde{x},\ G(\widetilde{x},\ \widetilde{y})),\ D(g\widetilde{y},\ G(\widetilde{y},\ \widetilde{x}))\right\}\right) - \varphi\left(\psi\left(\max\left\{D(g\widetilde{x},\ G(\widetilde{x},\ \widetilde{y})),\ D(g\widetilde{y},\ G(\widetilde{y},\ \widetilde{x}))\right\}\right)\right).$$

Since ψ is non-decreasing, therefore

$$\psi\left(\max\left\{D(g\widetilde{x},\ G(\widetilde{x},\ \widetilde{y})),\ D(g\widetilde{y},\ G(\widetilde{y},\ \widetilde{x}))\right\}\right) \leq \psi\left(\max\left\{D(g\widetilde{x},\ G(\widetilde{x},\ \widetilde{y})),\ D(g\widetilde{y},\ G(\widetilde{y},\ \widetilde{x}))\right\}\right) - \varphi\left(\psi\left(\max\left\{D(g\widetilde{x},\ G(\widetilde{x},\ \widetilde{y})),\ D(g\widetilde{y},\ G(\widetilde{y},\ \widetilde{x}))\right\}\right)\right),$$

which, by (ii_φ) and (ii_ψ), implies that $\max\left\{D(g\widetilde{x},\ G(\widetilde{x},\ \widetilde{y})),\ D(g\widetilde{y},\ G(\widetilde{y},\ \widetilde{x}))\right\} = 0$, it follows that $g\widetilde{x} \in G(\widetilde{x},\ \widetilde{y})$ and $g\widetilde{y} \in G(\widetilde{y},\ \widetilde{x})$, that is, $(\widetilde{x},\ \widetilde{y})$ is a coupled coincidence point of G and g. This proves (b).
Furthermore, from condition (c), we have f is $F-$weakly commuting at $(x,\ y)$, that is, $f^2x \in F(fx,\ fy)$ and $f^2y \in F(fy,\ fx)$, $f^2x = fx$ and $f^2y = fy$. Thus $fx = f^2x \in F(fx,\ fy)$ and $fy = f^2y \in F(fy,\ fx)$, that is, $u = fu \in F(u,\ v)$ and $v = fv \in F(v,\ u)$. This proves (c). A similar argument proves (d). Then (e) holds immediately. □

If we put $\theta(t) = 0$ in the Theorem 2.1, we get the following result:

Corollary 2.2. *Let $(X,\ d)$ be a metric space. Assume $F,\ G : X \times X \to CB(X)$ and $f,\ g : X \to X$ be mappings satisfying (i) of Theorem 2.1 and*

(i) for all $x,\ y,\ u,\ v \in X$, there exist some $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\psi\left(H(F(x,\ y),\ G(u,\ v))\right) \leq \psi\left(M(x,\ y,\ u,\ v)\right) - \varphi\left(\psi\left(M(x,\ y,\ u,\ v)\right)\right).$$

If (iii) of Theorem 2.1 holds. Then

(a) F and f have a coupled coincidence point,

(b) G and g have a coupled coincidence point,

(c) F and f have a common coupled fixed point, if f is $F-$weakly commuting at $(x,\ y)$ and $f^2x = fx$ and $f^2y = fy$ for $(x,\ y) \in C(F,\ f)$,

(d) G and g have a common coupled fixed point, if g is $G-$weakly commuting at $(\widetilde{x},\ \widetilde{y})$ and $g^2\widetilde{x} = g\widetilde{x}$ and $g^2\widetilde{y} = g\widetilde{y}$ for $(\widetilde{x},\ \widetilde{y}) \in C(G,\ g)$,

(e) $F,\ G,\ f$ and g have common coupled fixed point provided that both (c) and (d) are true.

If we put $\varphi(t) = t - t\widetilde{\varphi}(t)$ for all $t \geq 0$ in Corollary 2.2, then we get the following result:

Corollary 2.3. *Let $(X,\ d)$ be a metric space. Assume $F,\ G : X \times X \to CB(X)$ and $f,\ g : X \to X$ be mappings satisfying*
(i) of Theorem 2.1 and
(i) for all $x,\ y,\ u,\ v \in X$, there exist some $\psi \in \Psi$ and $\widetilde{\varphi} \in \Phi$ such that

$$\psi\left(H(F(x,\ y),\ G(u,\ v))\right) \leq \widetilde{\varphi}\left(\psi\left(M(x,\ y,\ u,\ v)\right)\right)\psi\left(M(x,\ y,\ u,\ v)\right).$$

If (iii) of Theorem 2.1 holds. Then
(a) F and f have a coupled coincidence point,
(b) G and g have a coupled coincidence point,
(c) F and f have a common coupled fixed point, if f is $F-$weakly commuting at $(x,\ y)$ and $f^2x = fx$ and $f^2y = fy$ for $(x,\ y) \in C(F,\ f)$,
(d) G and g have a common coupled fixed point, if g is $G-$weakly commuting at $(\widetilde{x},\ \widetilde{y})$ and $g^2\widetilde{x} = g\widetilde{x}$ and $g^2\widetilde{y} = g\widetilde{y}$ for $(\widetilde{x},\ \widetilde{y}) \in C(G,\ g)$,
(e) $F,\ G,\ f$ and g have common coupled fixed point provided that both (c) and (d) are true.

If we put $\psi(t) = 2t$ for all $t \geq 0$ in Corollary 2.3, then we get the following result:

Corollary 2.4. *Let $(X,\ d)$ be a metric space. Assume $F,\ G : X \times X \to CB(X)$ and $f,\ g : X \to X$ be mappings satisfying*
(i) of Theorem 2.1 and
(i) for all $x,\ y,\ u,\ v \in X$, there exists some $\widetilde{\varphi} \in \Phi$ such that

$$H(F(x,\ y),\ G(u,\ v)) \leq \widetilde{\varphi}\left(2\psi\left(M(x,\ y,\ u,\ v)\right)\right)M(x,\ y,\ u,\ v).$$

If (iii) of Theorem 2.1 holds. Then
(a) F and f have a coupled coincidence point,
(b) G and g have a coupled coincidence point,
(c) F and f have a common coupled fixed point, if f is $F-$weakly commuting at $(x,\ y)$ and $f^2x = fx$ and $f^2y = fy$ for $(x,\ y) \in C(F,\ f)$,
(d) G and g have a common coupled fixed point, if g is $G-$weakly commuting at $(\widetilde{x},\ \widetilde{y})$ and $g^2\widetilde{x} = g\widetilde{x}$ and $g^2\widetilde{y} = g\widetilde{y}$ for $(\widetilde{x},\ \widetilde{y}) \in C(G,\ g)$,
(e) $F,\ G,\ f$ and g have common coupled fixed point provided that both (c) and (d) are true.

If we put $\widetilde{\varphi}(t) = k$ where $0 < k < 1$, for all $t \geq 0$ in Corollary 2.4, then we get the following result:

Corollary 2.5. *Let $(X,\ d)$ be a metric space. Assume $F,\ G : X \times X \to CB(X)$ and $f,\ g : X \to X$ be mappings satisfying*
(i) of Theorem 2.1 and
(i) for all $x,\ y,\ u,\ v \in X$, where $0 < k < 1$,

$$H(F(x,\ y),\ G(u,\ v)) \leq kM(x,\ y,\ u,\ v).$$

If (iii) of Theorem 2.1 holds. Then
(a) F and f have a coupled coincidence point,
(b) G and g have a coupled coincidence point,
(c) F and f have a common coupled fixed point, if f is $F-$weakly commuting at $(x,\ y)$ and $f^2x = fx$ and $f^2y = fy$ for $(x,\ y) \in C(F,\ f)$,

(d) G and g have a common coupled fixed point, if g is $G-$weakly commuting at $(\widetilde{x},\ \widetilde{y})$ and $g^2\widetilde{x}=g\widetilde{x}$ and $g^2\widetilde{y}=g\widetilde{y}$ for $(\widetilde{x},\ \widetilde{y})\in C(G,\ g)$,

(e) F, G, f and g have common coupled fixed point provided that both (c) and (d) are true.

Theorem 2.6. *Let $(X,\ d)$ be a metric space. Assume $F,\ G: X\times X\to CB(X)$ and $f,\ g: X\to X$ be mappings satisfying (i), (ii) of Theorem 2.1 and*

(i) $\{F,\ f\}$ and $\{G,\ g\}$ are $w-$compatible.

(ii) Suppose that either

(a) $f(X)$ is a closed subset of X and $F(X\times X)\subseteq g(X)$ or

(b) $g(X)$ is a closed subset of X and $G(X\times X)\subseteq f(X)$.

Then F, G, f and g have a common coupled fixed point.

Proof. Since $\{F,\ f\}$ and $\{G,\ g\}$ satisfy the common (EA) property, there exist sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ in X, some u, v in X and A, B, C, D in $CB(X)$ satisfying (1) of Theorem 2.1. Suppose (a) holds, that is, $f(X)$ is a closed subset of X, then there exist $x,\ y\in X$, we have

$$u=fx \text{ and } v=fy. \tag{3}$$

As in Theorem 2.1, we can prove that

$$fx\in F(x,\ y) \text{ and } fy\in F(y,\ x),$$

that is, $(x,\ y)$ is a coupled coincidence point of F and f. Hence $(x,\ y)\in C(F,\ f)$. From $w-$compatibility of $\{F,\ f\}$, we have $fF(x,\ y)\subseteq F(fx,\ fy)$, hence $f^2x\in F(fx,\ fy)$ and $f^2y\in F(fy,\ fx)$, that is, $fu\in F(u,\ v)$ and $fv\in F(v,\ u)$. Now, by condition (ii) of Theorem 2.1 and (i_ψ), we get

$$\psi\left(H(F(u,\ v),\ G(u_n,\ v_n))\right)\le\psi\left(M(u,\ v,\ u_n,\ v_n)\right)-\varphi\left(\psi\left(M(u,\ v,\ u_n,\ v_n)\right)\right)+\theta\left(N(u,\ v,\ u_n,\ v_n)\right),$$

where

$$M(u,\ v,\ u_n,\ v_n)=\max\left\{\begin{array}{c} d(fu,\ gu_n),\ D(fu,\ F(u,\ v)),\ D(gu_n,\ G(u_n,\ v_n)),\\ \frac{D(fu,\ G(u_n,\ v_n))+D(gu_n,\ F(u,\ v))}{2},\\ d(fv,\ gv_n),\ D(fv,\ F(v,\ u)),\ D(gv_n,\ G(v_n,\ u_n)),\\ \frac{D(fv,\ G(v_n,\ u_n))+D(gv_n,\ F(v,\ u))}{2}\end{array}\right\},$$

and

$$N(u,\ v,\ u_n,\ v_n)=\max\left\{\begin{array}{c} D(fu,\ F(u,\ v)),\ D(gu_n,\ G(u_n,\ v_n)),\\ D(fv,\ F(v,\ u)),\ D(gv_n,\ G(v_n,\ u_n)),\\ D(fu,\ G(u_n,\ v_n)),\ D(gu_n,\ F(u,\ v)),\\ D(fv,\ G(v_n,\ u_n)),\ D(gv_n,\ F(v,\ u))\end{array}\right\}.$$

Letting $n\to\infty$ in the above inequality, by (i_ψ), (i_φ), (i_θ), (ii_θ), (1), (3), we get

$$\psi\left(H(F(u,\ v),\ B)\right)\le\psi\left(\max\{d(fu,\ u),\ d(fv,\ v)\}\right)-\varphi\left(\psi\left(\max\{d(fu,\ u),\ d(fv,\ v)\}\right)\right).$$

Since $fu \in F(u,\ v)$ and $u \in B$, therefore by triangle inequality, we have

$$\psi\left(d(fu,\ u)\right) \leq \psi\left(\max\left\{d(fu,\ u),\ d(fv,\ v)\right\}\right) - \varphi\left(\psi\left(\max\left\{d(fu,\ u),\ d(fv,\ v)\right\}\right)\right).$$

Similarly, we can obtain that

$$\psi\left(d(fv,\ v)\right) \leq \psi\left(\max\left\{d(fu,\ u),\ d(fv,\ v)\right\}\right) - \varphi\left(\psi\left(\max\left\{d(fu,\ u),\ d(fv,\ v)\right\}\right)\right).$$

Combining them, we get

$$\max\left\{\psi\left(d(fu,\ u)\right),\ \psi\left(d(fv,\ v)\right)\right\} \leq \psi\left(\max\left\{d(fu,\ u),\ d(fv,\ v)\right\}\right) - \varphi\left(\psi\left(\max\left\{d(fu,\ u),\ d(fv,\ v)\right\}\right)\right).$$

Since ψ is non-decreasing, we have

$$\psi\left(\max\left\{d(fu,\ u),\ d(fv,\ v)\right\}\right) \leq \psi\left(\max\left\{d(fu,\ u),\ d(fv,\ v)\right\}\right) - \varphi\left(\psi\left(\max\left\{d(fu,\ u),\ d(fv,\ v)\right\}\right)\right),$$

which, by (ii_φ) and (ii_ψ), implies that $\max\{d(fu,\ u),\ d(fv,\ v)\} = 0$. Hence, we have $d(fu,\ u) = d(fv,\ v) = 0$. Thus $u = fu \in F(u,\ v)$ and $v = fv \in F(v,\ u)$. Since $F(X \times X) \subseteq g(X)$, then there exist $\widetilde{x},\ \widetilde{y} \in X$ such that $g\widetilde{x} = u = fu \in F(u,\ v)$ and $g\widetilde{y} = v = fv \in F(v,\ u)$. Now, by condition (ii) of Theorem 2.1 and (i_ψ), we get

$$\begin{aligned}
\psi\left(D(u,\ G(\widetilde{x},\ \widetilde{y}))\right) &\leq \psi\left(H(F(u,\ v),\ G(\widetilde{x},\ \widetilde{y}))\right) \\
&\leq \psi\left(M(u,\ v,\ \widetilde{x},\ \widetilde{y})\right) - \varphi\left(\psi\left(M(u,\ v,\ \widetilde{x}, \widetilde{y})\right)\right) + \theta\left(N(u,\ v,\ \widetilde{x},\ \widetilde{y}\right)) \\
&\leq \psi\left(\max\left\{D(u,\ G(\widetilde{x},\ \widetilde{y})),\ D(v,\ G(\widetilde{y},\ \widetilde{x}))\right\}\right) - \varphi\left(\psi\left(\max\left\{D(u,\ G(\widetilde{x},\ \widetilde{y})),\ D(v,\ G(\widetilde{y},\ \widetilde{x}))\right\}\right)\right).
\end{aligned}$$

Thus

$$\psi\left(D(u,\ G(\widetilde{x},\ \widetilde{y}))\right) \leq \psi\left(\max\left\{D(u,\ G(\widetilde{x},\ \widetilde{y}),\ D(v,\ G(\widetilde{y},\ \widetilde{x})\right\}\right) - \varphi\left(\psi\left(\max\left\{D(u,\ G(\widetilde{x},\ \widetilde{y}),\ D(v,\ G(\widetilde{y},\ \widetilde{x})\right\}\right)\right).$$

Similarly, we can obtain that

$$\psi\left(D(v,\ G(\widetilde{y},\ \widetilde{x}))\right) \leq \psi\left(\max\left\{D(u,\ G(\widetilde{x},\ \widetilde{y})),\ D(v,\ G(\widetilde{y},\ \widetilde{x}))\right\}\right) - \varphi\left(\psi\left(\max\left\{D(u,\ G(\widetilde{x},\ \widetilde{y})),\ D(v,\ G(\widetilde{y},\ \widetilde{x}))\right\}\right)\right).$$

Combining them, we get

$$\begin{aligned}
\max\left\{\psi\left(D(u,\ G(\widetilde{x},\ \widetilde{y}))\right),\ \psi\left(D(v,\ G(\widetilde{y},\ \widetilde{x}))\right)\right\} &\leq \psi\left(\max\left\{D(u,\ G(\widetilde{x},\ \widetilde{y})),\ D(v,\ G(\widetilde{y},\ \widetilde{x}))\right\}\right) \\
&\quad - \varphi\left(\psi\left(\max\left\{D(u,\ G(\widetilde{x},\ \widetilde{y})),\ D(v,\ G(\widetilde{y},\ \widetilde{x}))\right\}\right)\right).
\end{aligned}$$

Since ψ is non-decreasing, we have

$$\begin{aligned}
\psi\left(\max\left\{D(u,\ G(\widetilde{x},\ \widetilde{y})),\ D(v,\ G(\widetilde{y},\ \widetilde{x}))\right\}\right) &\leq \psi\left(\max\left\{D(u,\ G(\widetilde{x},\ \widetilde{y})),\ D(v,\ G(\widetilde{y},\ \widetilde{x}))\right\}\right) \\
&\quad - \varphi\left(\psi\left(\max\left\{D(u,\ G(\widetilde{x},\ \widetilde{y})),\ D(v,\ G(\widetilde{y},\ \widetilde{x}))\right\}\right)\right),
\end{aligned}$$

which, by (ii_φ) and (ii_ψ), implies that $\max\{D(u,\ G(\widetilde{x},\ \widetilde{y})),\ D(v,\ G(\widetilde{y},\ \widetilde{x}))\} = 0$. Hence, we have $D(u,\ G(\widetilde{x},\ \widetilde{y}) = D(v,\ G(\widetilde{y},\ \widetilde{x}) = 0$. Thus $u = g\widetilde{x} \in G(\widetilde{x},\ \widetilde{y})$ and $v = g\widetilde{y} \in G(\widetilde{y},\ \widetilde{x})$, that is, $(\widetilde{x},\ \widetilde{y})$ is a coupled coincidence point of G and g. Hence

$(x,\, y) \in C(G,\, g)$. From $w-$compatibility of $\{G,\, g\}$, we have $gG(\widetilde{x},\, \widetilde{y}) \subseteq G(g\widetilde{x},\, g\widetilde{y})$, hence $g^2\widetilde{x} \in G(g\widetilde{x},\, g\widetilde{y})$ and $g^2\widetilde{y} \in G(g\widetilde{y},\, g\widetilde{x})$, that is, $gu \in G(u,\, v)$ and $gv \in G(v,\, u)$. Again, by condition (ii) of Theorem 2.1, we get

$$\begin{aligned}\psi\left(H(F(u,\ v),\ G(u,\ v))\right) &\leq \psi\left(M(u,\ v,\ u,\ v)\right) - \varphi\left(\psi\left(M(u,\ v,\ u,\ v)\right)\right) + \theta\left(N(u,\ v,\ u,\ v)\right)\\ &\leq \psi\left(\max\left\{d(u,\ gu),\ d(v,\ gv)\right\}\right) - \varphi\left(\psi\left(\max\left\{d(u,\ gu),\ d(v,\ gv)\right\}\right)\right).\end{aligned}$$

Since $u \in F(u,\, v)$ and $gu \in G(u,\, v)$. Therefore, by (i_ψ) and triangle inequality, we get

$$\psi\left(d(u,\ gu)\right) \leq \psi\left(\max\left\{d(u,\ gu),\ d(v,\ gv)\right\}\right) - \varphi\left(\psi\left(\max\left\{d(u,\ gu),\ d(v,\ gv)\right\}\right)\right).$$

Similarly, we can get

$$\psi\left(d(v,\ gv)\right) \leq \psi\left(\max\left\{d(u,\ gu),\ d(v,\ gv)\right\}\right) - \varphi\left(\psi\left(\max\left\{d(u,\ gu),\ d(v,\ gv)\right\}\right)\right).$$

Combining them, we get

$$\max\left\{\psi\left(d(u,\ gu)\right),\ \psi\left(d(v,\ gv)\right)\right\} \leq \psi\left(\max\left\{d(u,\ gu),\ d(v,\ gv)\right\}\right) - \varphi\left(\psi\left(\max\left\{d(u,\ gu),\ d(v,\ gv)\right\}\right)\right).$$

Since ψ is non-decreasing, we have

$$\psi\left(\max\left\{d(u,\ gu),\ d(v,\ gv)\right\}\right) \leq \psi\left(\max\left\{d(u,\ gu),\ d(v,\ gv)\right\}\right) - \varphi\left(\psi\left(\max\left\{d(u,\ gu),\ d(v,\ gv)\right\}\right)\right),$$

which, by (ii_φ) and (ii_ψ), implies that $\max\left\{d(u,\ gu),\ d(v,\ gv)\right\} = 0$. Hence, we have $d(u, gu) = d(v, gv) = 0$. Thus $u = gu \in G(u,\ v)$ and $v = gv \in G(v,\ u)$. Therefore $(u,\, v)$ is a common coupled fixed point of F, G, f and g. The proof is similar when (b) holds. □

If we put $\theta(t) = 0$ in the Theorem 2.6, we get the following result:

Corollary 2.7. *Let $(X,\, d)$ be a metric space. Assume $F,\, G : X \times X \to CB(X)$ and $f,\, g : X \to X$ be mappings satisfying (i) of Theorem 2.1, (i) of Corollary 2.2, (i) and (ii) of Theorem 2.6. Then F, G, f and g have a common coupled fixed point.*

If we put $\varphi(t) = t - t\widetilde{\varphi}(t)$ for all $t \geq 0$ in Corollary 2.7, then we get the following result:

Corollary 2.8. *Let $(X,\, d)$ be a metric space. Assume $F,\, G : X \times X \to CB(X)$ and $f,\, g : X \to X$ be mappings satisfying (i) of Theorem 2.1, (i) of Corollary 2.3, (i) and (ii) of Theorem 2.6. Then F, G, f and g have a common coupled fixed point.*

If we put $\psi(t) = 2t$ for all $t \geq 0$ in Corollary 2.8, then we get the following result:

Corollary 2.9. *Let $(X,\, d)$ be a metric space. Assume $F,\, G : X \times X \to CB(X)$ and $f,\, g : X \to X$ be mappings satisfying (i) of Theorem 2.1, (i) of Corollary 2.4, (i) and (ii) of Theorem 2.6. Then F, G, f and g have a common coupled fixed point.*

If we put $\widetilde{\varphi}(t) = k$ where $0 < k < 1$, for all $t \geq 0$ in Corollary 2.9, then we get the following result:

Corollary 2.10. *Let $(X,\ d)$ be a metric space. Assume $F,\ G : X \times X \to CB(X)$ and $f,\ g : X \to X$ be mappings satisfying (i) of Theorem 2.1, (i) of Corollary 2.5, (i) and (ii) of Theorem 2.6. Then $F,\ G,\ f$ and g have a common coupled fixed point.*

Theorem 2.11. *Let $(X,\ d)$ be a metric space. Assume $F,\ G : X \times X \to CB(X)$ and $f,\ g : X \to X$ be mappings satisfying (ii) of Theorem 2.1 and*

(i) $\{F,\ f\}$ and $\{G,\ g\}$ are occasionally $w-$compatible. Then $F,\ G,\ f$ and g have a common coupled fixed point.

Proof. Since the pairs $\{F,\ f\}$ and $\{G,\ g\}$ are occasionally $w-$compatible, therefore there exists some point $(x,\ y)$, $(\widetilde{x},\ \widetilde{y}) \in X \times X$ such that

$$\begin{aligned} fx &\in F(x,\ y),\ fy \in F(y,\ x) \text{ and } fF(x,\ y) \subseteq F(fx,\ fy), \\ g\widetilde{x} &\in G(\widetilde{x},\ \widetilde{y}),\ g\widetilde{y} \in G(\widetilde{y},\ \widetilde{x}) \text{ and } gG(\widetilde{x},\ \widetilde{y}) \subseteq G(g\widetilde{x},\ g\widetilde{y}). \end{aligned} \tag{4}$$

It follows that

$$\begin{aligned} f^2x &\in F(fx,\ fy) \text{ and } f^2y \in F(fy,\ fx), \\ g^2\widetilde{x} &\in G(g\widetilde{x},\ g\widetilde{y}) \text{ and } g^2\widetilde{y} \in G(g\widetilde{y},\ g\widetilde{x}). \end{aligned} \tag{5}$$

Now, we shall show that $u = fx = g\widetilde{x}$ and $v = fy = g\widetilde{y}$. Now, by condition (ii) of Theorem 2.1 and (i_ψ), we have

$$\psi\left(H(F(x,\ y),\ G(\widetilde{x},\ \widetilde{y}))\right) \le \psi\left(M(x,\ y,\ \widetilde{x},\ \widetilde{y})\right) - \varphi\left(\psi\left(M(x,\ y,\ \widetilde{x},\ \widetilde{y})\right)\right) + \theta\left(N(x,\ y,\ \widetilde{x},\ \widetilde{y})\right).$$

It follows, by (4), (i_ψ), (i_φ), (i_θ), (ii_θ) and triangle inequality, that

$$\psi\left(d(fx,\ g\widetilde{x})\right) \le \psi\left(\max\left\{d(fx,\ g\widetilde{x}),\ d(fy,\ g\widetilde{y})\right\}\right) - \varphi\left(\psi\left(\max\left\{d(fx,\ g\widetilde{x}),\ d(fy,\ g\widetilde{y})\right\}\right)\right).$$

Similarly, we can obtain that

$$\psi\left(d(fy,\ g\widetilde{y})\right) \le \psi\left(\max\left\{d(fx,\ g\widetilde{x}),\ d(fy,\ g\widetilde{y})\right\}\right) - \varphi\left(\psi\left(\max\left\{d(fx,\ g\widetilde{x}),\ d(fy,\ g\widetilde{y})\right\}\right)\right).$$

Combining them, we get

$$\max\left\{\psi\left(d(fx,\ g\widetilde{x})\right),\ \psi\left(d(fy,\ g\widetilde{y})\right)\right\} \le \psi\left(\max\left\{d(fx,\ g\widetilde{x}),\ d(fy,\ g\widetilde{y})\right\}\right) - \varphi\left(\psi\left(\max\left\{d(fx,\ g\widetilde{x}),\ d(fy,\ g\widetilde{y})\right\}\right)\right).$$

Since ψ is non-decreasing, therefore

$$\psi\left(\max\left\{d(fx,\ g\widetilde{x}),\ d(fy,\ g\widetilde{y})\right\}\right) \le \psi\left(\max\left\{d(fx,\ g\widetilde{x}),\ d(fy,\ g\widetilde{y})\right\}\right) - \varphi\left(\psi\left(\max\left\{d(fx,\ g\widetilde{x}),\ d(fy,\ g\widetilde{y})\right\}\right)\right),$$

which, by (ii_φ) and (ii_ψ), implies that $\max\left\{d(fx,\ g\widetilde{x}),\ d(fy,\ g\widetilde{y})\right\} = 0$, it follows that $d(fx,\ g\widetilde{x}) = d(fy,\ g\widetilde{y}) = 0$. Hence

$$u = fx = g\widetilde{x} \text{ and } v = fy = g\widetilde{y}. \tag{6}$$

Thus, by (5) and (6), we get

$$\begin{aligned} fu &\in F(u,\ v) \text{ and } fv \in F(v,\ u), \\ gu &\in G(u,\ v) \text{ and } gv \in G(v,\ u). \end{aligned} \tag{7}$$

Now, we shall show that $u = fu = gu$ and $v = fv = gv$. Again, by condition (ii) of Theorem 2.1 and (i_ψ), we have

$$\psi\left(H(F(u,\ v),\ G(\widetilde{x},\ \widetilde{y}))\right) \le \psi\left(M(u,\ v,\ \widetilde{x},\ \widetilde{y})\right) - \varphi\left(\psi\left(M(u,\ v,\ \widetilde{x},\ \widetilde{y})\right)\right) + \theta\left(N(u,\ v,\ \widetilde{x},\ \widetilde{y})\right).$$

It follows, by (7), (i_ψ), (i_φ), (i_θ), (ii_θ) and triangle inequality, that

$$\psi\left(d(fu,\ u)\right) \le \psi\left(\max\left\{d(fu,\ u),\ d(fv,\ v)\right\}\right) - \varphi\left(\psi\left(\max\left\{d(fu,\ u),\ d(fv,\ v)\right\}\right)\right).$$

Similarly, we can obtain that

$$\psi\left(d(fv,\ v)\right) \le \psi\left(\max\left\{d(fu,\ u),\ d(fv,\ v)\right\}\right) - \varphi\left(\psi\left(\max\left\{d(fu,\ u),\ d(fv,\ v)\right\}\right)\right).$$

Combining them, we get

$$\max\left\{\psi\left(d(fu,\ u)\right),\ \psi\left(d(fv,\ v)\right)\right\} \le \psi\left(\max\left\{d(fu,\ u),\ d(fv,\ v)\right\}\right) - \varphi\left(\psi\left(\max\left\{d(fu,\ u),\ d(fv,\ v)\right\}\right)\right).$$

Since ψ is non-decreasing, therefore

$$\psi\left(\max\left\{d(fu,\ u),\ d(fv,\ v)\right\}\right) \le \psi\left(\max\left\{d(fu,\ u),\ d(fv,\ v)\right\}\right) - \varphi\left(\psi\left(\max\left\{d(fu,\ u),\ d(fv,\ v)\right\}\right)\right),$$

which, by (ii_φ) and (ii_ψ), implies that $\max\left\{d(fu,\ u),\ d(fv,\ v)\right\} = 0$, it follows that $d(fu, u) = d(fv, v) = 0$. Thus

$$u = fu \text{ and } v = fv. \tag{8}$$

Similarly, we can show that

$$u = gu \text{ and } v = gv. \tag{9}$$

Thus, by (7), (8) and (9), we get

$$\begin{aligned} u &= fu \in F(u,\ v),\ v = fv \in F(v,\ u), \\ u &= gu \in G(u,\ v),\ v = gv \in G(v,\ u), \end{aligned}$$

that is, $(u,\ v)$ is a common coupled fixed point of F, G, f and g. □

If we put $\theta(t) = 0$ in the Theorem 2.11, we get the following result:

Corollary 2.12. *Let $(X,\ d)$ be a metric space. Assume $F,\ G : X \times X \to CB(X)$ and $f,\ g : X \to X$ be mappings satisfying (i) of Corollary 2.2 and (i) of Theorem 2.11. Then F, G, f and g have a common coupled fixed point.*

If we put $\varphi(t) = t - t\widetilde{\varphi}(t)$ for all $t \ge 0$ in Corollary 2.12, then we get the following result:

Corollary 2.13. *Let $(X,\ d)$ be a metric space. Assume $F,\ G : X \times X \to CB(X)$ and $f,\ g : X \to X$ be mappings satisfying (i) of Corollary 2.3 and (i) of Theorem 2.11. Then F, G, f and g have a common coupled fixed point.*

If we put $\psi(t) = 2t$ for all $t \ge 0$ in Corollary 2.13, then we get the following result:

Corollary 2.14. *Let $(X,\ d)$ be a metric space. Assume $F,\ G : X \times X \to CB(X)$ and $f,\ g : X \to X$ be mappings satisfying (i) of Corollary 2.4 and (i) of Theorem 2.11. Then F, G, f and g have a common coupled fixed point.*

If we put $\widetilde{\varphi}(t) = k$ where $0 < k < 1$, for all $t \geq 0$ in Corollary 2.14, then we get the following result:

Corollary 2.15. *Let (X, d) be a metric space. Assume $F, G : X \times X \to CB(X)$ and $f, g : X \to X$ be mappings satisfying (i) of Corollary 2.5 and (i) of Theorem 2.11. Then F, G, f and g have a common coupled fixed point.*

Example 2.1. *Suppose that $X = [0, 1]$, equipped with the metric $d : X \times X \to [0, +\infty)$ defined as $d(x, y) = \max\{x, y\}$ and $d(x, x) = 0$ for all $x, y \in X$. Let $F, G : X \times X \to CB(X)$ be defined as*

$$F(x, y) = \begin{cases} \{0\}, \text{ for } x, y = 1 \\ \left[0, \frac{x^4+y^4}{16}\right], \text{ for } x, y \in [0, 1) \end{cases}$$

and

$$G(x, y) = \begin{cases} \{0\}, \text{ for } x, y = 1 \\ \left[0, \frac{x^2+y^2}{32}\right], \text{ for } x, y \in [0, 1). \end{cases}$$

Suppose $f, g : X \to X$ be defined as

$$fx = \begin{cases} x^2, & x \neq 1, \\ \frac{3}{2}, & x = 1, \end{cases} \quad \text{for all } x \in X$$

and

$$gx = \begin{cases} \frac{x}{2}, & x \neq 1, \\ 1, & x = 1, \end{cases} \quad \text{for all } x \in X.$$

Define $\psi : [0, +\infty) \to [0, +\infty)$ by

$$\psi(t) = \ln(t+1), \text{ for all } t \geq 0,$$

and $\varphi : [0, +\infty) \to [0, +\infty)$ by

$$\varphi(t) = t - \ln\left(\frac{t}{2} + 1\right), \text{ for all } t \geq 0,$$

and $\theta : [0, +\infty) \to [0, +\infty)$ by

$$\theta(t) = \frac{t}{4}, \text{ for all } t \geq 0.$$

Now, for all $x, y, u, v \in X$ with $x, y, u, v \in [0, 1)$, we have

Case (a). If $\frac{x^4+y^4}{16} = \frac{u^2+v^2}{32}$, then

$$\begin{aligned} d(F(x, y), G(u, v)) &= \frac{1}{16}(x^4+y^4) \\ &\leq \frac{1}{4}\ln\left(x^2+1\right) + \frac{1}{4}\ln\left(y^2+1\right) \\ &\leq \frac{1}{4}\ln\left(\max\left\{x^2, \frac{u}{2}\right\}+1\right) + \frac{1}{4}\ln\left(\max\left\{y^2, \frac{v}{2}\right\}+1\right) \\ &\leq \frac{1}{4}\ln\left(d(fx, gu)+1\right) + \frac{1}{4}\ln\left(d(fy, gv)+1\right) \\ &\leq \frac{1}{4}\ln\left(M(x, y, u, v)+1\right) + \frac{1}{4}\ln\left(M(x, y, u, v)+1\right) \\ &\leq \frac{1}{2}\ln(M(x, y, u, v)+1), \end{aligned}$$

which implies that

$$\begin{aligned} \psi\left(d(F(x, y), G(u, v))\right) &= \ln\left(d(F(x, y), G(u, v))+1\right) \\ &\leq \ln\left(\frac{1}{2}\ln\left(M(x, y, u, v)+1\right)+1\right) \\ &\leq \ln\left(M(x, y, u, v)+1\right) - \left[\ln\left(M(x, y, u, v)+1\right) - \ln\left(\frac{1}{2}\ln\left(M(x, y, u, v)+1\right)+1\right)\right] \\ &\leq \psi\left(M(x, y, u, v)\right) - \varphi\left(\psi\left(M(x, y, u, v)\right)\right) + \theta\left(N(x, y, u, v)\right). \end{aligned}$$

Case (b). If $\frac{x^4+y^4}{16} \neq \frac{u^2+v^2}{32}$ *with* $\frac{x^4+y^4}{16} < \frac{u^2+v^2}{32}$, *then*

$$
\begin{aligned}
d(F(x,\ y),\ G(u,\ v)) &= \frac{1}{32}(u^2+v^2) \\
&\leq \frac{1}{4}\ln\left(\frac{u}{2}+1\right)+\frac{1}{4}\ln\left(\frac{v}{2}+1\right) \\
&\leq \frac{1}{4}\ln\left(\max\left\{x^2,\ \frac{u}{2}\right\}+1\right)+\frac{1}{4}\ln\left(\max\left\{y^2,\ \frac{v}{2}\right\}+1\right) \\
&\leq \frac{1}{4}\ln\left(d(fx,\ gu)+1\right)+\frac{1}{4}\ln\left(d(fy,\ gv)+1\right) \\
&\leq \frac{1}{4}\ln\left(M(x,\ y,\ u,\ v)+1\right)+\frac{1}{4}\ln\left(M(x,\ y,\ u,\ v)+1\right) \\
&\leq \frac{1}{2}\ln(M(x,\ y,\ u,\ v)+1),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\psi\left(d(F(x,\ y),\ G(u,\ v))\right) &= \ln\left(d(F(x,\ y),\ G(u,\ v))+1\right) \\
&\leq \ln\left(\frac{1}{2}\ln\left(M(x,\ y,\ u,\ v)+1\right)+1\right) \\
&\leq \ln\left(M(x,\ y,\ u,\ v)+1\right)-\left[\ln\left(M(x,\ y,\ u,\ v)+1\right)-\ln\left(\frac{1}{2}\ln\left(M(x,\ y,\ u,\ v)+1\right)+1\right)\right] \\
&\leq \psi\left(M(x,\ y,\ u,\ v)\right)-\varphi\left(\psi\left(M(x,\ y,\ u,\ v)\right)\right)+\theta\left(N(x,\ y,\ u,\ v)\right).
\end{aligned}
$$

Similarly, we obtain the same result for $\frac{u^2+v^2}{32} < \frac{x^4+y^4}{16}$. *Thus the contractive condition (ii) of Theorem 2.1 is satisfied for all* $x,\ y,\ u,\ v \in X$ *with* $x,\ y,\ u,\ v \in [0,\ 1)$. *Again, for all* $x,\ y,\ u,\ v \in X$ *with* $x,\ y \in [0,\ 1)$ *and* $u,\ v = 1$, *we have*

$$
\begin{aligned}
d(F(x,\ y),\ G(u,\ v)) &= \frac{1}{16}(x^4+y^4) \\
&\leq \frac{1}{4}\ln\left(x^2+1\right)+\frac{1}{4}\ln\left(y^2+1\right) \\
&\leq \frac{1}{4}\ln\left(\max\left\{x^2,\ \frac{u}{2}\right\}+1\right)+\frac{1}{4}\ln\left(\max\left\{y^2,\ \frac{v}{2}\right\}+1\right) \\
&\leq \frac{1}{4}\ln\left(d(fx,\ gu)+1\right)+\frac{1}{4}\ln\left(d(fy,\ gv)+1\right) \\
&\leq \frac{1}{4}\ln\left(M(x,\ y,\ u,\ v)+1\right)+\frac{1}{4}\ln\left(M(x,\ y,\ u,\ v)+1\right) \\
&\leq \frac{1}{2}\ln(M(x,\ y,\ u,\ v)+1),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\psi\left(d(F(x,\ y),\ G(u,\ v))\right) &= \ln\left(d(F(x,\ y),\ G(u,\ v))+1\right) \\
&\leq \ln\left(\frac{1}{2}\ln\left(M(x,\ y,\ u,\ v)+1\right)+1\right) \\
&\leq \ln\left(M(x,\ y,\ u,\ v)+1\right)-\left[\ln\left(M(x,\ y,\ u,\ v)+1\right)-\ln\left(\frac{1}{2}\ln\left(M(x,\ y,\ u,\ v)+1\right)+1\right)\right] \\
&\leq \psi\left(M(x,\ y,\ u,\ v)\right)-\varphi\left(\psi\left(M(x,\ y,\ u,\ v)\right)\right)+\theta\left(N(x,\ y,\ u,\ v)\right).
\end{aligned}
$$

Thus the contractive condition (ii) of Theorem 2.1 is satisfied for all $x,\ y,\ u,\ v \in X$ *with* $x,\ y \in [0,\ 1)$ *and* $u,\ v = 1$. *Similarly, we can see that the contractive condition (ii) of Theorem 2.1 is satisfied for all* $x,\ y,\ u,\ v \in X$ *with* $x,\ y,\ u,\ v = 1$. *Hence, the hybrid pairs* $\{F,\ f\}$ *and* $\{G,\ g\}$ *satisfy the condition (ii) of Theorem 2.1, for all* $x,\ y,\ u,\ v \in X$. *In addition, all the other conditions of Theorem 2.1, Theorem 2.6 and Theorem 2.11 are satisfied and* $z = (0,\ 0)$ *is a common coupled fixed point of* $F,\ G,\ f$ *and* g.

References

[1] M.Aamri and D.ElMoutawakil, *Some new common fixed point theorems under strict contractive conditions*, J. Math. Anal. Appl., 270(1)(2002), 181-188.

[2] M.Abbas, L.Ciric, B.Damjanovic and M.A.Khan, *Coupled coincidence point and common fixed point theorems for hybrid pair of mappings*, Fixed Point Theory Appl., 1687-1812-2012-4.

[3] M.Abbas and B.E.Rhoades, *Common fixed point theorems for hybrid pairs of occasionally weakly compatible mappings satisfying generalized contractive condition of integral type*, Fixed Point Theory Appl., 2007(2007), Article ID 54101.

[4] V.*Berinde, Coupled fixed point theorems for $\varphi-$contractive mixed monotone mappings in partially ordered metric spaces*, Nonlinear Anal., 75(2012), 3218-3228.

[5] T.G.Bhaskar and V.Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., 65(7)(2006), 1379-1393.

[6] L.Ciric, B.Damjanovic, M.Jleli and B.Samet, *Coupled fixed point theorems for generalized Mizoguchi-Takahashi contractions with applications*, Fixed Point Theory Appl., 51(2012).

[7] B.Deshpande and R.Pathak, *Fixed point theorems for noncompatible discontinuous hybrid pairs of mappings on 2-metric spaces*, Demonstr. Math., XLV(1)(2012), 143-154.

[8] B.Deshpande and S.Chouhan, *Common fixed point theorems for hybrid pairs of mappings with some weaker conditions in 2-metric spaces*, Fasc. Math., 46(2011), 37-55.

[9] B.Deshpande and S.Chouhan, *Fixed points for two hybrid pairs of mappings satisfying some weaker conditions on noncomplete metric spaces*, Southeast Asian Bull. Math., 35(2011), 851-858.

[10] B.Deshpande and A.Handa, *Nonlinear mixed monotone-generalized contractions on partially ordered modified intuitionistic fuzzy metric spaces with application to integral equations*, Afr. Mat., 26(3-4)(2015), 317-343.

[11] B.Deshpande and A.Handa, *Application of coupled fixed point technique in solving integral equations on modified intuitionistic fuzzy metric spaces*, Adv. Fuzzy Syst., 2014(2014), Article ID 348069.

[12] B.Deshpande and A.Handa, *Common coupled fixed point theorems for hybrid pair of mappings satisfying an implicit relation with application*, Afr. Mat., DOI 10.1007/s13370-015-0326-7.

[13] B.Deshpande and A.Handa, *Common coupled fixed point theorems for two hybrid pairs of mappings under $\varphi - \psi$ contraction*, ISRN, 2014(2014), Article ID 608725.

[14] B.Deshpande and A.Handa, *Common coupled fixed point for hybrid pair of mappings under generalized nonlinear contraction*, East Asian Math. J., 31(1)(2015), 77-89.

[15] B.Deshpande and A.Handa, *Common coupled fixed point theorems for hybrid pair of mappings under some weaker conditions satisfying an implicit relation*, Nonlinear Analysis Forum, 20(2015), 79-93.

[16] B.Deshpande and A.Handa, *Common coupled fixed point theorems for two hybrid pairs of mappings satisfying an implicit relation*, Sarajevo Journal of Mathematics, 11(23)(1)(2015), 85-100.

[17] B.Deshpande, A.Handa and C.Kothari, *Common coupled fixed point results for hybrid pair of mappings satisfying weak psi-phi contraction under new weaker condition*, IMF, 10(10)(2015), 457-465.

[18] B.Deshpande and A.Handa, *Common coupled fixed point theorem under generalized Mizoguchi-Takahashi contraction for hybrid pair of mappings*, J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math., 22(3)(2015), 199-214.

[19] H.S.Ding, L.Li and S.Radenovic, *Coupled coincidence point theorems for generalized nonlinear contraction in partially ordered metric spaces*, Fixed Point Theory Appl., 96(2012).

[20] M.E.Gordji, E.Akbartabar, Y.J.Cho and M.Ramezani, *Coupled common fixed point theorems for mixed weakly monotone mappings in partially ordered metric spaces*, Fixed Point Theory Appl., 95(2012).

[21] M.E.Gordji, H.Baghani and G.H.Kim, *Common fixed point theorems for* $(\psi, \varphi)-$*weak nonlinear contraction in partially ordered sets*, Fixed Point Theory Appl., 62(2012).

[22] D.Guo and V.Lakshmikantham, *Coupled fixed points of nonlinear operators with applications*, Nonlinear Anal., 11(5)(1987), 623-632.

[23] M.Jain, K.Tas, S.Kumar and N.Gupta, *Coupled common fixed point results involving a* $\varphi - \psi$ *contractive condition for mixed g-monotone operators in partially ordered metric spaces*, J. Inequal. Appl., (2012), 285.

[24] G.Jungck and B.E.Rhoades, *Fixed point theorems for occasionally weakly compatible mappings*, Fixed Point Theory, 7(2006), 286-296.

[25] T.Kamran, *Coincidence and fixed points for hybrid strict contractions*, J. Math. Anal. Appl., 299(1)(2004), 235-241.

[26] V.Lakshmikantham and L.Ciric, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal., 70(12)(2009), 4341-4349.

[27] Y.Liu, J.Wu and Z.Li, *Common fixed points of single-valued and multivalued mappings*, Int. J. Math. Math. Sci., 19(2005), 3045-3055.

[28] W.Long, S.Shukla and S.Radenovic, *Some coupled coincidence and common fixed point results for hybrid pair of mappings in 0-complete partial metric spaces*, Fixed Point Theory Appl., (2013), 145.

[29] W.Long, M.Abbas, T.Nazir and S.Radenovic, *Common fixed point for two pairs of mappings satisfying (EA) property in generalized metric spaces*, Abstr. Appl. Anal., 2012(2012), Article ID 394830.

[30] N.V.Luong and N.X.Thuan, *Coupled fixed points in partially ordered metric spaces and application*, Nonlinear Anal., 74(2011), 983-992.

[31] M.Mursaleen, S.A.Mohiuddine and R.P.Agarwal, *Coupled fixed point theorems for alpha-psi contractive type mappings in partially ordered metric spaces*, Fixed Point Theory Appl., (2012), 228.

[32] K.P.R.Rao, G.Ravi Babu and V.C.C.Raju, *A Common fixed point theorem for two pairs of occasionally weakly semi-compatible hybrid mappings under an implicit relation*, Mathematical Sciences, 1(3)(2007), 01-06.

[33] K.P.R.Rao and K.R.K.Rao, *A common fixed point theorem for two hybrid pairs of mappings in b-metric spaces*, International Journal of Analysis, 2013(2013), Article ID 404838.

[34] J.Rodriguez-Lopez and S.Romaguera, *The Hausdorff fuzzy metric on compact sets*, Fuzzy Sets Syst., 147(2004), 273-283.

[35] B.Samet, *Coupled fixed point theorems for generalized Meir-Keeler contraction in partially ordered metric spaces*, Nonlinear Anal., 72(2010), 4508-4517.

[36] B.Samet, E.Karapinar, H.Aydi and V.C.Rajic, *Discussion on some coupled fixed point theorems*, Fixed Point Theory Appl., (2013), 50.

[37] S.Sharma, B.Deshpande and R.Pathak, *Common fixed point theorems for hybrid pairs of mappings with some weaker conditions*, Fasc. Math., 39(2008), 71-84.

[38] N.Singh and R.Jain, *Coupled coincidence and common fixed point theorems for set-valued and single-valued mappings in fuzzy metric space*, Journal of Fuzzy Set Valued Analysis, 2012(2012), Article ID jfsva-00129.

[39] W.Sintunavarat, P.Kumam and Y.J.Cho, *Coupled fixed point theorems for nonlinear contractions without mixed monotone property*, Fixed Point Theory Appl., 2012(2012), 170.

[40] Z.Wu, C.Zhu and J.Li, *Common fixed point theorems for two hybrid pairs of mappings satisfying the common property (EA) in Menger PM-spaces*, Fixed Point Theory Appl., 2013(2013), 25.