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Harmonised Fractal dimensional Measure: A Special Case of a Haar Measure and Convenience With Martingales

Research Article

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Abstract: The Martingale's property is one of the fundamental mathematical properties which underline many important results in finance, obeying the principle of risk neutral pricing and a necessary condition for an efficient market. The aim of this paper is to show that the Harmonized fractal dimensional measure (HFDM) is a special case of a Haar measure and also convenience with martingale measure. To establish this, we first transform the measure into a solution then checkmate it under the Martingale properties. In this sense, the efficiency of harmonized fractal dimensional measure in capital market connotes that the measure is a martingale as it deals with wealth distribution that are highly skewed, curbs investment by neutralizing risky assets and diverting the wealth to consumption.

Keywords: Borel Sets, Haar Measure, Harmonized integral transform, Market efficiency and Martingales.

1. Introduction

Martingales are very convenient in pricing security and in fact a very powerful tool. Through a numeraire accounting we can obtain relative prices that are Martingale including historic probabilities. For example, the fundamental theorem of asset pricing states that if there are no arbitrage opportunities, then properly normalized security prices are Martingales under some probability measure. Furthermore, efficient markets are defined when relevant information is reflected in the market prices (Fama, 1970). This means that at any point in time, the current price fully represents all the information. The Martingales was long considered to be a necessary condition for an efficient asset market, one in which the information contained in past prices is instantly, fully and perpetually reflected in the asset's current price. If this is the case, how can a Martingale account for the tradeoff between risk and returns? A Haar measure is a special case of a Borel sets with an integral base, showing that mixing time of random walks on compact groups can be obtained on concentration inequalities respectively (Bondar, et al 1981). Here, we observed that the solution of Harmonized fractal dimensional measure is a special case of a Haar measure and also a Martingale hence, acting as if all expected returns equals the risk-free rate, which is the same as if all investors are risk neutral. This is called the principle of risk-neutral pricing. To price risk-neutrality, one must change the probability measure to what is called risk-neutral probability and such exists if there are no arbitrage opportunities in market which is a mild assumption. In this paper we established the fact that the efficiency of Harmonized fractal dimensional measure in capital market is a special case of a Haar measure and convenience with martingales as it deals with wealth distribution that are highly skewed by neutralizing risky assets and diverting the wealth to consumption.

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2. Borel Sets

Definition 2.1. *let $\mathcal{C}$ denote the collection of open subset of $\mathbb{R}$. Then $\Sigma(\mathcal{C})$ is called the Borel σ-algebra of $\mathbb{R}$, usually written $\mathcal{B}(\mathbb{R})$. The elements of $(\mathbb{R})$ are called Borel sets. Similarly, one defines $(\mathbb{R}^n)$ as the σ-algebra generated by the open subsets of $\mathbb{R}^n$.*

Proposition 2.2. *The following subsets of $\mathbb{R}$ belong to $\mathcal{B}(\mathbb{R})$:*

(a) (a, b) for any $a < b$;

(b) $(-\infty, a)$ for any $a \in \mathbb{R}$;

(c) (a, ∞) for any $a \in \mathbb{R}$;

(d) $[a, b]$ for any $a \leq b$;

(e) $(-\infty, a]$ for any $a \in \mathbb{R}$;

(f) $[a, \infty$ for any $a \in \mathbb{R}$;

(g) $(a, b]$ for any $a < b$;

(h) $[a, b]$ for any ab;

(i) any closed subset of $\mathbb{R}$;

Proof. Each of the sets in (a), (b), (c) is open and so belong to $\mathbf{B}(\mathbb{R})$ by construction.

(d) $[a, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n}) \in \mathbf{B}(\mathbb{R})$

(e) $(-\infty, b) = \bigcap_{n=1}^{\infty} (\infty, a + \frac{1}{n}) \in \mathbf{B}(\mathbb{R})$

(f) $[a, \infty) = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, \infty) \in \mathbf{B}(\mathbb{R})$

(g) $(a, b] = \bigcap_{n=1}^{\infty} (a, b + \frac{1}{n}) \in \mathbf{B}(\mathbb{R})$

(h) $[a, b) = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b) \in \mathbf{B}(\mathbb{R})$

(i) If F is closed $\mathbb{R}$, then F^c is open and so belongs to $(\mathbb{R})$. But then $F = (F^c) \in B(R)$.

□

In fact we will see that each of these families of subsets of $\mathbb{R}$ generates the σ-algebra $(\mathbb{R})$.

2.1. Haar Measure

Definition 2.3. *Let Γ be a compact group. A Haar measure on Γ is a measure $\mu : \Sigma \to [0, \infty)$, with Σ, a σ-algebra containing all borel subsets of Γ and is invariant under left and right multiplication and under inversion. i.e*

$$S \in \Sigma \Rightarrow \gamma S = \{\gamma\alpha | \alpha \in S\} \in \Sigma$$

$$S\gamma = \{\gamma\alpha | \alpha \in S\} \in \Sigma$$

$$S^{-1} = \{\alpha^{-1}\alpha \in S\} \in \Sigma.$$

And a measure μ on Σ such that *(a)* $\mu(\Gamma) = 1$

(b) $\mu(\gamma S) = \mu(S\gamma) = \mu(S^{-1}) = \mu(S)\ \forall\ \gamma \in \mathcal{F}, S \in \Sigma$

(c) We may associate to any measure on μ *on* Γ *a bounded linear functional* $E : L^1(\Gamma, \Sigma, \mu) \to R$ *by*

$$E(F) = \int_{\Gamma}^{\infty} f(\gamma) d\mu(\gamma). \tag{1}$$

Example, for $\mu(1) = \{e^{i\theta} | \ 0 \leq \theta < tan\pi/2\}$, then;

$$\int_{\mu(1)}^{\infty} f(\gamma) d\mu(\gamma) = \frac{1}{2\pi} \int_{0}^{tan\pi/2} f(e^{i\theta}) d(\theta). \tag{2}$$

The Haar measure was introduced by Alfred Haar in 1933, assigning an invariant volume to subsets of locally compact topological group, consequently defining an integral for functions of those groups. (Haar, 1933).

2.2. 2.2 The Harmonized Dimensional Transform Measure

According to Ying and Lawrence (2008), the average fractal dimension is the P^{th} fractal distribution and the basis of fractal geometry is the idea of self similarity hence fractal dimension average all over the fractal properties. A natural dimensional measure relevant for dynamic system is given as (Zhang, 2011)

$$\mu(\beta) = \lim_{T \to \infty} \frac{1}{T} \int_{o}^{T} I_{\beta}(\varphi_t(X_o)) dt \Rightarrow \frac{1}{\Delta\alpha} \int_{\min}^{\max} F(\alpha) d\alpha, \tag{3}$$

while a harmonized measure is given as(Osu and Ogwo, 2015)

$$F(t) = \lim_{T \to \infty} \frac{1}{T} \int_{o}^{T} I_{\beta}(\varphi_t(t_o))^{\lambda} dt \Rightarrow \frac{1}{\Delta\alpha} \int_{0}^{\infty} (S_T)^{\gamma} dt. \tag{4}$$

Where the evolution operator ϕ_t tells how the state of the system changes with time and it is a family of function ϕ_t : $R^m \to R^m$ that maps the current state of the system into the future state at a time units later and φ_t satisfy $\varphi_0(t) = t$ and $\varphi_{t+s}(t) = \varphi_t(\varphi_s(t))$.
φ_t can be defined either as a discrete map or in terms of ordinary differential equation. The evolution operator α tells how the state of the system changes with time. $\alpha \in (0, 1]$ is the extent of how non- overlapping price changes are uncorrelated at all lead and lag. Here γ it is the maximum displacement parameter which can be defined as the doubling indicator function or the utility function.

2.3. Harmonized Dimensional Measure a Special Case of the Haar Measure

The Harmonised fractal dimensional measure considered as a special case of a Haar measure is given as

$$\int_{\mu(1)}^{\infty} f(\gamma) d\mu(\gamma) = \frac{1}{2\pi} \int_{0}^{\infty} f(e^{i\theta}) d(\theta) = \frac{1}{\Delta\alpha} \int_{0}^{\infty} (S_T)^{\gamma} dt. \tag{5}$$

The specialty is due to the presence of the doubling indicator or utility function γ which is the risk neutral function(that is the principle of risk-neutral pricing. In pricing risk-neutrality, one must change the probability measure to what is called risk-neutral probability and such exists if there are no arbitrage opportunities in market which is a mild assumption). The future return of the asset S_T is the price of the security at time T and γ is calibrated to verify the risk neutral condition hence S_T evolves like a risk neutral asset and $\gamma = \frac{\mu - rc}{\beta} T$ (pricing kernel / the utility function).

2.4. Martingales

In probability theory, a martingale is a stochastic process (sequence of random variables) such that the conditional expected value of an observation at some time t, given all the observations up to some earlier time t, is equal to the observation at that earlier time t. It is a model of a fair wager where one makes profit by doubling the bet, but as the gamblers wealth and available time jointly approach infinity, his probability of eventually flipping head approaches 1. Which make the martingales betting seem like a sure thing. However, the exponential growth of the bets eventually bankrupts its users.

Definition 2.4. *Let $\{X(t), t \in T\}$ and $\{F(t), t \in T\}$ be a stochastic process $\{X(t), t \in T\}$ is said to be a martingale with respect to $\{X(t), t \in T\}$ if for all $t, X(t)$ is measurable with respect to the σ-algebra $\{F(s), s \leq t\}$ generated by the filtration $F(s), s \leq t$ and if in addition we have:*

(a) For each $t, X(t)$ is integrable and $E(X(t)) < +\infty$.

(b)

$$E\{X(t+u)|F(s), s \leq t\} = X(t) \text{ where } u \text{ is an integer.} \tag{6}$$

But if there are conditions or inequality imposed on the definition, it becomes sub-martingales or super martingales. $E\{X(t+u)|F(s), s \leq t\} \geq X(t)$. This is sub-martingale or favorable wager (gain) and $E\{X(t+u)|F(s), s \leq t\} \leq X(t)$, is super martingale(loss). Martingales are forward rates which are the best estimates of prices

$$Y(T,t) = E[X(t, t+T)|F(t)] \to E[X(T-1, t+1)|F(t)] = Y(T,t).$$

And rational expectation can be written by

$$E[Y(T-1, t+1)|F(t)] = E[E[X(t+T)|F(t+1)|F(t)].$$

Therefore

$$E[E[X(t+T)|F(t+1)|F(t)] = E[X(t+T)|F(t)] = Y(T,t).$$

The process $X(t) = exp\{\alpha W(t) - \frac{\alpha^2 t}{2}\}$ is a martingale. We have

$$E[X(t+s)|F(t)] = E\left[exp\left\{\alpha W(t+s) - \frac{\alpha^2(t+s)}{2}\right\}|F(t)\right] = E\left[X(t)exp\left\{\alpha W(t+s) - W(t)\frac{\alpha^2(t+s)}{2}\right\}|F(t)\right] \tag{7}$$

which leads to $E[X(t+s|F(t)] = X(t)$.

2.5. Measurable martingale

Lemma 2.5. *Let $x_{t+1} = x_t + \varepsilon_t, x_0 = 0$, where* $\varepsilon_t = \begin{cases} +1 \ w.p.P \\ 0 \ w.p.r \\ -1 \ w.p.q \end{cases}$ $, p \geq 0, q \geq 0, r \geq 0, p+q+r = 1.$
Then $(x_t - t(p-q); t \geq 0)$ is an $F(t)$ measurable martingale.

$$\begin{aligned} E[x_{t-1} - (t+1)(p-q)F(t)] - E[x_t + \varepsilon_t - (t+1)(p-q)F(t)] &= x_t - (t-1)(p-q) + E(\varepsilon_t) = x_t - (t-1)(p-q) + (p-q) \\ &= x_t - t(p-q) - (p-q) + (p-q) = x_t - t(p-q). \end{aligned}$$

The Weiner process is a martingale.

Proof.

$$E\{W(t+h)|F(t)\} = E\{W(t+h) - W(t)|F(t)\} + E\{W(t)|F(t)\}.$$

And

$$E\{W(t+h) - W(T)|F(t)\} = E\{W(t+h) - W(t)\} = 0$$

$X(t) = W(t)^2$ is a Martingale

$$\begin{aligned} E[X(t+s)F(t)] &= E[W(t+s)^2 F(t)] - (t+s) \\ &= E[\{W(t+s) - W(t)\}^2 + 2W(t+s)W(t) - W(t)^2|F(t)] - (t-s) \\ &= E[\{W(t+s) - W(t)\}^2|F(t)] + 2E[W(t+s)|F(t)] - E[W(t)^2|F(t)] - (t-s) \end{aligned}$$

Due to independence of increments of the filtration F we can write

$$\begin{aligned} E[\{W(t+s) - W(t)\}^2 F(t)] &= E[\{W(t+s)| - W(t)\}^2] = s \\ E[\{W(t+s)W(t)\}|F(t)] &= Z(t)E[W(t+s)|F(t)] \\ E[\{W(t+s)W(t)\}|F(t)] &= W(t)^2 \end{aligned}$$

By conditional expectation, $E[\{W(t)^2\}|F(t)] = W(t)^2$, which lead to

$$E[X(t+s)F(t)] = s + 2W(t)^2 - W(t)^2 - (t+s) = W(t)^2 - t = X(t)$$
$$X(t) = \exp\left\{\alpha W(t) - \frac{\alpha^2 t}{2}\right\}.$$

We have

$$\begin{aligned} E[X(t+s)F(t)] &= E\left[exp\left\{\alpha W(t+s) - \frac{\alpha^2(t+s)}{2}\right\}|F(t)\right] \\ &= E\left[X(t)exp\left\{\alpha W(t+s) - W(t)\frac{\alpha^2(t+s)}{2}\right\}|F(t)\right] \end{aligned}$$

Independence and conditional expectation makes it possible

$$\begin{aligned} E[X(t+s)F(t)] &= X(t)E\left[X(t)\exp\left\{\alpha(W(t+s) - W(t) - \frac{\alpha^2 s}{2}\right\}|F(t)\right] \\ &= X(t)\exp\left(-\frac{\alpha^2 s}{2}\right)E[\exp\{\alpha(W(t+s) - W(t))\}] \end{aligned} \tag{8}$$

Which leads to $E[X(t+s)F(t)] = X(t)$. □

2.6. Market Efficiency

According to Fama (1970), market is efficient if the expected future price equals current price hence efficient / complete markets is equated to the existence of a martingale.

$$E[P(t+T)|F(t)] = P(t), \text{ where } P = X. \tag{9}$$

On the other hand, if the market model be described by a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ and Suppose that trading takes place in continuous time, and that there is one risky security. Let $h > 0, t \geq 0$ and let $r_h(t+h)$ denote the return of security from t to $t+h$, where h is a fixed time lag and let $S(t)$ be the price of the risky security at time t. Also let $\mathcal{F}(t)$ be the collection of historical information available to every market participant at time t. Then the market is weakly efficient if

(a) if $x\mathcal{F}(t)$ exist $\forall\, x \in \mathbb{R}$.

(b) $\mathbb{P}[(r_h(t+h) \leq xF(t)] \;\forall\, x \in\; R.h > 0, t \geq 0$.

And strongly efficient if

(a) if $x\mathcal{F}(t)$ exist $\forall\, x \in R$.

(b) $\mathbb{P}[(r_h(t+h) \geq x\mathbb{F}(t)] \;\forall\, x \in\; R.h > 0, t \geq 0$.

Here, the information $\mathcal{F}(t)$ which is publicly available at time t is nothing other than the generated σ-algebra of the price returns with the random variable x. Where

$$X(t) = \log \frac{S(t+h)}{S(t)}. \tag{10}$$

is the rate of return. Note that Dimension is defined as $\lim\limits_{size \to 0} \frac{\log\ bulk}{\log\ size}$ which is equivalent to the rate of return $X(t) = \log \frac{S(t+h)}{S(t)}$.

3. Market Instruments

Market instruments are defined in terms of other underlying quantities such as stock, indices, currency and interest rates or volatilities. Here $S_t = S(t)$ is the price of underlying asset, subscripts are used to emphasize on evolutions of underlying asset process through time and due to the time dependency, where $h = T - t$ is the remaining time of maturity. S_T is normally distributed $(X \sim N(\mu, \sigma^2))$ if it has a density

$$\begin{aligned} S_T = S(t, \mu, \sigma, \pi) &= \frac{1}{\sqrt{2\pi\sigma}} e^{(-\frac{(x-\mu)^2}{2\sigma^2})}, \quad -\infty < x < \infty, \\ &= \frac{1}{\sqrt{2\pi\sigma}} e^{(-\frac{(InS_t)^2}{4\theta})}. \end{aligned} \tag{11}$$

Where, $lnS_t = x - \mu$ and $\theta = 2$ where $\sigma^2 = 4$. The cumulative distribution function for which there is a close form expression is $X(t) = P(X \leq x) =$ where $\Phi(z)$ is the probability that a standard normal random variable is less than or equal to z.
A given pay off function $g : R^+ \to R$ written as, can be regarded as a financial instrument that pays the holder $g(S(T))$ at expiry T for all $S \in R^+$. We then take S_T to be

$$S_T = S_t e^{((\mu - q - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t))} \;\; \forall\, t \in [0, T]. \tag{12a}$$

Where $(T - t) = h$ and $W_T - W_t = \beta$. So that equation (12a) reduces to

$$S_T = S(\mu, \sigma.q, h, \beta) = S_t e^{\left((\mu - q - \frac{1}{2}\sigma^2)h + \sigma(\beta)\right)} \;\; \forall\, t \in [0, T]. \tag{12b}$$

Here, $S(t)$ is the price of the risky security at time t, μ is the appreciation rate of price or riskless interest rate, q is the dividend yield and $\sigma > 0$ is the volatility and it is well known that the logarithm of S grows linearly in the long-run. The Radon Nicodyn derivative of S_T is given as

$$S_T = e^{-\left(\frac{1}{2}\frac{(\mu-q)^2}{\sigma^2}T - \frac{\mu-q}{\sigma}\beta\right)} \;\; \forall\, t \in [0, T]. \tag{13}$$

Let

$$S_t = \begin{cases} aS_{t-1} & \text{w.p. } \frac{1-b}{a-b} \\ bS_{t-1} & \text{w.p. } \frac{a-1}{a-b} \end{cases}.$$

Then

$$
\begin{aligned}
E(S_{t+1}|S_t, S_{t-1}, \ldots, S_0) &= E(S_{t+1}|S_t) = S_t \\
E(S_{t+1}|S_t) &= aS_t\frac{1-b}{a-b} + bS_t\frac{a-1}{a-b} \\
&= S_t\frac{b(1-b)+a(1-b)}{a-b} = S_t \text{ is a Martingale.}
\end{aligned} \tag{14}
$$

4. Applications

Sequel to the martingale, the following theorem and there proofs are thus

Theorem 4.1. *Consider a market where the risk neutral asset price $S_t, t \in [0,T]$ is governed by a HFDM with initial value $S_t \in (0,\infty)$. Let $F(t)$ follow a Gaussian distribution and a fractal function $(S_T)^\gamma$ then the optimal strategy has the fractal law distribution as;*

$$
\left(\frac{\left(\frac{\left(InS_t\frac{b(1-b)+a(1-b)}{a-b}\right)^2}{4\theta}\right)^{1-\gamma}}{\gamma\left(\sqrt{2\pi\sigma}\right)^\gamma\Delta\alpha}\right) \geq -\frac{\left(InS_t\frac{b(1-b)+a(1-b)}{a-b}\right)^2}{4\theta},
$$

which is sub- Martingale.

Proof. By equations (4) and (11) we have

$$
\begin{aligned}
\mathcal{F}(t) &= \frac{1}{\Delta\alpha}\int_0^\infty (S_T)^\gamma dT = E\{X(t+u)|F(s), s \leq t\} \\
&= \frac{1}{\Delta\alpha}\int_0^\infty \left(\frac{1}{\sqrt{2\pi\sigma}}e^{\left(-\frac{(InS_t)^2}{4\theta}\right)}\right)^\gamma dx.
\end{aligned}
$$

Let $x = \frac{(InS_t)^2}{4\theta}$, then it follows that

$$
\begin{aligned}
\mathcal{F}(t) &= \frac{1}{\Delta\alpha}\int_0^\infty \left(\frac{1}{\sqrt{2\pi\sigma}}e^{-x}\right)^\gamma dx \\
&= \frac{1}{\gamma(\sqrt{2\pi\sigma})^\gamma\Delta\alpha}x^{1-\gamma} \\
&= \frac{\left(\frac{(InS_t)^2}{4\theta}\right)^{1-\gamma}}{\gamma(\sqrt{2\pi\sigma})^\gamma\Delta\alpha},
\end{aligned} \tag{15}
$$

using $X(t) = \log\frac{S(t+h)}{S(t)} = -\frac{(InS_t)^2}{4\theta}$. Here, the information $\mathcal{F}(t)$ which is publicly available at time t is nothing other than the generated σ-algebra of the price returns. If $E\{X(t+u)|F(s), s \leq t\} \geq X(t)$. This is sub-martingale or favorable wager (gain) and $E\{X(t+u)|F(s), s \leq t\} \leq X(t)$, is super martingale (loss). $X(t)$ is assumed to be today's rate of return. Hence $\left(\frac{\left(\frac{(InS_t)^2}{4\theta}\right)^{1-\gamma}}{\gamma(\sqrt{2\pi\sigma})^\gamma\Delta\alpha}\right) \geq -\frac{(InS_t)^2}{4\theta}$ a. s. a sub-martingale.

Substituting equation (14) into (15) gives;

$$
\left(\frac{\left(\frac{\left(InS_t\frac{b(1-b)+a(1-b)}{a-b}\right)^2}{4\theta}\right)^{1-\gamma}}{\gamma(\sqrt{2\pi\sigma})^\gamma\Delta\alpha}\right) \geq -\frac{(InS_t\frac{b(1-b)+a(1-b)}{a-b})^2}{4\theta}.
$$

Where $\left(\frac{\left(\frac{\left(InS_t\frac{b(1-b)+a(1-b)}{a-b}\right)^2}{4\theta}\right)^{1-\gamma}}{\gamma(\sqrt{2\pi\sigma})^\gamma\Delta\alpha}\right)$, is the variance of risk neutral returns. $\square$

Theorem 4.2. *Let $F(t)$ follow a one dimensional Brownian motion and a fractal function $(S_T)^\gamma$ then the optimal strategy has the fractal law distribution as;*

$$-\frac{\left(S_t\frac{b(1-b)+a(1-b)}{a-b}\right)^\gamma}{\gamma\left(\left(\mu-q-\frac{1}{2}\sigma^2\right)h+\sigma(\beta)\right)^{\gamma-1}\Delta\alpha}\leq\left(\left(\mu-q-\frac{1}{2}\sigma^2\right)h+\sigma(\beta)\right),$$

which is super martingale.

Proof. From Equation (4) and (12b) we have

$$F(t)=\frac{1}{\Delta\alpha}\int_0^\infty(S_te^x)^\gamma dx=E\{X(t+u)|F(s),s\leq t\}$$

Let $x=((\mu-q-\frac{1}{2}\sigma^2)h+\sigma(\beta))$ it follows that

$$\begin{aligned}\mathcal{F}(t)&=-\frac{(S_t)^\gamma}{\gamma x^{\gamma-1}\Delta\alpha}\\&=-\frac{(S_t)^\gamma}{\gamma\left(\left(\mu-q-\frac{1}{2}\sigma^2\right)h+\sigma(\beta)\right)^{\gamma-1}\Delta\alpha}\end{aligned}\tag{16}$$

using

$$X(t)=\left(\left(\mu-q-\frac{1}{2}\sigma^2\right)h+\sigma(\beta)\right)\tag{12}$$

If $E\{X(t+u)|F(s),s\leq t\}\geq X(t)$. This is sub-martingale or favorable wager (gain) and $E\{X(t+u)|F(s),s\leq t\}\leq X(t)$, is super martingale (loss).

$$-\frac{(S_t)^\gamma}{\gamma\left(\left(\mu-q-\frac{1}{2}\sigma^2\right)h+\sigma(\beta)\right)^{\gamma-1}\Delta\alpha}\leq\left(\left(\mu-q-\frac{1}{2}\sigma^2\right)h+\sigma(\beta)\right)\text{ a. s. a super-martingale.}$$

Substituting (14) into (16). We have

$$-\frac{\left(S_t\frac{b(1-b)+a(1-b)}{a-b}\right)^\gamma}{\gamma\left(\left(\mu-q-\frac{1}{2}\sigma^2\right)h+\sigma(\beta)\right)^{\gamma-1}\Delta\alpha}\leq\left(\left(\mu-q-\frac{1}{2}\sigma^2\right)h+\sigma(\beta)\right).$$

Where

$$-\frac{\left(S_t\frac{b(1-b)+a(1-b)}{a-b}\right)^\gamma}{\gamma\left(\left(\mu-q-\frac{1}{2}\sigma^2\right)h+\sigma(\beta)\right)^{\gamma-1}\Delta\alpha},$$

is the variance of risk neutral returns. □

Theorem 4.3. *Let $F(t)$ follow a Radon Nicodyn distribution and a fractal function $(S_T)^\gamma$ then the optimal strategy has the fractal law distribution as;*

$$\frac{1}{\gamma\left(\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}T-\frac{\mu-r}{\sigma}\beta\right)^{\gamma-1}\Delta\alpha}\geq-\left(\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}T-\frac{\mu-r}{\sigma}\beta\right)a.s.$$

which is sub-martingale.

Proof. From Equation (4) and (13) we have

$$\mathcal{F}(t)=\frac{1}{\Delta\alpha}\int_0^\infty(e^{-x})^\gamma dx=E\{X(t+u)|\mathcal{F}(s),s\leq t\}.$$

Let $x = \left(\frac{1}{2}\frac{(\mu-q)^2}{\sigma^2}T - \frac{\mu-q}{\sigma}\beta\right)$, it follows that

$$\begin{aligned}\mathcal{F}(t) &= \frac{1}{\gamma x^{\gamma-1}\Delta\alpha}.\\ &= \frac{1}{\gamma\left(\frac{1}{2}\frac{(\mu-q)^2}{\sigma^2}T - \frac{\mu-q}{\sigma}\beta\right)^{\gamma-1}\Delta\alpha}\end{aligned} \tag{17}$$

using

$$X(t) = \log\frac{S(t+h)}{S(t)} = -\left(\frac{1}{2}\frac{(\mu-q)^2}{\sigma^2}T - \frac{\mu-q}{\sigma}\beta\right). \tag{18}$$

If $E\{X(t+u)|F(s), s \leq t\} \geq X(t)$. This is sub-martingale or favorable wager (gain) and $E\{X(t+u)|F(s), s \leq t\} \leq X(t)$, is super martingale (loss).

Hence $\frac{1}{\gamma\left(\frac{1}{2}\frac{(\mu-q)^2}{\sigma^2}T-\frac{\mu-q}{\sigma}\beta\right)^{\gamma-1}\Delta\alpha} \geq -\left(\frac{1}{2}\frac{(\mu-q)^2}{\sigma^2}T - \frac{\mu-q}{\sigma}\beta\right)$ a.s which is sub-martingale. And $\frac{1}{\gamma\left(\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}T-\frac{\mu-r}{\sigma}\beta\right)^{\gamma-1}\Delta\alpha}$. is the variance of risk neutral returns. □

The proofs have established the fact in Proposition 4.4 below

Proposition 4.4. *Let $X(t), t \in T$ be a martingale with respect to the filtration $\mathcal{F}(t)$ and let Φ be a negative function; then the process $\{\Phi(X(t)\}$ is a Sub- martingale with respect to $\mathcal{F}(t)$. This is a direct outcome of a Jensen's inequality: $E\{\Phi(X(t+1))|F(t)\} \geq \Phi\{E(X(t+1))|F(t)\} = \Phi(X(t)$.*
But if Φ be a positive function; then the process $\{\Phi(X(t)\}$ is a super-martingale with respect to $\mathcal{F}(t)$. $E\{\Phi(X(t+1))|F(t)\} \leq \Phi\{E(X(t+1))|\mathcal{F}(t)\} = \Phi(X(t)$.

5. Conclusion

Using HFDM in presuming a certain behavior of market we observed that when the $X(t)$ is negative, doubling it results to gain but when positive the reverse becomes the case. Meaning that, doubling the exponential growth of a positive bet (wealth) eventually bankrupts the investor and doubling the negative results to gain (sub-martingale) which is a direct outcome of a Jensen inequality (Proposition 4.4) connotes that the measure is a martingale as it deals with wealth distribution that are highly skewed,curbs investment by neutralizing risky assets and diverting the wealth to consumption.

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