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On Non-Vacuum Static Spherically Symmetric Solution in $f(R)$ Theory of Gravity in V_5

Research Article

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Abstract: In this paper, we studied non-vacuum static spherically symmetric solution of the field equation in metric gravity using dust matter and constant Ricci scalar curvature in five-dimensions. The density of dust matter and the function of the Ricci scalar curvature will also be evaluated at constant scalar curvature.

Keywords: $f(R)$ theory of gravity, Spherically symmetric solutions, non-vacuum solution, higher dimension.

1. Introduction

The $f(R)$ gravity is an extension of Einstein's General Relativity derived from relaxing the hypothesis that the Hilbert-Einstein action for the gravitational field is strictly linear in the Ricci curvature scalar R, i.e. $f(R) = R$. In this sense, $f(R)$ gravity represents a class of theories defined as arbitrary functions of R. It can be considered as the simplest example of Extended Theory of Gravity (Capozziello and De Laurentis, 2011). The study of the solutions in $f(R)$ theory of gravity is an important source of inspiration for all reserachers in the field of general theory of relativity. In $f(R)$ theory of gravity there are two approaches to find out the solutions of modified Einstein's field equations. The first approach is called metric approach and second one is known as Palatini formalism.

The vacuum solutions of the field equations in metric $f(R)$ gravity has attracted many reseachers. Since the spherically symmetry plays a fundamental role in understanding the nature of gravity, most of the solutions are discussed in this context. The static spherically symmetric vacuum solutions of the field equation in $f(R)$ theory of gravity have been ibtained by Multamaki and Vilja (2006).

It is shown that solution with constant scalar curvature corresponds to Schwarzschild de Sitter spacetime for a specific choice of constants of integration. Carames and Bezerra (2009) discussed spherically symmetric vacuum solutions in higher dimensions. Capozziello et al. (2007) analyzed spherically symmetric solution using Noether symmetry.

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In a past few years there have been many attempts to construct a unified field theory based on the idea of multidimensional space-time. The idea that space-time should be extended from four to higher dimension was introduced by Kaluza and Klein (1921, 26) to unify gravity and electromagnetism. Several aspects of five dimensional space-time have been studied in different theories by many authors [Wesson (1983, 84), Reddy D.R.K. (1999), Khadekar et al. (2001), Ghosh and Dadhich (2001), Adhao (1994),etc.]. Recently Sharif and Kausar (2011) have studied non vacuum static spherically symmetric solutions of the field equations in $f(R)$ theory of gravity in the presence of dust fluid in four-dimensional space-time.

In this note we attempt to solve five dimensional field equations in $f(R)$ theory of gravity using metric approach with constant scalar curvature and obtain non-vacuum static spherically symmetric solutions in the presence of dust fluid. The density ρ of dust matter and the Ricci scalar curvature function $f(R)$ will also be evaluated at constant scalar curvature.

The paper is organized as follows. In section 2, we present spherically symmetric field equations and some of the relevant quantities. Section 3 is devoted to study the non-trivial solution of the field equations.

2. Field Equations in $f(R)$ Gravity

The field equations in $f(R)$ theory of gravity in V_5 are given by

$$F(R)R_{ij} - \frac{1}{2}f(R)g_{ij} - \nabla_i\nabla_j F(R) + g_{ij}\Box F(R) = kT_{ij}, \quad (i, j = 1, 2, 3, 4, 5) \tag{1}$$

where $F(R) \equiv \frac{df(R)}{dR}$, $\Box \equiv \nabla^i\nabla_i$ with ∇_i representing the covariant derivative, $k(= 8\pi)$ is the coupling constant in gravitational units and T_{ij} is the standard matter energy momentum tensor. Taking trace of above equation, we obtained

$$F(R) - \frac{5}{2}f(R) + 4\Box F(R) = 8\pi T \tag{2}$$

Here R and T are related differentially and not algebraically as in general relativity ($R = T$). This indicates that the field equations of $f(R)$ gravity will admit a larger variety of solutions than does general relativity. Further, $T = 0$ does no longer implies $R = 0$ in this theory. The Ricci scalar curvature function $f(R)$ can expressed in terms of its derivatives as under

$$f(R) = \frac{2}{5}\left(-8\pi T + F(R)R + 4\Box F(R)\right) \tag{3}$$

Substituting this value of $f(R)$ in in (1.1), we obtain

$$\frac{1}{5}\left[F(R)R - \Box F(R) - 8\pi T\right] = \frac{1}{g_{ij}}\left[F(R)R_{ij} - \nabla_i\nabla_j F(R) - 8\pi T_{ij}\right] \tag{4}$$

In the above equation, the expression on the left hand side is independent of the index i, so the field equations can be written as

$$A_i = \frac{1}{g_{ij}}\left[F(R)R_{ij} - \nabla_i\nabla_j F(R) - 8\pi T_{ij}\right] \tag{5}$$

Notice that A_i is not a 5-vector rather just a notation for the traced quantity.

3. Spherically symmetric space-time V_5

Pokley et. al. (2008) have obtained the spherically symmetric line element in V_5 in a narrow sense by transformation method given as,

$$ds^2 = -Adr^2 - B(d\theta^2 + \sin^2\theta\ d\phi^2) + C\ dt^2 - D\ du^2 \tag{6}$$

where A, B, C and D are functions of r, t, u and $x^i = (r, \theta, \phi, t, u)$ and V_5 is of the type (-,-,-,+,-). We take the following five-dimensional static spherically symmetric space-time

$$ds^2 = -A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta\ d\phi^2) + C(r)\ dt^2 - D(r)\ du^2 \tag{7}$$

where A, C and D are the function of radial coordinate r. The components of the Ricci tensor are

$$R_{11} = \frac{1}{4}\left(\frac{4A'}{rA} + \frac{A'C'}{AC} + \frac{A'D'}{AD} + \frac{C'^2}{C^2} - \frac{2C''}{C} + \frac{D'^2}{D^2} - \frac{2D''}{D}\right) \tag{8}$$

$$R_{22} = \frac{1}{2}\left(2 + \frac{rA'}{A^2} - \frac{2}{A} - \frac{rC'}{AC} - \frac{rD'}{AD}\right) \tag{9}$$

$$R_{33} = sin^2\theta R_{22} \tag{10}$$

$$R_{44} = \frac{1}{4}\left(-\frac{A'C'}{A^2} - \frac{C'^2}{AC} + \frac{4C'}{rA} + \frac{C'D'}{AD} + \frac{2C''}{A}\right) \tag{11}$$

$$R_{55} = \frac{1}{4}\left(\frac{A'D'}{A^2} - \frac{4D'}{rA} - \frac{C'D'}{AC} + \frac{D'^2}{AD} - \frac{2D''}{A}\right) \tag{12}$$

The corresponding Ricci scalar is

$$R = \frac{1}{2r^2A}\left[4 - 4A - \frac{4r^2A'C'}{AC} - \frac{4rA'}{A} - \frac{r^2D'}{AD} - \frac{r^2C'^2}{C^2} + \frac{r^2C'D'}{CD} + \frac{4rC'}{C} + \frac{2r^2C''}{C} - \frac{r^2D'^2}{D^2} + \frac{4rD'}{D} + \frac{2r^2D''}{D}\right] \tag{13}$$

where prime denotes derivative with respect to the radial coordinate r. The dust energy-momentum tensor is given as $T_{ij} = \rho u_i u_j$, where $u_i = \delta_i^5$ is five-velocity in co-moving ccordinates and ρ is the density of dust matter. Since Eq.(2.4) is independent of index i, so $A_i - A_j = 0$ for all i, j, from (2.5) we have $A_5 - A_1 = 0, A_5 - A_2 = 0, A_5 - A_3 = 0$ and $A_5 - A_4 = 0$. We get following three independent equations

$$-\frac{F''(R)}{A} + \frac{F'(R)}{2A}\left[\frac{A'}{A} + \frac{D'}{D}\right] + \frac{F(R)}{4A}\left[\frac{4D'}{rD} + \frac{C'D'}{CD} + \frac{4A'}{rA} + \frac{A'C'}{AC} + \frac{C'^2}{C^2} - \frac{2C''}{C}\right] - \frac{8\pi\rho}{D} = 0 \tag{14}$$

$$\frac{F'(R)}{A}\left[\frac{D'}{2D} - \frac{1}{r}\right] + \frac{F(R)}{4A}\left[-\frac{A'D'}{AD} + \frac{2D'}{rD} + \frac{C'D'}{CD} - \frac{D'^2}{D^2} + \frac{2D''}{D} + \frac{4A}{r^2} + \frac{2A'}{rA} - \frac{4}{r^2} - \frac{2C'}{rC}\right] - \frac{8\pi\rho}{D} = 0 \tag{15}$$

$$\frac{F'(R)}{2A}\left[\frac{D'}{D} - \frac{C'}{C}\right] + \frac{F(R)}{4A}\left[-\frac{A'D'}{AD} + \frac{4D'}{rD} - \frac{D'^2}{D^2} + \frac{2D''}{D} + \frac{A'C'}{AC} + \frac{C'^2}{C^2} - \frac{4C'}{rC} - \frac{2C''}{C}\right] - \frac{8\pi\rho}{D} = 0 \tag{16}$$

Thus we get a system of three non-linear differential equations with five unknown functions, namely, $F(r), \rho(r), A(r), C(r)$ and $D(r)$.

4. Solution of the Field Equations

In this section we discuss the solution of the field equations by assuming constant scalar curvature and thereby obtain the non-trivial solution of field equations in V_5. The conservation law of energy-momentum tensor, $T^j_{i;j} = 0$, for dust matter gives

$D=$ constant $= D_0$ (say). Thus the system of field equations (3.8) to (3.10) is reduced to four unknowns $F(r), \rho(r), A(r)$ and $C(r)$ with the following three non-linear differential equations

$$-\frac{F''(R)}{A}+\frac{F'(R)A'}{2A^2}+\frac{F(R)}{4A}\left[\frac{4A'}{rA}+\frac{A'C'}{AC}+\frac{C'^2}{C^2}-\frac{2C''}{C}\right]-\frac{8\pi\rho}{D_0}=0 \tag{17}$$

$$-\frac{F'(R)}{rA}+\frac{F(R)}{4A}\left[\frac{4A}{r^2}+\frac{2A'}{rA}-\frac{4}{r^2}-\frac{2C'}{rC}\right]-\frac{8\pi\rho}{D_0}=0 \tag{18}$$

$$-\frac{F'(R)C'}{2AC}+\frac{F(R)}{4A}\left[\frac{A'C'}{AC}+\frac{C'^2}{C^2}-\frac{4C'}{rC}-\frac{2C''}{C}\right]-\frac{8\pi\rho}{D_0}=0 \tag{19}$$

Using the assumption of constant scalar curvature ($R = R_0$), i.e., $F(R_0) =$ constant, the field equations become

$$\frac{F(R)}{4A}\left[\frac{4A'}{rA}+\frac{A'C'}{AC}+\frac{C'^2}{C^2}-\frac{2C''}{C}\right]-\frac{8\pi\rho}{D_0}=0 \tag{20}$$

$$\frac{F(R)}{4A}\left[\frac{4A}{r^2}+\frac{2A'}{rA}-\frac{4}{r^2}-\frac{2C'}{rC}\right]-\frac{8\pi\rho}{D_0}=0 \tag{21}$$

$$\frac{F(R)}{4A}\left[\frac{A'C'}{AC}+\frac{C'^2}{C^2}-\frac{4C'}{rC}-\frac{2C''}{C}\right]-\frac{8\pi\rho}{D_0}=0 \tag{22}$$

Now we have three differential equations with three unknowns, $A(r), C(r)$ and $\rho(r)$. Subtracting Eq.(4.6) from Eq.(4.4) yields $A(r)C(r) = a$, where a is constant. Again subtracting Eq.(4.5) from Eq.(4.6) and using $A(r)C(r) = a$, we get

$$r^2C'' - 2C = 2a \tag{23}$$

which has the following solution $C(r) = \frac{b_1}{r} + b_2r^2 + a$ and hence

$$A(r) = a\left(\frac{b_1}{r}+b_2r^2+a\right)^{-1} \tag{24}$$

where b_1 and b_2 are constants. Using these values of $A(r)$ and $C(r)$, we have density of dust matter as

$$\rho = -\frac{3b_2D_0F(R_0)}{8a\pi} = \rho_0 = \text{constant}. \tag{25}$$

The space-time for constant curvature solution takes the following form

$$ds^2 = -a\left(\frac{b_1}{r}+b_2r^2+a\right)^{-1}dr^2 - r^2(d\theta^2+\sin^2\theta\, d\phi^2)+\left(\frac{b_1}{r}+b_2r^2+a\right)dt^2 - D_0du^2 \tag{26}$$

This is the required solution. Using (4.8) the scalar curvature becomes $R_0 = \frac{12b_2}{a}$ and therefore Eq.(2.3) yields

$$f(R_0) = \frac{2}{5}\left[-\frac{8\pi\rho_0}{D_0}+F(R_0)R_0\right] \tag{27}$$

Putting the value of ρ_0 and R_0 in Eq.(4.12), we have Ricci scalar curvature function $f(R)$ such that

$$f(R_0) = \frac{18b_2}{5a}f'(R_0). \tag{28}$$

5. Conclusion

We have investigated five dimensional dust static spherically symmetric non-vaccume solutions (4.10) in $f(R)$ theory of gravity with the assumption of constant scalar curvature. From metric (4.10), we observed that, becouse of presence of the term $b_2 r^2$, metric does not become asymptotically flat as $r \to \infty$. This behaves like a contribution that comes from cosmological constant if Einstein theory was considered with cosmological constant, that is, $R_i^j - \frac{1}{2}\delta_i^j R = -\Lambda\delta_i^j$. This contribution is small.

In particular, if $a = 1, b_1 = -2M$ and $b_2 = 0$. This follows the metric $ds^2 = -\left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \left(1 - \frac{2M}{r}\right) dt^2 - D_0 du^2$. Above form of metric is Schwarzscild like metric in V_5. Where M is the total mass of sphere given by $M = 4\pi \int\limits_0^{R_b} \rho(r) r^2 dr$, where R_b is the radius of the fluid sphere. If $b_1 << r << \frac{1}{\sqrt{b_2}}$, the metric (4.10) is nearly flat. The scalar curvatue for the solution (4.10) obtained as non-zero constant which leads to constant density of dust matter.

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