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Radius of p-valent Strong Starlikeness for Certain Class of Analytic Functions

Research Article

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Abstract: This paper deals with p-valent strongly starlikeness of the class $SP(\alpha, A, B)$ satisfying the subordination

$$e^{i\alpha}\frac{zf'(z)}{f(z)} \prec cos\ z\frac{1+Az}{1+Bz} + i\ sin\ \alpha,$$

$f \in A$, $z \in \Delta$, $0 \leq \alpha < 1$, $-1 \leq B < A \leq 1$. We are concerned with computing the radius results for the above mentioned class and the results that we obtained are generalizations of earlier results obtained previously by different authors.

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1. Introduction

Let A denote the class of all functions $f(z)$ analytic functions $f(z)$ defined on the open unit disk $\Delta = \{z; |z| < 1\}$ normalized by $f(0) = 0$ and $f'(0) = 1$.

Let S denote the subclass of A consisting of univalent functions in Δ. Let A_p be the class of functions $f(z) = z^p + \sum_{n=k+p}^{\infty} a_n z^n$, $p \geq 1$ which are analytic and p-valent in the unit disk Δ. Also let $SP(\alpha, A, B)$ denote the class of functions in A satisfying the subordination condition

$$e^{i\alpha}\frac{zf'(z)}{f(z)} \prec cos\ z\frac{1+Az}{1+Bz} + i\ sin\ \alpha, \tag{1}$$

$z \in \Delta$, $0 \leq \alpha < 1$, $-1 \leq B < A \leq 1$.

Gangadharan et al. [1] obtained radius of strongly starlikeness of functions in $SP(\alpha, A, B)$, $ST[A, B]$ and some more classes of functions.

Motivated by earlier works, we compute the radii of strongly starlikeness of order γ for some other class of functions.

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2. Radius of p-valent Strongly Starlikeness

To prove our main results, we need the following lemmas.

Lemma 2.1 ([1]). *If $Ra \leq (Re\ a)sin\left(\frac{\pi\gamma}{2}\right) - (Im\ a)cos\left(\frac{\pi\gamma}{2}\right)$, $Im\ a > 0$, the disc $|w - a| \leq Ra$ is contained in the sector $|arg\ w| \leq \frac{\pi\gamma}{2}$, $0 < \gamma \leq 1$.*

Lemma 2.2 ([1]). *For $|z| \leq r < 1$, $|z_k| = R > r$, $\left|\frac{z}{z-z_k} + \frac{r^2}{R^2-r^2}\right| \leq \frac{Rr}{R^2-r^2}$*

Lemma 2.3 ([1]). *Suppose $g \in SP(\alpha, A, B)$, then,*

$$\left|\frac{zg'(z)}{g(z)} - \left[\frac{1 - B[(A-B)e^{i\alpha}cos\alpha + B]r^2}{1 - B^2r^2}\right]\right| \leq \frac{(A-B)r\ cos\alpha}{1 - B^2r^2}, \quad for\ |z| = r < 1.$$

Lemma 2.4 (MacGregor [2]). *If $Re\ p(z) > 0$ and $p(z) = 1 + c_nz^n + \ldots$ then $\left|\frac{zp'(z)}{p(z)}\right| \leq \frac{2nr^n}{1-r^{2n}}$.*

Lemma 2.5 (Ratti [3]). *If $\phi(z)$ is analytic in U and $|\phi(z)| \leq 1$, then for $|z| = r < 1$ $\left|\frac{z\phi'(z)+\phi(z)}{1+z\phi(z)}\right| \leq \frac{1}{1-r}$.*

Theorem 2.6. *Suppose $F(z) = f(z)[Q(z)]^{\beta/n}$ where β is real and $Q(z)$ is a polynomial of degree $n > 0$ with no zeros in $|z| < R$, $R \geq 1$ and if $f \in A_p$ satisfies $Re\left(\frac{f(z)}{g(z)}\right)^{1/\delta} > 0$, $0 < \delta \leq 1$, $z \in \Delta$ for some $g \in SP(\alpha, A, B)$, then $F(z)$ is p-valent strongly starlike in $|z| < R_1$ where R_1 is the smallest root of the equation*

$$\left.\begin{aligned}
&r^6\{-B[(A-B)cos^2\alpha + B]sin\left(\tfrac{\pi\gamma}{2}\right) - B(A-B)\ sin\ \alpha\ cos\ \alpha\ cos\left(\tfrac{\pi\gamma}{2}\right)\\
&\qquad\qquad -\beta B^2 sin\tfrac{\pi\gamma}{2}\}\\
&+r^5\{-2\delta B^2 - |\beta|RB^2 - (A-B)\ cos\alpha\}\\
&+r^4\{B[(A-B)cos^2\alpha + B]sin\tfrac{\pi\gamma}{2}(1+R^2) + \beta(1+B^2)sin\left(\tfrac{\pi\gamma}{2}\right)\\
&\qquad +B(A-B)\ sin\ \alpha\ cos\ \alpha\ cos\left(\tfrac{\pi\gamma}{2}\right)(1+R^2) + sin\left(\tfrac{\pi\gamma}{2}\right)\}\\
&+r^3\{2\delta(1+B^2R^2) + |\beta|R(1+B^2) + (A-B)(1+R^2)\ cos\alpha\}\\
&+r^2\{-B[(A-B)cos^2\alpha + B]R^2 sin\left(\tfrac{\pi\gamma}{2}\right) - \beta sin\left(\tfrac{\pi\gamma}{2}\right)\\
&\qquad -B(A-B)\ sin\ \alpha\ cos\ \alpha\ cos\left(\tfrac{\pi\gamma}{2}\right)R^2 - (1+R^2)sin\left(\tfrac{\pi\gamma}{2}\right)\}\\
&+r\{-2\delta R^2 - |\beta|R - (A-B)R^2\ cos\alpha\} + R^2 sin\left(\tfrac{\pi\gamma}{2}\right)
\end{aligned}\right\} = 0$$

Proof. Let $p(z) = \left(\frac{f(z)}{g(z)}\right)^{\frac{1}{\delta}} \in P$ then by the conditions given in the theorem,

$$F(z) = p(z)^{\delta} g(z)[Q(z)]^{\beta/n} \tag{2}$$

Suppose $Q(z) = a_0 \prod_{k=1}^{n}(z - z_k)$ where z_k's are the roots of $Q(z)$ such that $|z_k| \geq R$ for $1 \leq k \leq n$ then,

$$z\frac{Q'(z)}{Q(z)} = \sum_{k=1}^{n} \frac{z}{z - z_k} \tag{3}$$

Taking the logarithmic differentiation $F(z)$ yields

$$z\frac{F'(z)}{F(z)} = \delta z\frac{p'(z)}{p(z)} + z\frac{g'(z)}{g(z)} + \frac{\beta}{n} z\frac{Q'(z)}{Q(z)} \tag{4}$$

Using (3) in (4), we get

$$z\frac{F'(z)}{F(z)} = \delta z\frac{p'(z)}{p(z)} + z\frac{g'(z)}{g(z)} + \frac{\beta}{n}\sum_{k=1}^{n}\frac{z}{z - z_k}$$

Using Lemmas 2.4, 2.3 and 2.2 we get

$$\left|z\frac{F'(z)}{F(z)}-\left[\frac{1-B[(A-B)e^{i\alpha}cos\ \alpha+B]r^2}{1-B^2r^2}+\frac{\beta r^2}{R^2-r^2}\right]\right| \leq \frac{2\delta r}{1-r^2}+\frac{(A-B)r\ cos\ \alpha}{1-B^2r^2}+\frac{|\beta|Rr}{R^2-r^2}$$

By Lemma 2.1, the above disk will be contained in the sector $|arg\ w|\leq\frac{\pi\gamma}{2}$ if

$$\begin{aligned}&2\frac{\delta r}{1-r^2}+\frac{(A-B)rcos\alpha}{1-B^2r^2}+\frac{|\beta|Rr}{R^2-r^2}\\ &\leq\left[\frac{1-B[(A-B)cos^2\alpha+B]r^2}{1-B^2r^2}-\frac{\beta r^2}{R^2-r^2}\right]sin\left(\frac{\pi\gamma}{2}\right)\\ &\qquad-\frac{B[(A-B)\ sin\ \alpha\ cos\ \alpha]r^2}{1-B^2r^2}cos\left(\frac{\pi\gamma}{2}\right)\end{aligned}$$

is satisfied. This reduces to $\chi_1(r)\geq 0$ where

$$\chi_1(r)=\begin{cases} r^6\{-B[(A-B)cos^2\alpha+B]sin\left(\frac{\pi\gamma}{2}\right)-B(A-B)\ sin\ \alpha\ cos\ \alpha\ cos\left(\frac{\pi\gamma}{2}\right)\\ \qquad -\beta B^2 sin\frac{\pi\gamma}{2}\}\\ +r^5\{-2\delta B^2-|\beta|RB^2-(A-B)\ cos\alpha\}\\ +r^4\{B[(A-B)cos^2\alpha+B]sin\frac{\pi\gamma}{2}(1+R^2)+\beta(1+B^2)sin\left(\frac{\pi\gamma}{2}\right)\\ \qquad +B(A-B)\ sin\ \alpha\ cos\ \alpha\ cos\left(\frac{\pi\gamma}{2}\right)(1+R^2)+sin\left(\frac{\pi\gamma}{2}\right)\}\\ +r^3\{2\delta(1+B^2R^2)+|\beta|R(1+B^2)+(A-B)(1+R^2)\ cos\alpha\}\\ +r^2\{-B[(A-B)cos^2\alpha+B]R^2sin\left(\frac{\pi\gamma}{2}\right)-\beta sin\left(\frac{\pi\gamma}{2}\right)\\ \qquad -B(A-B)\ sin\ \alpha\ cos\ \alpha\ cos\left(\frac{\pi\gamma}{2}\right)R^2-(1+R^2)sin\left(\frac{\pi\gamma}{2}\right)\}\\ +r\{-2\delta R^2-|\beta|R-(A-B)R^2\ cos\alpha\}+R^2sin\left(\frac{\pi\gamma}{2}\right)\end{cases}$$

It can be seen that $\chi_1(0)>0$ and $\chi_1(1)=2\delta(1-B^2)(1-R^2)<0$ (since $R>1$). Therefore there exists a real root of $\chi_1(r)=0$ in the interval $(0,1)$. If R_1 is the smallest positive root of $\chi_1(r)=0$ in $(0,1)$ then $F(z)$ is p-valent strongly starlike in $|z|<R_1$. □

Theorem 2.7. *Suppose $F(z)$ is as in Theorem 2.6 and $f\in A_p$ satisfies $\left|\left(\frac{f(z)}{g(z)}\right)^{\frac{1}{\delta}}-1\right|<1$, $0<\delta\leq 1$ and $Re\left(\frac{g(z)}{h(z)}\right)>0$, $z\in\Delta$, for some $g\in A_p$ and $h\in SP(\alpha,A,B)$, the $F(z)$ is p-valent strongly starlike for $|z|<R_2$, when R_2 is the smallest positive root of the equation*

$$\left.\begin{aligned} &r^6\Big\{-B[(A-B)cos^2\alpha+B]sin\left(\tfrac{\pi\gamma}{2}\right)-\beta B^2 sin\left(\tfrac{\pi\gamma}{2}\right)\\ &\qquad +B(A-B)\ sin\ \alpha\ cos\ \alpha\ cos\left(\tfrac{\pi\gamma}{2}\right)\Big\}\\ &+r^5\Big\{-2B^2-(A-B)\ cos\ \alpha+|\beta|RB^2-\beta B^2\Big\}\\ &+r^4\Big\{sin\tfrac{\pi\gamma}{2}+B(1+R^2)[(A-B)cos^2\alpha+B]sin\left(\tfrac{\pi\gamma}{2}\right)+\beta(1+B^2)sin\left(\tfrac{\pi\gamma}{2}\right)\\ &\qquad -B(A-B)(1+R^2)\ sin\ \alpha\ cos\ \alpha\ cos\left(\tfrac{\pi\gamma}{2}\right)\Big\}\\ &+r^3\Big\{2(1+B^2R^2)+(A-B)(1+R^2)\ cos\ \alpha-|\beta|R(1+B^2)+\beta(1+B^2R^2)\Big\}\\ &+r^2\Big\{-sin\left(\tfrac{\pi\gamma}{2}\right)(1+R^2)-BR^2[(A-B)cos^2\alpha+B]sin\left(\tfrac{\pi\gamma}{2}\right)-\beta sin\left(\tfrac{\pi\gamma}{2}\right)\\ &\qquad +B(A-B)R^2\ sin\ \alpha\ cos\ \alpha\ cos\left(\tfrac{\pi\gamma}{2}\right)\Big\}\\ &+r\Big\{-2R^2-R^2(A-B)\ cos\ \alpha+|\beta|R-\beta R^2\Big\}+R^2 sin\left(\tfrac{\pi\gamma}{2}\right)\end{aligned}\right\}=0$$

Proof. Choosing that branch of $\left(\frac{f(z)}{g(z)}\right)^{\frac{1}{\delta}}$ so that it is analytic in Δ and its value at $z=0$ is 1, it is seen that $F(z)=g(z)(1+w(z))^{\delta}$ where $w(z)$ is a Schartz function, therefore $F(z)=p(z)h(z)(1+z\phi(z))^{\delta}Q(z)^{\beta/n}$, where $Q(z)$ is analytic in Δ and satisfies $|\phi(z)|<1$ for $z\in\Delta$. By a simple computation,

$$z\frac{F'(z)}{F(z)}=z\frac{p'(z)}{p(z)}+z\frac{h'(z)}{h(z)}+\delta\left[\frac{z\phi'(z)+\phi(z)}{1+z\phi(z)}\right]+\frac{\beta}{n}\sum_{k=1}^{n}\frac{z}{z-z_k}$$

Proceeding as in Theorem 2.6, by applying Lemmas 2.2, 2.3, 2.4 and 2.5

$$\begin{aligned}&\left|z\frac{F'(z)}{F(z)}-\frac{1-B[(A-B)e^{i\alpha}cos\ \alpha+B]r^2}{1-B^2r^2}-\frac{\beta r^2}{R^2-r^2}\right|\\&\leq\frac{2r}{1-r^2}+\frac{(A-B)r\ cos\ \alpha}{1-B^2r^2}-\frac{|\beta|Rr}{R^2-r^2}+\frac{Br}{1-r^2}\end{aligned}$$

Application of Lemma 2.1 gives that the above disk will be contained in the sector $|arg\ w|\leq\frac{\pi\gamma}{2}$ if

$$\begin{aligned}&\left(\frac{2r}{1-r^2}+\frac{(A-B)r\ cos\ \alpha}{1-B^2r^2}-\frac{|\beta|Rr}{R^2-r^2}+\frac{\beta r}{1-r^2}\right)\\&\leq\left\{\frac{[1-B[(A-B)cos^2\alpha+B]r^2]}{1-B^2r^2}-\frac{\beta r^2}{R^2-r^2}\right\}sin\left(\frac{\pi\gamma}{2}\right)\\&\qquad\qquad+\frac{r^2B(A-B)sin\ \alpha\ cos\alpha}{1-B^2r^2}cos\left(\frac{\pi\gamma}{2}\right)\end{aligned}$$

is satisfied. This reduces to $\chi_2(r)\geq 0$, where

$$\chi_2(r)=\begin{cases}r^6\Big\{-B[(A-B)cos^2\alpha+B]sin\left(\frac{\pi\gamma}{2}\right)-\beta B^2sin\left(\frac{\pi\gamma}{2}\right)\\\qquad\qquad+B(A-B)\ sin\ \alpha\ cos\ \alpha\ cos\left(\frac{\pi\gamma}{2}\right)\Big\}\\+r^5\Big\{-2B^2-(A-B)\ cos\ \alpha+|\beta|RB^2-\beta B^2\Big\}\\+r^4\Big\{sin\frac{\pi\gamma}{2}+B(1+R^2)[(A-B)cos^2\alpha+B]sin\left(\frac{\pi\gamma}{2}\right)+\beta(1+B^2)sin\left(\frac{\pi\gamma}{2}\right)\\\qquad\qquad-B(A-B)(1+R^2)\ sin\ \alpha\ cos\ \alpha\ cos\left(\frac{\pi\gamma}{2}\right)\Big\}\\+r^3\Big\{2(1+B^2R^2)+(A-B)(1+R^2)\ cos\ \alpha-|\beta|R(1+B^2)+\beta(1+B^2R^2)\Big\}\\+r^2\Big\{-sin\left(\frac{\pi\gamma}{2}\right)(1+R^2)-BR^2[(A-B)cos^2\alpha+B]sin\left(\frac{\pi\gamma}{2}\right)-\beta sin\left(\frac{\pi\gamma}{2}\right)\\\qquad\qquad+B(A-B)R^2\ sin\ \alpha\ cos\ \alpha\ cos\left(\frac{\pi\gamma}{2}\right)\Big\}\\+r\Big\{-2R^2-R^2(A-B)\ cos\ \alpha+|\beta|R-\beta R^2\Big\}+R^2sin\left(\frac{\pi\gamma}{2}\right)\end{cases}$$

As $\chi_2(0)=R^2sin\left(\frac{\pi\gamma}{2}\right)$ is positive and $\chi_2(1)=(1-B^2)(2+\beta)(1-R^2)$ is negative (since $R>1$) there exists a real root of $\chi_2(r)=0$ in the interval $(0,1)$. If R_2 is the smallest positive root of $\chi_2(r)=0$ in $(0,1)$ then $F(z)$ is p-valent strongly starlike in $|z|<R_2$. □

Theorem 2.8. *Suppose $F(z)$ is as in Theorem 2.6 and if $f\in A_p$ satisfies $\left|\left(\frac{f(z)}{g(z)}\right)^{\frac{1}{\delta}}-1\right|<1$, $0<\delta\leq 1$ for some $g\in SP(\alpha,A,B)$, then $F(z)$ is p-valent strongly starlike in $|z|<R_3$ where R_3 is the smallest positive root of the equation*

$$\left.\begin{aligned}&r^6\{-B[(A-B)cos^2\alpha+B]sin\left(\tfrac{\pi\gamma}{2}\right)-\beta B^2sin\left(\tfrac{\pi\gamma}{2}\right)\\&\qquad\qquad+B(A-B)cos\left(\tfrac{\pi\gamma}{2}\right)sin\ \alpha\ cos\ \alpha\}\\&+r^5\{-(A-B)\ cos\alpha-|\beta|B^2R-B^2\beta\}\\&+r^4\{sin\tfrac{\pi\gamma}{2}+B[(A-B)cos^2\alpha+B](1+R^2)sin\left(\tfrac{\pi\gamma}{2}\right)+\beta(1+B^2)sin\left(\tfrac{\pi\gamma}{2}\right)\\&\qquad\qquad-B(A-B)(1+R^2)cos\left(\tfrac{\pi\gamma}{2}\right)sin\ \alpha\ cos\ \alpha\}\\&+r^3\{(A-B)(1+R^2)\ cos\alpha+|\beta|(1+B^2)R+\beta(1+B^2R^2)\}\\&+r^2\{-(1+R^2)sin\left(\tfrac{\pi\gamma}{2}\right)-B[(A-B)cos^2\alpha+B]R^2sin\left(\tfrac{\pi\gamma}{2}\right)-\beta sin\left(\tfrac{\pi\gamma}{2}\right)\\&\qquad\qquad+B(A-B)R^2sin\ \alpha\ cos\ \alpha\ cos\left(\tfrac{\pi\gamma}{2}\right)\}\\&+r\{-(A-B)R^2\ cos\alpha-|\beta|R-\beta R^2\}+R^2sin\left(\tfrac{\pi\gamma}{2}\right)\end{aligned}\right\}=0$$

Proof. Choose the same branch of $\left(\frac{f(z)}{g(z)}\right)^{\frac{1}{\delta}}$ as in the earlier theorem. Since $f\in A_p$ satisfies

$$\left|\left(\frac{f(z)}{g(z)}\right)^{\frac{1}{\delta}}-1\right|<1$$

for some $g(z) \in SP(\alpha, A, B)$, $F(z) = g(z)[1+z\phi(z)]^{\delta}[Q(z)]^{\beta/n}$ where $\phi(z)$ is analytic in Δ and satisfies $|\phi(z)| \leq 1$ for $z \in \Delta$. A simple computation gives

$$z\frac{F'(z)}{F(z)} = z\frac{g'(z)}{g(z)} + \delta\left[\frac{[z\phi'(z)+\phi(z)]}{1+z\phi(z)}\right] + \frac{\beta}{n}\sum_{k=1}^{n}\frac{z}{z-z_k}$$

Applying Lemma 2.5, 2.3 and 2.2,

$$\left|z\frac{F'(z)}{F(z)} - \frac{1-B[(A-B)e^{i\alpha}cos\ \alpha+B]r^2}{1-B^2r^2} - \frac{\beta r^2}{R^2-r^2}\right| \leq \frac{(A-B)r\ cos\ \alpha}{1-B^2r^2} + \frac{\beta r}{1-r^2} + \frac{|\beta|Rr}{R^2-r^2}$$

By applying Lemma 2.1, the above disk will be contained in the sector $|arg\ w| \leq \frac{\pi\gamma}{2}$ if

$$\frac{(A-B)r\ cos\ \alpha}{1-B^2r^2} + \frac{|\beta|Rr}{R^2-r^2} + \frac{\beta r}{1-r^2} \leq \left\{\frac{[1-B[(A-B)cos^2\alpha+B]r^2]}{1-B^2r^2} - \frac{\beta r^2}{R^2-r^2}\right\}sin\left(\frac{\pi\gamma}{2}\right) + \frac{B(A-B)r^2 sin\ \alpha\ cos\alpha}{1-B^2r^2}cos\left(\frac{\pi\gamma}{2}\right)$$

which reduces to

$$\left.\begin{aligned}
&r^6\{-B[(A-B)cos^2\alpha+B]sin\left(\tfrac{\pi\gamma}{2}\right) - \beta B^2 sin\left(\tfrac{\pi\gamma}{2}\right) \\
&\qquad +B(A-B)cos\left(\tfrac{\pi\gamma}{2}\right)sin\ \alpha\ cos\ \alpha\} \\
&+r^5\{-(A-B)\ cos\alpha - |\beta|B^2R - B^2\beta\} \\
&+r^4\{sin\tfrac{\pi\gamma}{2} + B[(A-B)cos^2\alpha+B](1+R^2)sin\left(\tfrac{\pi\gamma}{2}\right) + \beta(1+B^2)sin\left(\tfrac{\pi\gamma}{2}\right) \\
&\qquad -B(A-B)(1+R^2)cos\left(\tfrac{\pi\gamma}{2}\right)sin\ \alpha\ cos\ \alpha\} \\
&+r^3\{(A-B)(1+R^2)\ cos\alpha + |\beta|(1+B^2)R + \beta(1+B^2R^2)\} \\
&+r^2\{-(1+R^2)sin\left(\tfrac{\pi\gamma}{2}\right) - B[(A-B)cos^2\alpha+B]R^2 sin\left(\tfrac{\pi\gamma}{2}\right) - \beta sin\left(\tfrac{\pi\gamma}{2}\right) \\
&\qquad +B(A-B)R^2 sin\ \alpha\ cos\ \alpha\ cos\left(\tfrac{\pi\gamma}{2}\right)\} \\
&+r\{-(A-B)R^2\ cos\alpha - |\beta|R - \beta R^2\} + R^2 sin\left(\tfrac{\pi\gamma}{2}\right)
\end{aligned}\right\} \geq 0$$

On taking the left hand side of this inequality as $\chi_3(r)$, it is seen that $\chi_3(0) = R^2 sin\left(\frac{\pi\gamma}{2}\right) > 0$ and $\chi_3(1) = \beta(1-B^2)(1-R^2) < 0$ (since $R > 1$) and hence there exists a positive real root of $\chi_3(r) = 0$ in $(0, 1)$. Let R_3 be the least such root. Then $F(z)$ is p-valent strongly starlike in $|z| < R_3$. □

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