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Connected Boundary Domination in Graphs

Research Article

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Abstract: Let $G = (V; E)$ be a connected graph. A dominating set $S \subseteq V$ is called a connected dominating set if $\langle S \rangle$ is connected subgraph. In this paper we introduce the connected boundary dominating set and analogous to the connected boundary domination number, we define the connected boundary domatic number in graphs. Exact values of some standard graphs are obtained and some other interesting results are established.

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1. Introduction

The graph $G = (V; E)$ we mean a finite, undirected and connected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [1] and Haynes et.al [3].

Let $v \in V$. The open neighborhood and the closed neighborhood of v are denoted by $N(v)$ and $N[v] = N(v) \cup \{v\}$ respectively. If $S \subseteq V$, then $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. Sampathkumar E. and Walikar H.B. [7] introduced the concept of connected domination in graphs.

A dominating set S of G is called a connected dominating set if the induced subgraph $\langle S \rangle$ is connected the minimum cardinality of a connected dominating set of G is called the connected domination number of G and is denoted by $\gamma_c(G)$. A partition $P = V_1, V_2, ..., V_l$ of a vertex set $V(G)$ of a graph is called connected domatic partition of G if V_i is connected dominating set for every $1 \leq i \leq l$. The connected domatic number of G is the maximum cardinality of connected domatic partition of G and denoted by $d_c(G)$.

Let G be a simple graph $G = (V, E)$ with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$. For $i \neq j$, a vertex v_i is a boundary vertex of v_j if $d(v_j, v_t) \leq d(v_j, v_i)$ for all $v_t \in N(v_i)$ [2]. A vertex v is called a boundary neighbor of u if v is a nearest boundary of u. If $u \in V$, then the boundary neighbourhood of u denoted by $N_b(u)$ is defined as $N_b(u) = \{v \in V : d(u, w) \leq d(u, v)$ for all $w \in N(u)\}$.

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The cardinality of $N_b(u)$ is denoted by $deg_b(u)$ in G. The maximum and minimum boundary degree of a vertex in G are denoted respectively by $\Delta_b(G)$ and $\delta_b(G)$. That is $\Delta_b(G) = \max_{u\in V} |N_b(u)|$, $\delta_b(G) = \min_{u\in V} |N_b(u)|$. A vertex u boundary dominate a vertex v if v is a boundary neighbor of u.

A subset S of $V(G)$ is called a boundary dominating set if every vertex of $V - S$ is boundary dominated by some vertex of S. The minimum taken over all boundary dominating sets of a graph G is called the boundary domination number of G and is denoted by $\gamma_b(G)$. KM. Kathiresan, G. Marimuthu and M. Sivanandha Saraswathy [4] introduced the concept of Boundary domination in graphs. Puttaswamy and Mohammed Alatif [6] introduced the concept of Boundary edge domination in graphs. We need the following theorems.

Theorem 1.1 ([4]).

(a). For any path P_n , $n \geq 3$, $\gamma_b(P_n) = n - 2$.

(b). For any complete graph K_n, $n \geq 4$, $\gamma_b(K_n) = 1$.

(c). For any complete bipartition graph $K_{m,n}$, $m, n \geq 2$, $\gamma_b(K_{m,n}) = 2$.

Theorem 1.2 ([5]).

(a). For any path graph P_n,

$$\gamma_c(P_n) = \begin{cases} n-2 & \text{if } n \geq 3, \\ 1 & \text{otherwise.} \end{cases}$$

(b). For any cycle C_n , $\gamma_c(C_n) = n - 2$.

(c). For a complete bipartition graph $K_{m,n}$, $\gamma_c(K_{m,n}) = \min(m, n)$.

Theorem 1.3 ([7]). *For any graph G, $n \geq 3$, $\gamma_c(G) \leq n - 2$.*

Theorem 1.4 ([5]). *For any connected graph G of order n, $\gamma_c(G) \leq n - \Delta$.*

2. Connected Boundary Domination In Graphs

Definition 2.1. *A boundary dominating set S of a connected graph G is called the connected boundary dominating set (cb-set) if the induced subgraph $\langle S\rangle$ of G is connected. The minimum cardinality of a cb-set is called the connected boundary domination number (cb-number) and is denoted by $\gamma_{cb}(G)$.*

We supposed that G is connected because if the graph has more than one component the boundary dominating set has at least one vertex from every component of G and then $\langle S\rangle$ is not connected, and conversely if G has a minimum connected boundary dominating set S and hence connected boundary number then $\langle S\rangle$ is connected that means G is connected according to that we state the following observation.

Theorem 2.2. *A connected boundary dominating set exist for a graph G if and only if G is connected.*

Example 2.3. *In Figure 1, $\{v_2, v_4, v_7\}$ is the minimum dominating set of G, $\{v_2, v_3, v_7, v_9\}$ is the minimum connected dominating set of G, $\{v_5, v_9\}$ is the minimum boundary dominating set of G and $\{v_1, v_2, v_{10}\}$ is the minimum connected boundary dominating set of G, then $\gamma(G) = 3$, $\gamma_c(G) = 4$, $\gamma_b(G) = 2$, and $\gamma_{cb}(G) = 3$.*

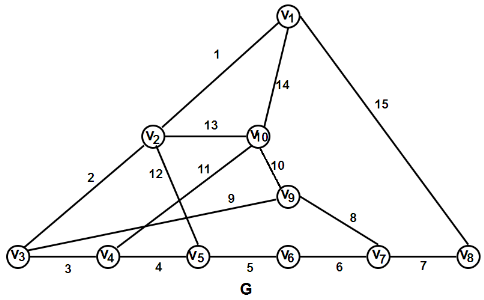

Figure 1. G

In the following Theorem the proof is straightforward from the definition of the connected boundary domination number of a graph.

Theorem 2.4.

(1). $\gamma_{cb}(K_n) = 1$.

(2). $\gamma_{cb}(W_n) = 1$.

(3). $\gamma_{cb}(P_n) = \begin{cases} 1 & \text{if } n = 3, \\ 2 & \text{if } n = 4, 5, \\ n-4 & \text{if } n \geq 6. \end{cases}$.

(4). $\gamma_{cb}(C_n) = \begin{cases} 1 & \text{if } n = 3, \\ 2 & \text{if } n = 4, 5, \\ n-4 & \text{if } n \geq 6. \end{cases}$.

Observation 2.1. *For any connected graph G with $n \geq 3$,*

(1). $\gamma_{cb}(G) = 1$ *if and only if* $\gamma_b(G) = 1$

(2). $\gamma_{cb}(G) \leq \gamma_c(G) \leq n - 2$.

Theorem 2.5. *For any graph G , $\gamma_{cb}(G) = 2$ if and only if $G \cong K_{m,n}$ or $B_{m,n}; m, n \geq 2$.*

Proof. Let $G \cong K_{m,n}$ and let (V_1, V_2) be the bipartition of $K_{m,n}$ with $V_1 = \{v_1, v_2, ..., v_m\}$ and $V_2 = \{u_1, u_2, ..., u_n\}$. Let $v_i \in V_1$. Then $d(v_i, v_j) = 2$ for all $v_j \in V_1 - \{v_i\}; i \neq j$ and every vertex v in V_1 is a boundary neighbour of v_i except v_i. Similarly if $u_i \in V_2$, then every vertex of $V_2 - \{u_i\}$ is a boundary neighbour of u_i except u_i. Then $S = \{v_i, u_i\}$, v_i is adjacent to u_i and $\langle S \rangle = S$ is connected. Hence $\gamma_{cb}(G) = 2$.

Conversely, suppose $\gamma_{cb}(G) = 2$ let S denote the set of all connected boundary dominating of G such that $|S| = 2$, then S contains two vertices say $\{v_i, u_i\}$, since $d(v_i, u_i) \leq d(v_i, v_j)$ and $d(u_i, v_i) \leq d(u_i, u_j)$ for all $i \neq j$ and $u_i \in N(v_i), v_j \in N_b(v_i)$, $v_j \in N(u_i), u_j \in N_b(u_i)$, then there exist (V_1, V_2) such that $v_i) \in V_1, u_i) \in V_2, N_b(v_i) = V_1 - \{v_i\}$ and $N_b(u_i) = V_2 - \{u_i\}$. Hence $G \cong K_{m,n}$. Similarly we can prove that if $G \cong B_{m,n}$. □

Corollary 2.6. *If G is a star graph $K_{1,n}$ then $\gamma_{cb}(G) = 1$.*

Theorem 2.7. *If G is a tree graph then $\gamma_{cb}(G) = n - n_1$.*

Proof. Let V_1 be the set of all pendant vertices of the tree T of order n_1. Then every vertex in $V - V_1$ has a boundary neighbour in V_1 and $S = V - V_1$. Since $V - V_1$ is a path of order $n - n_1$ then S is connected graph and $\langle S \rangle$ is connected. Hence $\gamma_{cb}(G) = n - n_1$. □

Theorem 2.8. *For any connected graph* G, $\gamma_b(G) \leq \gamma_{cb}(G)$.

Proof. From the definition of the connected boundary dominating set of a graph G, it is clearly that for any graph G any connected boundary dominating set S is also a boundary dominating set. Hence $\gamma_b(G) \leq \gamma_{cb}(G)$. □

Theorem 2.9 ([4]). *For any connected graph G with n vertices,* $\lceil \frac{n}{1+\Delta_b(G)} \rceil \leq \gamma_b(G) \leq n - \Delta_b(G)$.

The following Theorem is straightforward from the definition of the connected boundary domination.

Theorem 2.10. *For any graph G with n vertices,*

(1). $\gamma_{cb}(G) = n$ *if and only if G has n boundary isolated vertices.*

(2). $\gamma_{cb}(G) = 1$ *if there exist at least one vertex $v \in G$, such that $deg_b(v) = n - 1$.*

Theorem 2.11. *For any (n, m) connected graph G with $\delta_b \geq 1$,*

$$\lceil \frac{n}{1 + \Delta_b(G)} \rceil \leq \gamma_{cb}(G) \leq 2m - n$$

.

Proof. Let G be any (n, m) connected graph, then by Observation 2.5.(ii) we have $\gamma_{cb}(G) \leq n - 2$, then $\gamma_{cb}(G) \leq n - 2 = 2(n-1) - n \leq 2m - n$. And from Theorem 2.9 we have $\gamma_b(G) \leq \gamma_{cb}(G)$ and by Theorem 2.10 we get $\lceil \frac{n}{1+\Delta_b(G)} \rceil \leq \gamma_{cb}(G)$. Hence

$$\lceil \frac{n}{1 + \Delta_b(G)} \rceil \leq \gamma_{cb}(G) \leq 2m - n.$$

□

Theorem 2.12. *Let G be a graph with n vertices and without any boundary isolated vertices. Then* $\gamma_{cb}(G) \leq n - \Delta_b(G)$.

Theorem 2.13. *For any (n, m)-graph G,* $\gamma_c(G) + \gamma_{cb}(G) \leq n + 1$.

Proof. Let $v \in V(G)$, then $N_c(v) \cup N_{cb}(v) \cup \{v\} = V$, $|N_c(v)| + |N_{cb}(v)| + 1 = n$ and $\Delta_c + \Delta_{cb} + 1 = n$. But we have $\gamma_c \leq n - \Delta_c$ and $\gamma_{cb} \leq n - \Delta_{cb}$. Therefor $\gamma_c + \gamma_{cb} \leq 2n - (\Delta_c + \Delta_{cb} = 2n - n + 1 = n + 1$. Hence $\gamma_c(G) + \gamma_{cb}(G) \leq n + 1$. □

Theorem 2.14. *Let G be a graph without any boundary isolated vertices and with diameter two. Then* $\gamma_{cb}(G) \leq \delta_b(G) + 1$.

Proof. Let v be any vertex with $deg_b(v) = \delta_b(G)$. Then obviously $N_b[v]$ is connected boundary dominating set and hence $\gamma_{cb}(G) \leq \delta_b(G) + 1$. □

3. Connected Boundary Domatic Number

Definition 3.1. *Let $G = (V, E)$ be a connected graph. The maximal order of partition of the vertices V into connected boundary dominating sets is called the connected boundary domatic number of G and denoted by* $d_{cb}(G)$.

We first determine the connected boundary domatic number of some standard graphs. We observe that

Theorem 3.2.

(1). $d_b(C_n) = \begin{cases} 3 & \text{if } n = 6 \\ 2 & \text{if } n = 4,5,7,8 \\ 1 & \text{if } n \geq 9. \end{cases}$,

(2). $d_{cb}(P_n) = \begin{cases} 2 & \text{if } n = 4, \\ \\ 1 & \text{if } n \geq 5. \end{cases}$.

Theorem 3.3. *For any complete graph K_n , $d_{cb}(K_n) = n$.*

Proof. Let Since $\gamma_{cb}(K_n) = 1$, it follows that $d_{cb}(K_n) \leq \lfloor \frac{n}{\gamma_{cb}(K_n)} \rfloor = n$.
To prove the reverse inequality, let $S_1 = \{v_1\}, S_2 = \{v_2\}, ..., S_n = \{v_n\}$. Clearly $\{S_1, S_2, ..., S_n\}$ is a connected boundary domatic partition of K_n so that $d_{cb}(K_n) \geq n$, Hence $d_{cb}(K_n) = n$. □

Corollary 3.4. *For any wheel graph W_n, $d_{cb}(W_n) = n$.*

Theorem 3.5. *For a complete bipartite graph $K_{m,n}$, $2 \leq m \leq n$, $d_{cb}(K_{m,n}) = m$.*

Proof. Let X, Y be a bipartition of $K_{m,n}$ with $| X |= m$ and $| Y |= n$ and let $m \leq n$. For each $v \in X$, let $S(v)$ denote the set of all boundary vertices with v. Then $\{S(v); v \in X\}$ forms a connected boundary domatic partition of $K_{m,n}$ so that $d_{cb}(K_{m,n}) \geq m$. Further $\delta_b(K_{m,n}) = m - 1$ and hence $d_{cb}(K_{m,n}) \leq \delta_b + 1 = m - 1 + 1 = m$, so that $d_{cb}(K_{m,n}) \leq m$. Thus $d_{cb}(K_{m,n}) = m$. □

Corollary 3.6. *For any star graph $K_{1,n}$, $d_{cb}(K_{1,n}) = n$.*

Proposition 3.7. *For any connected graph G with n vertices,*

(1). $d_{cb}(G) \leq \lfloor \frac{n}{\gamma_{cb}(G)} \rfloor$.

(2). $d_{cb}(G) \leq \delta_b(G) + 1$.

Theorem 3.8. *For any connected graph G with n vertices,*

(1). $d_{cb}(G) \leq d_b(G)$.

(2). $d_c(G) \leq d_{cb}(G)$.

Proof.

Case 1: Since $d_{cb}(G) \leq \lfloor \frac{n}{\gamma_{cb}(G)} \rfloor \leq \frac{n}{\gamma_{cb}(G)}$ and $d_b(G) \leq \lfloor \frac{n}{\gamma_b(G)} \rfloor \leq \frac{n}{\gamma_b(G)}$, from the Theorem 2,9 we have $\gamma_b(G) \leq \gamma_{cb}(G)$ and $\frac{n}{\gamma_{cb}} \leq \frac{n}{\gamma_b(G)}$, then $d_{cb}(G) \leq d_b(G)$.

Case 2: Since $d_{cb}(G) \leq \lfloor \frac{n}{\gamma_{cb}(G)} \rfloor \leq \frac{n}{\gamma_{cb}(G)}$ and $d_c(G) \leq \lfloor \frac{n}{\gamma_c(G)} \rfloor \leq \frac{n}{\gamma_c(G)}$, from the Observation 2.5.(ii) we have $\gamma_{cb}(G) \leq \gamma_c(G)$ and $\frac{n}{\gamma_c} \leq \frac{n}{\gamma_{cb}(G)}$, then $d_c(G) \leq d_{cb}(G)$. □

Theorem 3.9. *For any connected graph G, $\gamma_{cb}(G) + d_{cb}(G) \leq n + 1$ and equality holds if and only if G is isomorphic to K_n or $K_{1,n}$.*

Proof. Since $d_{cb}(G) \leq \delta_b(G) + 1$ and $\gamma_{cb}(G) \leq n - \Delta_b(G)$, we have $\gamma_{cb} + d_{cb} \leq n - \Delta_b + \delta_b + 1 \leq n + 1$. Further $\gamma_{cb} + d_{cb} = n + 1$ if and only if $\gamma_{cb} = n - \Delta_b$ and $d_{cb} = \delta_b + 1$ and $\Delta_b = \delta_b$. We claim that $\gamma_{cb} = 1$. If $\gamma_{cb} \geq 2$, then $d_{cb} \leq \frac{n}{2}$. Since $\gamma_{cb} + d_{cb} = n + 1$, we have $\gamma_{cb} > \frac{n}{2}$. It follows that $d_{cb} = 1$ so that $\gamma_{cb} = n$ which is a contradiction. Hence $\gamma_{cb} = 1$ and $d_{cb} = n$ so that G is isomorphic to K_n or $K_{1,n}$. □

Theorem 3.10. *For any connected graph G, $d_{cb}(G) + d_c(G) \leq 2n$ if and only if $\gamma_{cb}(G) = \gamma_c(G) = 1$*

Proof. Assume that $\gamma_{cb}(G) = \gamma_c(G) = 1$, then $\delta_{cb} = \Delta_{cb} = \delta_c = \Delta_c = n-1$ and $d_{cb}(G) + d_c(G) \leq \delta_{cb} + \delta_c + 2 = 2n - 2 + 2 = 2n$.

Conversely, suppose $d_{cb}(G)+d_c(G) \leq 2n$, since $d_{cb} \leq \lfloor \frac{n}{\gamma_{cb}} \rfloor$ and $d_c \leq \lfloor \frac{n}{\gamma_c} \rfloor$, then $d_{cb} \leq \frac{n}{\gamma_{cb}}$, $d_c \leq \frac{n}{\gamma_c}$ and $d_{cb}+d_c \leq \frac{n}{\gamma_{cb}} + \frac{n}{\gamma_c} = \frac{n(\gamma_{cb}+\gamma_c)}{\gamma_{cb}\gamma_c}$, therefore $\frac{n(\gamma_{cb}+\gamma_c)}{\gamma_{cb}\gamma_c} \leq 2n$ and equality holds if and only if $\gamma_{cb}(G) = \gamma_c(G) = 1$. □

Theorem 3.11. *For any graph G with n vertices, $d_{cb}(G) \geq \lfloor \frac{n}{n-\delta_b(G)} \rfloor$.*

Proof. Assume that $d_{cb}(G) = l$ and $\{S_1, S_2, ..., S_l\}$ is a partition of V into l connected boundary dominating sets, clearly $\mid S_i \mid \geq \gamma_{cb}(G)$ for $i = 1, 2, ..., l$ and we have $n = \sum_{i=1}^{l} \mid S_i \mid \geq l\delta_b(G)$. Hence $d_{cb}(G) \geq \lfloor \frac{n}{n-\delta_b(G)} \rfloor$. □

4. Conclusion

In this paper we computed the exact value of the connected boundary domination number and the connected boundary domatic number for some standard graphs and some special graphs. Also we found some upper and lower bounds for connected boundary domination number and connected boundary domatic number of graph.

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