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$M*$-open Sets in Topological Spaces

Research Article

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Abstract: The aim of this paper is to introduce a new class of open sets called $M*$-Open sets and investigate some properties of these sets in topological spaces.

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1. Introduction

In 1968, N.V.Velicko exhibited and studied some new types of open sets called θ-open sets [12] and δ-open sets [12]. N.Levine in 1963 initiated a new type of open set called semi-open set [6]. In 1993, S.Raychaudhuri and N.Mukherjee defined δ-pre-open sets [10]. In 1997, δ-semi-open sets was obtained by J.H.Park [9], and M.Caldas obtained θ-semi-open sets in 2008 [1]. E.Ekici in 2008 introduced e-open sets [2] and also later in 2008 he introduced a-open sets [3]. In the year 2008 E.Ekici invented e^*-open sets [3]. The notion of M-open sets was introduced by A.I.El.Maghrabi and M.A.Al.Juhani in 2011 [5]. This paper is devoted to introduce and investigate a new class of open set namely $M*$-open sets.

1.1. Preliminaries

Throughout this paper (X,τ) and (Y,σ) (Simply X and Y) represent topological spaces on which no separation axioms are assumed unless or otherwise mentioned. For a subset A of a space (X,τ) the closure of A,the interior of A, and the complement of A are represented by $cl(A)$,$int(A)$, and $X\backslash A$ respectively. A subset A of a space X is said to be regular open [11] if $A = int(cl(A)$. A point $x \in X$ is said to be θ-interior point of A [12] if there exists an open set U containing x such that $U \subseteq cl(U) \subseteq A$. The set of all θ-interior points of A is said to be the θ-interior of A and denoted by $int_\theta(A)$. A subset A of X is said to be θ-open if $A = int_\theta(A)$.

Definition 1.1. *A subset A of X is said to be,*

(1). pre-open if $A \subseteq int(cl(A))$[7].

(2). semi-open if $A \subseteq cl(int(A))$[6].

(3). α-open if $A \subseteq int(cl(int(A)))$.[8]

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(4). θ*-semi-open if* $A \subseteq cl(int_\theta(A))$.[1]

(5). M*-open if* $A \subseteq cl(int_\theta(A) \cup int(cl_\delta(A))$.[5]

Definition 1.2. *The complement of a pre-open(resp. semi-open,* α*-open,* θ*-semi-open,* M*-open) set is called pre-closed(resp. semi-closed,* α*-closed,* θ*-semi-closed,* M*-closed).*

Definition 1.3. *The intersection of all pre-closed(resp. semi-closed,* α*-closed,* θ*-semi-closed,* M*-closed) sets containing* A *is called the pre-closure(resp.semi-closure,* α*-closure,* θ*-semi-closure,* M*-closure) of* A *and is denoted by* $pcl(A)$*(resp.* $scl(A)$, α*-*$cl(A)$, $scl_\theta(A)$, M*cl-(A).).*

Definition 1.4. *The union of all pre-open(resp. semi-open,* α*-open,* θ*-semi-open,* M*-open)sets contained in* A *is called the pre-interior(resp. semi-interior,* α*-interior,* θ*-semi-interior,* M*-interior) of* A *and is denoted by* $pint(A)$*(resp.* $sint(A)$, α*-*$int(A)$, $sint_\theta(A)$, M*-int(A).).*

Lemma 1.5 ([5])**.**

(1). A *is open if and only if* $A = int_\theta(A)$*.*

(2). $int_\theta(A)$ *is the union of all* θ*-open sets of* X *whose closures are contained in* A*.*

(3). For any subset A *of* X $A \subseteq Cl(A) \subseteq Cl_\delta(A) \subseteq Cl_\theta(A)$*(resp.* $int_\theta(A) \subseteq int_\delta(A) \subseteq int(A) \subseteq A$*).*

(4). $int_\theta(A \cap B) = int_\theta(A) \cap int_\theta(B)$*.* $int_\theta(A) \cup int_\theta(B) \subseteq int_\theta(A \cup B)$*.*

(5). $Cl_\theta(A \cap B) = Cl_\theta(A) \cap Cl_\theta(B)$*.* $Cl_\theta(A \cup B) = Cl_\theta(A) \cup Cl_\theta(B)$*.*

Lemma 1.6 ([5])**.** *Let* A *be a subset of a space* (X, τ)*. Then the following statements are hold.*

(1). $pint(\delta\text{-}pcl(A) = \delta\text{-}pcl(A) \cap int(cl(A))$ *and*

$pcl(\delta\text{-}pint(A) = \delta\text{-}pint(A) \cap cl(int(A))$

(2). $pint_\theta(\delta\text{-}pcl(A) = \delta\text{-}pcl(A) \cap int(cl_\theta(A))$ *and*

$pcl_\theta(\delta\text{-}pint(A) = \delta\text{-}pint(A) \cap cl(int_\theta(A))$

(3). $scl_\theta(int_\theta(A)) = scl(int_\theta(A)) = int(cl(int_\theta(A)))$

(4). $sint_\theta(cl_\theta(A)) = sint(cl_\theta(A)) = cl(int(cl_\theta(A)))$

2. M*-open Sets

Definition 2.1. *Let* (X, τ) *be topological space. Then a subset* A *of a space* (X, τ) *is said to be,*

(1). an M***-open** *set if* $A \subseteq int(cl(int_\theta(A)))$*.*

(2). an M***-closed** *set if* $A \supseteq cl(int(cl_\theta(A)))$*.*

Lemma 2.2. *Let* A *be a subset of a space* (X, τ)*. Then the following statements are hold:*

(1). Every θ*-open set is an* M**-open set.*

(2). Every M**-open set is a* θ*-semi-open set.*

(3). Every M-open set is an M-open set.*

Proof.

(1). Let A be an θ-open set. Then $A = int_\theta(A)$ and by Lemma 1.5 $int_\theta(A) \subseteq int(A) \subseteq A$. Hence, $A = int(A)$. Since $A = int_\theta(A) \subseteq cl(int_\theta(A))$, then $A = int(A) \subseteq int(cl(int_\theta(A)))$. Thus A is M*-open.

(2). Obvious from the definition.

(3). Let A be M*-open. Then $A \subseteq int(cl(int_\theta(A))) \subseteq cl(int_\theta(A)) \subseteq cl(int_\theta(A)) \cup int(cl_\delta(A))$.

Hence A is an M-open set. □

But the converse of the above results (2) and (3) need not be true as shown by the following examples.

Example 2.3. *Let $X = \{a, b, c\}$ with topology $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then $A = \{a, c\}$ is an M-open set and θ-semi-open-set but it is not M*-open.*

Lemma 2.4. *Let (X, τ) be a topological space. Then the following statements are hold:*

(1). The arbitrary union of M-open sets is M*-open.*

(2). The arbitrary intersection of M-closed sets is M*-closed.*

Proof. (1). Let $\{A_i : i \in I\}$ be a family of M*-open sets. Then $A_i \subseteq int(cl(int_\theta(A_i))$ for all $i \in I$. Then, $\cup_{i\in I} A_i \subseteq \cup_{i\in I} int(cl(int_\theta(A_i))) \subseteq int(cl(int_\theta(\cup_{i\in I} A_i)))$. Hence $\cup_{i\in I} A_i$ is M*-open. □

Lemma 2.5. *For a topological space (X, τ) the family of all M*-open sets of X forms a topology denoted by τ_{M^*} for X.*

Proof. It is obvious that X,ϕ are in $M^*O(X)$ and from Lemma 2.5 we've arbitrary union of M*-open sets is M*-open. Let A and B be M*-open sets. Then, $A \subseteq int(cl(int_\theta(A)))$ and $B \subseteq int(cl(int_\theta(B)))$. And hence,

$$\begin{aligned} A \cap B &\subseteq int(cl(int_\theta(A))) \cap int(cl(int_\theta(B))) \\ &\subseteq int(cl(int_\theta(A)) \cap cl(int_\theta(B))) \\ &\subseteq int(cl(int_\theta(A) \cap int_\theta(B))) \\ &\subseteq int(cl(int_\theta(A \cap B))) \end{aligned}$$

Hence the finite intersection of M*-open sets is M*-open and hence τ_{M^*} is a topology for X. □

Definition 2.6. *Let A be a subset of a space (X, τ). Then,*

(1). The intersection of all M-closed sets containing A is called the M*-closure of A and is denoted by M*-cl(A).*

(2). The union of all M-open sets contained in A is called the M*-interior of A and is denoted by M*-int(A).*

Theorem 2.7. *The following hold for a subset of a space (X, τ):*

(1). A is M-open if and only if $A = A \cap int(cl(int_\theta(A)))$.*

(2). A is M-closed if and only if $A = A \cup cl(int(cl_\theta(A)))$.*

Proof. (1). Let A be an M*-open. Then $A \subseteq int(cl(int_\theta(A)))$. Hence $A \cap int(cl(int_\theta(A))) = A$. Conversely let $A = A \cap int(cl(int_\theta(A)$. Then, $A \subseteq int(cl(int_\theta(A)))$. Hence A is M*-open. □

Theorem 2.8. *The following hold for a subset of a space* (X, τ)*:*

(1). M*-$int(A) = A \cap int(cl(int_\theta(A)))$

(2). M*-$cl(A) = A \cup cl(int(cl_\theta(A)))$

Proof. (1). Since M*-int(A) is M*-open, $M^* - int(A) \subseteq int(cl(int_\theta(M^* - int(A)))) \subseteq int(cl(int_\theta(A)))$ Also, $A \cap M^* - int(A) \subseteq A \cap int(cl(int_\theta(A)))$. Hence, $M^* - int(A) \subseteq A \cap int(cl(int_\theta(A)))$.
Conversely since,

$$\begin{aligned} int(cl(int_\theta(A \cap int(cl(int_\theta(A)))))) &\supseteq int(cl(int_\theta(A \cap int_\theta(cl(int_\theta(A)))))) \\ &\supseteq int(cl(int_\theta(A) \cap int_\theta(cl(int_\theta(A))))) \\ &\supseteq int(cl(int_\theta(A) \cap int_\theta(int_\theta(A)))) \\ &= int(cl(int_\theta(A \cap int_\theta(A)))) \\ &= int(cl(int_\theta(A))) \\ &= A \cap int(cl(int_\theta(A))) \end{aligned}$$

Hence, $int(cl(int_\theta(A \cap int(cl(int_\theta(A)))))) \supseteq A \cap int(cl(int_\theta(A)))$. This implies that, $A \cap int(cl(int_\theta(A)))$ is an M*-open set contained in A. Hence, $A \cap int(cl(int_\theta(A))) \subseteq M^* - int(A)$. Therefore M*-$int(A) = A \cap int(cl(int_\theta(A)))$. □

Theorem 2.9. *For a subset A of a topological space* (X, τ)*,*

(1). A is an M-open set if and only if $A = M$*-$int(A)$.*

(2). A is an M-closed set if and only if $A = M$*-$cl(A)$.*

Theorem 2.10. *Let A and B be subsets of a space (X, τ). Then the following are hold:*

(1). M*-$cl(X \backslash A) = X \backslash M$*-$int(A)$

(2). M*-$int(X \backslash A) = X \backslash M$*-$cl(A)$

(3). If $A \subseteq B$ then M-$cl(A) \subseteq M$*-$cl(B)$ and M*-$int(A) \subseteq M$*-$int(B)$.*

(4). M*-$cl(M$*-$cl(A)) = M$*-$cl(A)$ *and* M*-$int(M$*-$int(A)) = M$*-$int(A)$.

(5). M*-$cl(A) \cup M$*-$cl(B) \subseteq M$*-$cl(A \cup B)$ *and* M*-$int(A) \cup M$*-$int(B) \subseteq M$*-$int(A \cup B)$.

(6). M*-$cl(A) \cap M$*-$cl(B) \supseteq M$*-$cl(A \cap B)$ *and* M*-$int(A) \cap M$*-$int(B) \supseteq M$*-$int(A \cap B)$.

Proof. (1). By Theorem 2.9,

$$\begin{aligned} M^* - cl(X \backslash A) &= (X \backslash A) \cup (cl(int(cl_\theta(X \backslash A))) \\ &= (X \backslash A) \cup ((X \backslash int(cl(int_\theta(A))))) \\ &= X \backslash (A \cap int(cl(int_\theta(A)))) \\ &= X \backslash M^* - int(A) \end{aligned}$$

(2)and (3) follows from the definition.

(4). By Theorem 2.9(1),

$$\begin{aligned} M^* - cl(M^* - cl(A)) &= cl(int(cl_\theta(M^* - cl(A) \\ &= cl(int(cl_\theta(A \cup cl(int(cl_\theta(A)))))) \\ &\subseteq cl(int(cl_\theta(A) \cup cl_\theta(int(cl_\theta(A))))) \\ &\subseteq cl(int(cl_\theta(A))) \\ &\subseteq M^* - cl(A) \end{aligned}$$

But M*-$cl(A) \subseteq M$*-$cl(M$*-$cl(A))$.

Hence M*-$cl(A) = M$*-$cl(M$*-$cl(A))$.

(5). By Theorem 2.9(2),

$$\begin{aligned} M^* - cl(A) \cup M^* - cl(B) &= (A \cup cl(int(cl_\theta(A)))) \cup (B \cup cl(int(cl_\theta(B)))) \\ &= (A \cup B) \cup (cl(int(cl_\theta(A))) \cup cl(int(cl_\theta(B)))) \\ &= (A \cup B) \cup cl(int(cl_\theta(A \cup B))) \\ &= M^* - cl(A \cup B) \end{aligned}$$

(6). By Theorem 2.9(2),

$$\begin{aligned} M^* - int(A \cap B) &= (A \cap B) \cap int(cl(int_\theta(A \cap B))) \\ &= (A \cap B) \cap int(cl(int_\theta(A) \cap int_\theta(B))) \\ &\subseteq (A \cap int(cl(int_\theta(A)))) \cap (B \cap int(cl(int_\theta(B)))) \\ &= M^* - int(A) \cap M^* - int(B) \end{aligned}$$

□

Lemma 2.11. *Let A be a subset of a space (X, τ). Then,*

(1). M-$cl(A) = A \cup sint_\theta(cl_\theta(A))$*

(2). M-$int(A) = A \cap scl_\theta(int_\theta(A))$*

Proof. (1). From lemma 1.6(4), $A \cup sint_\theta(cl_\theta(A)) = A \cup (cl(int(cl_\theta(A)))) = M^* - cl(A)$

(2). From Lemma 1.6(3), $A \cap scl_\theta(int_\theta(A)) = A \cap (int(cl(int_\theta(A)))) = M^* - int(A)$ □

Theorem 2.12. *The following are equivalent for a subset A of (X, τ):*

(1). A is an M-open set.*

(2). $A \subseteq scl_\theta(int_\theta(A))$

(3). $scl_\theta(A)) = scl_\theta(int_\theta(A))$

Proof. (1) $\Rightarrow$ (2): Let A be an M*-open set. Then by Theorem 2.10, $A = M$*-$int(A)$. By lemma 2.12, $A = A \cap scl_\theta(int_\theta(A)) \subseteq scl_\theta(int_\theta(A))$. Hence $A \subseteq scl_\theta(int_\theta(A))$.

(2) $\Rightarrow$ (1): Let $A \subseteq scl_\theta(int_\theta(A))$. This implies that $A \subseteq A \cap scl_\theta(int_\theta(A)) = M^* - int(A)$. Hence $A \subseteq M$*-$int(A)$ and hence $A = M$*-$int(A)$ and M*-open.

(2) $\Rightarrow$ (3): Let $A \subseteq scl_\theta(int_\theta(A))$. Then $scl_\theta(A) \subseteq scl_\theta(int_\theta(A))$. But $int_\theta(A) \subseteq A$. Hence $scl_\theta(int_\theta(A)) \subseteq scl_\theta(A))$. Hence $scl_\theta(A)) = scl_\theta(int_\theta(A))$.

(3) $\Rightarrow$ (2): Let $scl_\theta(A)) = scl_\theta(int_\theta(A))$. Then $scl_\theta(A)) \subseteq scl_\theta(int_\theta(A))$. But $A \subseteq scl_\theta(A)$. And therefore $A \subseteq scl_\theta(int_\theta(A))$. □

Theorem 2.13. *Let A be a subset of a space (X, τ). Then the following are equivalent:*

(1). A is an M-closed set.*

(2). $A \supseteq sint_\theta(cl_\theta(A))$

(3). $sint_\theta(A)) = sint_\theta(cl_\theta(A))$

Definition 2.14. *A subset A of a topological space (X, τ) is said to be locally M*-closed if $A = U \cap F$ for each $U \in \tau$ and $F \in M^*C(X)$.*

Theorem 2.15. *Let H be a subset of a space (X, τ). Then H is locally M*-closed if and only if $H = U \cap M$*-$cl(H)$.*

Proof. Let H be an locally M*-closed set. Then $H = U \cap F$ for each $U \in \tau$ and $F \in M^*C(X)$. Hence $H \subseteq M$*-$cl(H) \subseteq M$*-$cl(F) = F$. Thus $U \cap H \subseteq U \cap M$*-$cl(H) \subseteq U \cap M$*-$cl(F) = H$. This implies that $H \subseteq U \cap M$*-$cl(H) \subseteq U \cap M$*-$cl(F) = H$. Hence $H = U \cap M$*-$cl(H)$

Converse is obvious, since M*-$cl(H) \in M$*$C(X)$. □

Theorem 2.16. *Let A be a locally M^*-closed subset of a topological space (X, τ).Then the following are hold:*

(1). M-$cl(A) \backslash A$ is an M*-closed set.*

(2). $(A \cup (X \backslash M$-$cl(A)))$ is an M*-open set.*

(3). $A \subseteq M$-$int(A \cup (X \backslash M$*-$cl(A)))$.*

Definition 2.17. *Let (X, τ) be a topological space and $A \subseteq X$. Then the M*-boundary of A(Briefly M*-$b(A)$) is given by M*-$b(A) = M$*-$cl(A) \cap M$*-$cl(X \backslash A)$.*

Theorem 2.18. *If A is a subset of a space (X, τ) then the following statements hold:*

(1). M-$b(A) = M$*-$b(X \backslash A)$*

(2). M-$b(A) = M$*-$cl(A) \backslash M$*-$int(A)$*

(3). M-$b(A) \cap M$*-$int(A) = \phi$*

(4). M-$b(A) \cup M$*-$int(A) = M$*-$cl(A)$*

Definition 2.19. *Let (X, τ) be a topological space and $A \subseteq X$. Then the set $X \backslash M$*-$cl(A)$ is called the M*-exterior of A and is denoted by M*-$ext(A)$. Each point $p \in X$ is called an M*-exterior point of A if it is a M*-interior point of $X \backslash A$.*

Theorem 2.20. *If A and B are two subsets of a space (X, τ). Then the following statements hold:*

(1). M-$ext(A) = M$*-$int(X \backslash A)$*

(2). M-$ext(A) \cap M$*-$b(A) = \phi$*

(3). M-$ext(A) \cup M$*-$b(A) = M$*-$cl(X \backslash A)$*

(4). $\{M$-$int(A), M$*-$b(A)$, and M*-$ext(A)\}$ form a partition of X*

(5). If $A \subseteq B$ then M-$ext(B) \subseteq M$*-$ext(A)$*

(6). M-$ext(A \cup B) \subseteq M$*-$ext(A) \cup M$*-$ext(B)$*

(7). M-$ext(A \cap B) \supseteq M$*-$ext(A) \cap M$*-$ext(B)$*

(8). M-$ext(\phi) = X$ and M*-$ext(X) = \phi$*

Definition 2.21. *If A is a subset of a space (X, τ). Then a point $x \in X$ is called M*-limit point of a set $A \subseteq X$ if every M*-open set $G \subseteq X$ containing x contains a point other than x.*

Definition 2.22. *The set of all M*-limit points of A is called M*-derived set of A and is denoted by M*-d(A).*

Theorem 2.23. *If A and B are two subsets of a space X. Then the following statements hold:*

(1). If $A \subseteq B$ then M-d$(A) \subseteq M$*-d(B).*

(2). A is an M-closed set if and only if it contains each of its limit points.*

(3). M-cl$(A) = A \cup M$*-d(A).*

Definition 2.24. *A subset A of a space (X, τ) is said to be a M*-neighbourhood(Briefly M*-nbd) of a point $p \in X$ if there exists an M*-open set W such that $p \in W \subseteq N$. The class of M*-neighbourhoods of $p \in X$ is called the M*-neighbourhood system of p and denoted by M*-N_p.*

Theorem 2.25. *A subset G of a space X is M*-open if and only if it is M*-nbd for every point $p \in G$.*

Proof. Let G be an M*-open set. Then G is a M*-nbd for each $p \in G$. Conversely let G be an M*-nbd for each $p \in G$. Then there exists an M*-open set W_p containing p such that $p \in W_p \subseteq G$. So $G = \cup_{p \in G} W_p$. Therefore G is an M*-open set. □

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