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New Results on Edge Pair Sum Graphs

Research Article

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Abstract: Let G be a (p,q) graph. An injective map $f : E(G) \to \{\pm 1, \pm 2, \cdots, \pm q\}$ is said to be an edge pair sum labeling if the induced vertex function $f^* : V(G) \to Z - \{0\}$ defined by $f^*(v) = \sum_{e \epsilon E_v} f(e)$ is one- one where E_v denotes the set of edges in G that are incident with a vertex v and $f^*(V(G))$ is either of the form $\left\{\pm k_1, \pm k_2, \cdots, \pm k_{\frac{p}{2}}\right\}$ or $\left\{\pm k_1, \pm k_2, \cdots, \pm k_{\frac{p-1}{2}}\right\} \bigcup \left\{\pm k_{\frac{p+1}{2}}\right\}$ according as p is even or odd. A graph that admits an edge pair sum labeling is called an edge pair sum graph. In this paper we prove that the graphs jelly fish, Y-tree, theta, the subdivision of spokes in wheel $SS(W_n)$, $P_m + 2K_1$, $C_4 \times P_m$, $P_n \odot K_m^c$ admit edge pair sum labeling.

MSC: 05C78.

Keywords: Edge pair sum labeling, edge pair sum graph, jelly fish, Y-tree, theta graphs, subdivision of spokes in wheel.

1. Introduction

Throughout this paper we consider finite, simple and undirected graph $G = (V(G), E(G))$ with p vertices and q edges. G is also called a (p, q) graph. We follow the basic notations and terminology of graph theory as in [2]. Ponraj et al. introduced the concept of pair sum labeling in [3]. An injective map $f : V(G) \to \{\pm 1, \pm 2, \ldots, \pm p\}$ is said to be a pair sum labeling of a graph $G(p, q)$ if the induced edge function $f_e : E(G) \to Z - \{0\}$ defined by $f_e(uv) = f(u) + f(v)$ is one-one and $f_e(E(G))$ is either of the form $\left\{\pm k_1, \pm k_2, \ldots, \pm k_{\frac{q}{2}}\right\}$ or $\left\{\pm k_1, \pm k_2, \ldots, \pm k_{\frac{q-1}{2}}\right\} \cup \left\{\pm k_{\frac{q+1}{2}}\right\}$ according as q is even or odd. A graph that admits a pair sum labeling is called a pair sum graph. Analogous to pair sum labeling we define a new labeling called an edge pair sum labeling in [5] and further studied in [6-12]. In this paper we prove that the graphs jelly fish, Y-tree, theta, the subdivision of spokes in wheel $SS(W_n)$, $P_m + 2K_1$, $C_4 \times P_m$, $P_n \odot K_m^c$ admit edge pair sum labeling. We use the following definitions in the subsequent section.

Definition 1.1. *A Y-tree Y_{n+1} is a graph obtained from the path P_n by appending an edge to a vertex of the path P_n adjacent to an end point [4].*

Definition 1.2. *The jelly fish graph $J(m, n)$ is obtained from a 4-cycle v_1,v_2, v_3,v_4 by joining v_1 and v_3 with an edge and appending m pendent edges to v_2 and n pendent edges to v_4.*

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Definition 1.3. *Take k paths of length $l_1, l_2, l_3, ..., l_k$ where $k \geq 3$ and $l_i = 1$ for at most one i. Identify their end points to form a new graph. The new graph is called a generalized theta graph, and it is denoted by $\Theta(l_1, l_2, l_3, ..., l_k)$. In other words, $\Theta(l_1, l_2, l_3, ..., l_k)$ consists $k \geq 3$ pair wise internally disjoint paths of length $l_1, l_2, l_3, ..., l_k$ that share a pair of common end points u and v. If each $l_i (i = 1, 2, ..., k)$ is equal to l, we will write $\Theta(l^{[k]})$.*

2. Main results

Theorem 2.1. *For any positive integers m and n, the jelly fish graph $J(m, n)$ has an edge pair sum labeling.*

Proof. Let $V(J(m,n)) = V_1 \bigcup V_2$ where $V_1 = \{x, u, y, v\}$ and $V_2 = \{u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$. $E(J(m,n)) = E_1 \bigcup E_2$, where $E_1 = \left\{e_1^{''} = xu, e_2^{''} = uy, e_3^{''} = yv, e_4^{''} = vx, e_5^{''} = xy\right\}$ and $E_2 = \left\{e_i = uu_i, e_j^{'} = vv_j : 1 \leq i \leq m, 1 \leq j \leq n\right\}$. Define $f : E\left(J(m,n)\right) \to \{\pm 1, \pm 2, ..., \pm\left(m + n + 5\right)\}$ as follows:

Case(i). m and n are odd.

Label the edges $e_1^{''}, e_2^{''}, e_3^{''}, e_4^{''}, e_5^{''}$ by 1,-3,4,2,-4. Define $f\left(e_1\right) = 5 = -f\left(e_1^{'}\right)$, for $1 \leq i \leq \frac{m-1}{2}$ $f\left(e_{i+1}\right) = (5+i) = -f\left(e_{\frac{m+1}{2}+i}\right)$ and for $1 \leq i \leq \frac{n-1}{2}$ $f\left(e_{1+i}^{'}\right) = \frac{m+9+2i}{2} = -f\left(e_{\frac{n+1}{2}+i}^{'}\right)$. For each edge label f, the induced vertex label f^* is calculated as follows: $f^*\left(x\right) = -1 = -f^*\left(v\right)$, $f^*\left(u\right) = 3 = -f^*\left(y\right)$, $f^*\left(u_1\right) = 5 = -f^*\left(v_1\right)$, for $1 \leq i \leq \frac{m-1}{2}$ $f^*\left(u_{1+i}\right) = (5+i) = -f^*\left(u_{\frac{m+1}{2}+i}\right)$ and for $1 \leq i \leq \frac{n-1}{2}$ $f^*\left(v_{1+i}\right) = \frac{m+9+2i}{2} = -f^*\left(v_{\frac{n+1}{2}+i}\right)$. Then $f^*(V(J(m,n))) = \{\pm 1, \pm 3, \pm 5, \pm 6, \pm 7, \pm 8, ..., \pm(\frac{m+9}{2}), \pm(\frac{m+11}{2}), \pm(\frac{m+13}{2}), \pm(\frac{m+15}{2}), ..., \pm(\frac{m+n+8}{2})\}$. Hence f is an edge pair sum labeling. The example for the edge pair sum graph labeling of $J(3,5)$ is shown in Figure 1.

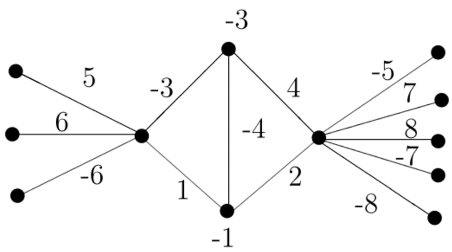

Figure 1. **Edge pair sum labeling of** $J(3,5)$

Case(ii). m and n are even.

Label the edges $e_1^{''}, e_2^{''}, e_3^{''}, e_4^{''}, e_5^{''}$ by -1,-8,-4,-6,5. Define $f\left(e_1\right) = 1$, $f\left(e_2\right) = 4$, $f\left(e_1^{'}\right) = 2$, $f\left(e_2^{'}\right) = 7$, for $1 \leq i \leq \frac{m-2}{2}$ $f\left(e_{2+i}\right) = 8 + i = -f\left(e_{\frac{m+2}{2}+i}\right)$ and for $1 \leq i \leq \frac{n-2}{2}$ $f\left(e_{2+i}^{'}\right) = \frac{m+14+2i}{2} = -f\left(e_{\frac{n+2}{2}+i}^{'}\right)$. For each edge label f, the induced vertex label f^* is calculated as follows: $f^*\left(x\right) = -2 = -f^*\left(v_1\right)$, $f^*\left(u\right) = -4 = -f^*\left(u_2\right)$, $f^*\left(y\right) = -7 = -f^*\left(v_2\right)$, $f^*\left(v\right) = -1 = -f^*\left(u_1\right)$, for $1 \leq i \leq \frac{m-2}{2}$ $f^*\left(u_{2+i}\right) = 8 + i = -f^*\left(u_{\frac{m+2}{2}+i}\right)$ and for $1 \leq i \leq \frac{n-2}{2}$ $f^*\left(v_{2+i}\right) = \frac{m+14+2i}{2} = -f^*\left(v_{\frac{n+2}{2}+i}\right)$. Then we get $f^*(V(J(m,n))) = \{\pm 1, \pm 2, \pm 4, \pm 7, \pm 9, \pm 10, \pm 11, ..., \pm(\frac{m+14}{2}), \pm(\frac{m+16}{2}), \pm(\frac{m+18}{2}), \pm(\frac{m+20}{2}), ..., \pm(\frac{m+n+12}{2})\}$. Hence f is an edge pair sum labeling.

Case(iii). m is odd and n is even or m is even and n is odd.

Label the edges $e_1^{''}, e_2^{''}, e_3^{''}, e_4^{''}, e_5^{''}$ by 3,4,-4,1,-1. Define $f\left(e_1\right) = -6$, for $1 \leq i \leq \frac{m-1}{2}$ $f\left(e_{1+i}\right) = 6 + i = -f\left(e_{\frac{m+1}{2}+i}\right)$ and for $1 \leq i \leq \frac{n}{2}$ $f\left(e_i^{'}\right) = \frac{m+11+2i}{2} = -f\left(e_{\frac{n}{2}+i}^{'}\right)$. For each edge label f, the induced vertex label f^* is calculated as follows: $f^*\left(x\right) = 3 = -f^*\left(v\right)$, $f^*\left(u\right) = 1 = -f^*\left(y\right)$, $f^*\left(u_1\right) = -6$, for $1 \leq i \leq \frac{m-1}{2}$ $f^*\left(u_{1+i}\right) = 6 + i = -f^*\left(u_{\frac{m+1}{2}+i}\right)$ and for $1 \leq i \leq \frac{n}{2}$ $f^*\left(v_i\right) = \frac{m+11+2i}{2} = -f^*\left(v_{\frac{n}{2}+i}\right)$. Therefore we get $f^*\left(V\left(J(m,n)\right)\right) =$

$\{\pm1,\pm3,\pm7,\pm8,\pm9,...,\pm\left(\frac{m+11}{2}\right),\pm\left(\frac{m+13}{2}\right),\pm\left(\frac{m+15}{2}\right),\pm\left(\frac{m+17}{2}\right),...,\pm\left(\frac{m+n+11}{2}\right)\}\bigcup\{-6\}$. Hence f is an edge pair sum labeling. □

Theorem 2.2. *For $n \geq 4$, the Y-tree $G = Y_{n+1}$ is an edge pair sum graph.*

Proof. Let $V(G) = \{v, u_i : 1 \leq i \leq n\}$ and $E(G) = \{e_1^{'} = vu_2, e_i = u_iu_{i+1} : 1 \leq i \leq n-1\}$ are the vertices and edges of the graph G. Define $f : E(G) \rightarrow \{\pm1, \pm2, ..., \pm n\}$ as follows:

Case(i). $n = 4$.

Let $f(e_1^{'}) = -4 = -f(e_1)$, $f(e_2) = -1$ and $f(e_3) = 2$. For each edge label f, the induced vertex label f^* is calculated as follows: $f^*(v) = -4 = -f^*(u_1)$, $f^*(u_2) = -1 = -f^*(u_3)$ and $f^*(u_4) = 2$. Then $f^*(V(G)) = \{\pm1, \pm4\}\bigcup\{2\}$. Hence f is an edge pair sum labeling if n =4.

Case(ii). $n = 5$.

Let $f(e_1^{'}) = 4 = -f(e_1)$, $f(e_2) = -2$, $f(e_3) = -1$ and $f(e_4) = 3$. For each edge label f, the induced vertex label f^* is calculated as follows: $f^*(v) = 4 = -f^*(u_1)$, $f^*(u_2) = -2 = -f^*(u_4)$ and $f^*(u_3) = -3 = -f^*(u_5)$. Then we get $f^*(V(G)) = \{\pm2, \pm3, \pm4\}$. Hence f is an edge pair sum labeling.

Case(iii). n is odd, take $n = 2k+1$, $k \geq 3$.

Let $f(e_k) = -2, f(e_{k+1}) = -1, f(e_{k+2}) = 3$, $f(e_1) = 4 = -f(e_1^{'})$, for $1 \leq i \leq k-2$ $f(e_{1+i}) = (2k+1-2i)$ and for $k+2 \leq i \leq 2k-1$ $f(e_{1+i}) = (2k-1-2i)$. For each edge label f, the induced vertex label f^* is calculated as follows: $f^*(v) = -4 = -f^*(u_1)$, $f^*(u_2) = (2k-1)$, $f^*(u_k) = 3 = -f^*(u_{k+1})$, $f^*(u_{k+2}) = 2 = -f^*(u_{k+3})$, $f^*(u_n) = -(2k-1)$, for $2 \leq i \leq k-2$ $f^*(u_{1+i}) = 4(k+1-i)$ and for $k+3 \leq i \leq 2k-1$ $f^*(u_{1+i}) = 4(k-i)$. Then the vertex labeling are $f^*(V(G)) = \{\pm2, \pm3, \pm4, \pm(2k-1), \pm12, \pm16, ...,$
$\pm 4(k-1)\}$. Hence f is an edge pair sum labeling.

Case(iv). n is even, take $n = 2k$, $k \geq 3$.

Let $f(e_{k+1}) = 1$, $f(e_k) = 2$, $f(e_{k-1}) = -5 = -f(e_{k+2})$, $f(e_1) = 4 = -f(e_1^{'})$, for $2 \leq i \leq k-2$ $f(e_i) = -(2k+3-2i)$ and for $k+3 \leq i \leq 2k-1$ $f(e_i) = (-2k+1+2i)$. For each edge label f, the induced vertex label f^* is calculated as follows: $f^*(v) = -4 = -f^*(u_1)$, $f^*(u_2) = -(2k-1) = -f^*(u_n)$, $f^*(u_k) = -3 = -f^*(u_{k+1})$, $f^*(u_{k+2}) = 6$, for $3 \leq i \leq k-1$ $f^*(u_i) = 4(-k+i-2)$ and for $k+3 \leq i \leq (2k-1)$ $f^*(u_i) = -4(k-i)$. Then the vertex labeling are $f^*(V(G)) = \{\pm3, \pm4, \pm(2k-1), \pm12, \pm16, ..., \pm4(k-1)\}\bigcup\{6\}$. Hence f is an edge pair sum labeling. The example for the edge pair sum graph labeling of Y_{6+1} is shown in Figure 2.

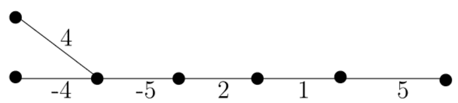

Figure 2. **Edge pair sum labeling of Y_{6+1}**

□

Theorem 2.3. *The theta graph $\Theta(l^{[m]})$ is an edge pair sum graph.*

Proof. Let $G(V,E) = \Theta(l^{[m]})$. Then $|V(G)| = m(l-1)+2$ and $|E(G)| = ml$ are the vertices and edges of G. Where $V(G) = \{u, v, u_i^j : 1 \leq i \leq m, 1 \leq j \leq l-1\}$ and $E(G) = \{e_i^j : 1 \leq i \leq m, 1 \leq j \leq l\}$.

Case(i). m is odd and l is even.

For $1 \leq j \leq \frac{l-2}{2}$ $f(e_1^j) = l+3-2j$, $f(e_1^{\frac{l}{2}}) = -2$, $f(e_1^{\frac{l+2}{2}}) = -1$, for $\frac{l+4}{2} \leq j \leq l$ $f(e_1^j) = l-1-2j$ and for $1 \leq i \leq \frac{m-1}{2}; 1 \leq j \leq l$

$f(e^j_{1+i}) = 2l(i-1)+2j+2 = -f(e^j_{\frac{m+1}{2}+i})$. For each edge label f, the induced vertex label f^* is calculated as follows: $f^*(u) = l+1 = -f^*(v)$, $f^*(u_1^{\frac{l}{2}-1}) = 3$, $f^*(u_1^{\frac{l}{2}}) = -3$, $f^*(u_1^{\frac{l}{2}+1}) = -6$, for $1 \le j \le \frac{l-4}{2}$ $f^*(u_1^j) = 2l+4-4j$, for $1 \le j \le \frac{l-4}{2}$ $f^*(u_1^{\frac{l+2}{2}+j}) = -(8+4j)$ and for $1 \le i \le \frac{m-1}{2}; 1 \le j \le (l-1)$ $f^*(u^j_{1+i}) = 4l(i-1)+6+4j = -f^*(u^j_{\frac{m+1}{2}+i})$. Then $f^*(V(G)) = \{\pm3, \pm(l+1), \pm12, \pm16, \pm20, ..., \pm2l\} \bigcup \{\pm(4l(i-1)+6+4j) | 1 \le i \le \frac{m-1}{2}, 1 \le j \le (l-1)\} \bigcup \{-6\}$. Hence f is an edge pair sum labeling.

Case(ii). m and l are odd.

For $1 \le j \le \frac{l-3}{2}$ $f(e_i^j) = -(l+2-2j)$, $f(e_1^{\frac{l-1}{2}}) = 2$, $f(e_1^{\frac{l+1}{2}}) = 1$, $f(e_1^{\frac{l+3}{2}}) = -3$, for $1 \le j \le \frac{l-3}{2}$ $f(e_1^{\frac{l+3}{2}+i}) = 3+2j$ and for $1 \le i \le \frac{m-1}{2}; 1 \le j \le l$ $f(e^j_{1+i}) = 2l(i-1)+2j+2 = -f(e^j_{\frac{m+1}{2}+i})$. For each edge label f, the induced vertex label f^* is calculated as follows: $f^*(u) = -l = -f^*(v)$, $f^*(u_1^{\frac{l-3}{2}}) = -3 = -f^*(u_1^{\frac{l-1}{2}})$, $f^*(u_1^{\frac{l+1}{2}}) = -2 = -f^*(u_1^{\frac{l+3}{2}})$, for $1 \le j \le \frac{l-5}{2}$ $f^*(u_1^j) = -(2(l-1)+4-4j)$, for $1 \le j \le \frac{l-5}{2}$ $f^*(u_1^{\frac{l+3}{2}+j}) = 8+4j$ and for $1 \le i \le \frac{m-1}{2}; 1 \le j \le (l-1) f^*(u^j_{1+i}) = 4l(i-1)+6+4j = -f^*(u^j_{\frac{m+1}{2}+i})$. Then $f^*(V(G)) = \{\pm2, \pm3, \pm l, \pm12, \pm16, ..., \pm2(l-1)\} \bigcup \{\pm(4l(i-1)+6+4j) | 1 \le i \le \frac{m-1}{2}, 1 \le j \le (l-1)\}$. Hence f is an edge pair sum labeling. The example for the edge pair sum graph labeling of $\Theta(5^{[5]})$ is shown in Figure 3.

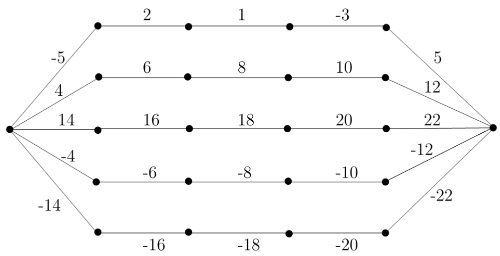

Figure 3. **Edge pair sum graph labeling of** $\Theta(5^{[5]})$

Case(iii). m is even and l is odd.

For $1 \le i \le \frac{m}{2}$; $1 \le j \le l$ $f(e_i^j) = 2l(i-1)+2j$ and $f(e^j_{\frac{m}{2}+i}) = -(l(m-2i+2)+2-2j)$. For each edge label f, the induced vertex label f^* is calculated as follows: for $1 \le i \le \frac{m}{2}$; $1 \le j \le (l-1)$ $f^*(u_i^j) = 4l(i-1)+2+4j$ and $f^*(u^j_{\frac{m}{2}+i}) = -(l(2m-4i+4)-4j+2)$, $f^*(u) = -m(l-1) = f^*(v)$. Then $f^*(V(G)) = \{\pm m(l-1)\} \bigcup \{\pm(4l(i-1)+2+4j) | 1 \le i \le \frac{m}{2}, 1 \le j \le (l-1)\}$. Hence f is an edge pair sum labeling.

Case(iv). m and l are even if $m \ge 4$.

For $1 \le i \le \frac{m}{4}$; $1 \le j \le l$, $f(e_i^j) = 4l(i-1)+4j-3$, $f(e^j_{\frac{m}{4}+i}) = 4l(i-1)+4j-1$, $f(e^j_{\frac{m}{2}+i}) = -(l(m-4i+4)-4j+1)$ and $f(e^j_{\frac{3m}{4}+i}) = -(l(m-4i+4)-4j+3)$. For each edge label f, the induced vertex label f^* is calculated as follows: $f^*(u) = -(2m(l-1)) = -f^*(v)$, for $1 \le i \le \frac{m}{4}$; $1 \le j \le (l-1)$ $f^*(u_i^j) = 8l(i-1)+6+(8j-8)$, $f^*(u^j_{\frac{m}{4}+i}) = 8l(i-1)+10+(8j-8)$, $f^*(u^j_{\frac{m}{2}+i}) = -(l(2m-8i+8)-8j-2)$ and $f^*(u^j_{\frac{3m}{4}+i}) = -(l(2m-8i+8)-8j+2)$. Then $f^*(V(G)) = \{\pm(2lm-2m)\} \bigcup \{\pm(8l(i-1)+6+(8j-8))$ and $(8l(i-1)+10+(8j-8)) | 1 \le i \le \frac{m}{4}, 1 \le j \le (l-1)\}$. Hence f is an edge pair sum labeling. □

Theorem 2.4. *The subdivision of spokes in wheel $SS(W_n)$ graph admits edge pair sum labeling.*

Proof. Let $V(SS(W_n)) = \{u_0, u_i, v_i : 1 \le i \le n\}$ and $E(SS(W_n)) = \{e_i = u_iu_{i+1} : 1 \le i \le (n-1), e_n = u_nu_1, e_i^{'} = u_iv_i$ and $e_i^{''} = u_0v_i : 1 \le i \le n\}$ are the vertices and edges of the graph $SS(W_n)$. Define the edge labeling $f : E(SS(W_n)) \rightarrow \{\pm1, \pm2, \ldots, \pm3n\}$ by considering the following two cases:

Case (i) n is even.

For $1 \le i \le n$ $f(e_i^{'}) = -i$ and $f(e_i^{''}) = -(3n-2i+1)$, for $1 \le i \le n-1$ $f(e_i) = n+i$ and $f(e_n) = 2n$. For each edge label

f, the induced vertex label f^* is calculated as follows: $f^*(u_1) = 3n$, for $1 \le i \le n-1$ $f^*(u_{1+i}) = 2n+i$, for $1 \le i \le n$ $f^*(v_i) = -(3n-i+1)$ and $f^*(u_0) = -2n^2$. From the above vertex labeling $f^*(V(SS(W_n))) = \{\pm(2n+1), \pm(2n+2), \pm(2n+3), ..., \pm 3n\} \bigcup \{-2n^2\}$. Hence f is an edge pair sum labeling. The example for the edge pair sum graph labeling of $SS(W_n)$ for $n = 4$ is shown in Figure 4.

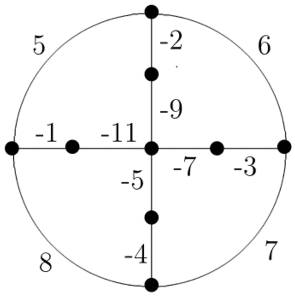

Figure 4. **Edge pair sum labeling of** $SS(W_4)$

Case (ii) n is odd.

For $1 \le i \le n$ $f(e_i^{'}) = 2i-1$ and $f(e_i^{''}) = 2i$, for $1 \le i \le \frac{n-1}{2}$ $f(e_{n-2i+1}) = -(\frac{n+1}{2} + 2i - 1)$, $f(e_{n-2i}) = -(\frac{3n+1}{2} + 2i)$ and $f(e_n) = -(\frac{3n+1}{2})$. For each edge label f, the induced vertex label f^* is calculated as follows: $f^*(u_0) = n(n+1)$, for $1 \le i \le n$ $f^*(u_i) = -(4n - 4i + 3)$ and $f^*(v_i) = 4i - 1$. From the above vertex labeling $f^*(V(SS(W_n))) = \{\pm 3, \pm 7, \pm 11, ..., \pm(4n-1)\} \bigcup \{n(n+1)\}$. Hence $SS(W_n)$ is an edge pair sum graph. The example for the edge pair sum graph labeling of $SS(W_n)$ for $n = 5$ is shown in Figure 5.

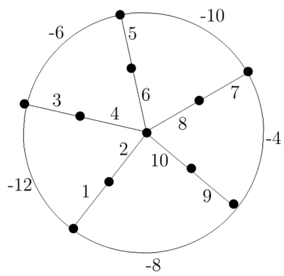

Figure 5. **Edge pair sum labeling of** $SS(W_5)$

□

Theorem 2.5. *The graph* $P_m + 2K_1$ *is an edge pair sum graph if* $m \ge 3$.

Proof. Let $V(P_m + 2K_1) = \{u_0, v_0, u_i : 1 \le i \le m\}$ and $E(P_m + 2K_1) = \{e_i = u_0u_i, e_i^{'} = v_0u_i : 1 \le i \le m, e_i^{''} = u_iu_{i+1} : 1 \le i \le m-1\}$ are the vertices and edges of the graph $P_m + 2K_1$. Define $f : E(P_m + 2K_1) \to \{\pm 1, \pm 2, \ldots, \pm(3m-1)\}$ as follows:

Case (i) m is even.

Subcase (a). $m = 4$.

For $1 \le i \le 4$ $f(e_i) = 2 + 2i = -f(e_i^{'})$, $f(e_1^{''}) = -2$, $f(e_2^{''}) = -1$ and $f(e_3^{''}) = 3$. For each edge label f, the induced vertex label f^* is calculated as follows: $f^*(u_1) = -2 = -f^*(u_3)$, $f^*(u_2) = -3 = -f^*(u_4)$, $f^*(u_0) = 28 = -f^*(v_0)$. Then $f^*(V(P_m + 2K_1)) = \{\pm 2, \pm 3, \pm 28\}$. Hence f is an edge pair sum labeling. The example for the edge pair sum graph labeling of $P_m + 2K_1$ for $m = 4$ is shown in Figure 6.

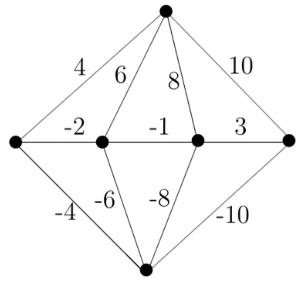

Figure 6. **Edge pair sum labeling of $P_4 + 2K_1$**

Subcase (b). m is even, $m \geq 6$.

Define $f(e''_{\frac{m}{2}-1}) = -2$, $f(e''_{\frac{m}{2}}) = -1$, $f(e''_{\frac{m}{2}+1}) = 3$, for $1 \leq i \leq \frac{m}{2} - 2$ $f(e''_i) = m + 1 - 2i$, for $\frac{m}{2} + 2 \leq i \leq m - 1$ $f(e''_i) = m - 1 - 2i$ and for $1 \leq i \leq m$ $f(e_i) = (2 + 2i) = -f(e'_i)$. For each edge label f, the induced vertex label f^* is calculated as follows: $f^*(u_1) = m - 1 = -f^*(u_m)$, $f^*(u_{\frac{m}{2}-1}) = 3 = -f^*(u_{\frac{m}{2}})$, $f^*(u_{\frac{m}{2}+1}) = 2 = -f^*(u_{\frac{m}{2}+2})$, $f^*(u_0) = m^2 + 3m = -f^*(v_0)$, for $2 \leq i \leq \frac{m}{2} - 2$ $f^*(u_i) = 4(\frac{m}{2} + 1 - i)$ and for $\frac{m}{2} + 3 \leq i \leq m - 1$ $f^*(u_i) = 4(\frac{m}{2} - i)$. Then $f^*(V(P_m + 2K_1)) = \{\pm 2, \pm 3, \pm(m - 1), \pm(m^2 + 3m), \pm 12, \pm 16, \pm 20, ..., \pm 2(m - 2)\}$. Hence f is an edge pair sum labeling.

Case (ii) m is odd.

Subcase (a). $m = 3$.

Define $f(e''_1) = -1$, $f(e''_2) = 2$ and for $1 \leq i \leq 3$ $f(e_i) = 2 + 2i = -f(e'_i)$. For each edge label f, the induced vertex label f^* is calculated as follows: $f^*(u_1) = -1 = -f^*(u_2)$, $f^*(u_3) = 2$ and $f^*(u_0) = 18 = -f^*(v_0)$. Then $f^*(V(P_m + 2K_1)) = \{\pm 1, \pm 18\} \bigcup \{2\}$. Hence f is an edge pair sum labeling.

Subcase (b). m is odd , $m \geq 5$.

Define $f(e''_{\frac{m+1}{2}}) = 1$, $f(e''_{\frac{m-1}{2}}) = 2$, $f(e''_{\frac{m-3}{2}}) = -5 = -f(e''_{\frac{m+3}{2}})$, for $1 \leq i \leq \frac{m-5}{2}$ $f(e''_i) = -(m+2-2i)$, for $\frac{m+5}{2} \leq i \leq m-1$ $f(e''_i) = (-m + 2 + 2i)$ and for $1 \leq i \leq m$ $f(e'_i) = -(2 + 2i) = -f(e_i)$. For each edge label f, the induced vertex label f^* is calculated as follows: $f^*(u_1) = -m = -f^*(u_m)$, $f^*(u_{\frac{m-1}{2}}) = -3 = -f^*(u_{\frac{m+1}{2}})$, $f^*(u_{\frac{m+3}{2}}) = 6$, for $2 \leq i \leq \frac{m-3}{2}$ $f^*(u_i) = 2(-m - 3 + 2i)$, for $\frac{m+5}{2} \leq i \leq m - 1$ $f^*(u_i) = -2(m - 1 - 2i)$ and $f^*(u_0) = m^2 + 3m = -f^*(v_0)$. Then $f^*(V(P_m + 2K_1)) = \{\pm 3, \pm(m^2 + 3m), \pm m, \pm 12, \pm 16, \pm 20, ..., \pm 2(m - 1)\} \bigcup \{6\}$. Hence f is an edge pair sum labeling. The example for the edge pair sum graph labeling of $P_m + 2k_1$ for $m = 7$ is shown in Figure 7.

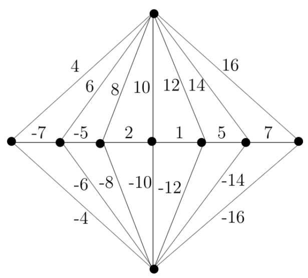

Figure 7. **Edge pair sum labeling of $P_7 + 2K_1$**

□

Theorem 2.6. *The graph $C_4 \times P_m$ is an edge pair sum graph.*

Proof. Let $V(C_4 \times P_m) = \{u_{ij} : 1 \leq i \leq m, 1 \leq j \leq 4\}$ and $E(C_4 \times P_m) = \{e_{ij} = u_{ij}u_{i,j+1} : 1 \leq i \leq m, 1 \leq j \leq 3; e_{i4} =$

$u_{i4}u_{i1} : 1 \leq i \leq m; e^{'}_{ij} = u_{ij}u_{i+1,j} : 1 \leq i \leq m-1, 1 \leq j \leq 4\}$ are the vertices and edges of the graph $C_4 \times P_m$. Define $f : E(C_4 \times P_m) \to \{\pm 1, \pm 2, \ldots, \pm(8m-4)\}$ as follows:

Case (i) $m = 2$.

Define $f(e_{11}) = 1 = -f(e_{13})$, $f(e_{12}) = 2 = -f(e_{14})$, $f(e_{21}) = -4 = -f(e_{23})$, $f(e_{22}) = 3 = -f(e_{24})$, $f(e^{'}_{11}) = -6 = -f(e^{'}_{13})$ and $f(e^{'}_{12}) = 5 = -f(e^{'}_{14})$. For each edge label f, the induced vertex label f^* is calculated as follows: $f^*(u_{11}) = -7 = -f^*(u_{13})$, $f^*(u_{12}) = 8 = -f^*(u_{14})$, $f^*(u_{21}) = -13 = -f^*(u_{23})$ and $f^*(u_{22}) = 4 = -f^*(u_{24})$. Then we get $f^*(V(C_4 \times P_m)) = \{\pm 4, \pm 7, \pm 8, \pm 13\}$. Hence f is an edge pair sum labeling.

Case (ii) $m \geq 3$.

Define $f(e_{11}) = 1 = -f(e_{13})$, $f(e_{12}) = 2 = -f(e_{14})$, for $2 \leq i \leq m-1$ $f(e_{i1}) = -2i = -f(e_{i3})$ and $f(e_{i2}) = 2i-1 = -f(e_{i4})$, $f(e_{m1}) = 2m-1 = -f(e_{m3})$, $f(e_{m2}) = 2m = -f(e_{m4})$, for $1 \leq i \leq m-1$ $f(e^{'}_{i1}) = -(2m+2i) = -f(e^{'}_{i3})$ and $f(e^{'}_{i2}) = 2m-1+2i = -f(e^{'}_{i4})$. For each edge label f, the induced vertex label f^* is calculated as follows: $f^*(u_{11}) = -(2m+3) = -f^*(u_{13})$, $f^*(u_{12}) = 2m+4 = -f^*(u_{14})$ for $2 \leq i \leq m-1$ $f^*(u_{i1}) = -(4m-3+8i) = -f^*(u_{i3})$ and $f^*(u_{i2}) = 4m-5+4i = -f^*(u_{i4})$, $f^*(u_{m1}) = -(4m-1) = -f^*(u_{m3})$ and $f^*(u_{m2}) = 8m-4 = -f^*(u_{m4})$. From the above labeling we get $f^*(V(C_4 \times P_m)) = \{\pm(2m+3), \pm(2m+4), \pm(4m-1), \pm(8m-4), \pm(4m+13), \pm(4m+21), \pm(4m+29), ..., \pm(12m-11), \pm(4m+3), \pm(4m+7), \pm(4m+11), ..., \pm(8m-9)\}$. Hence $C_4 \times P_m$ is an edge pair sum graph. The example for the edge pair sum graph labeling of $C_4 \times P_m$ for $m = 4$ is shown in Figure 8.

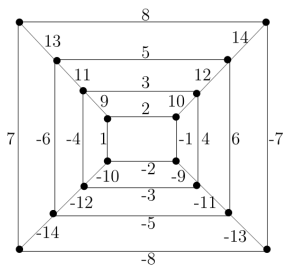

Figure 8. **Edge pair sum labeling of** $C_4 \times P_4$

□

Theorem 2.7. *The graph* $P_n \odot K^c_m$ *is an edge pair sum graph if* m *is odd.*

Proof. Let $V(P_n \odot K^c_m) = \{u_i, v_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(P_n \odot K^c_m) = \{e_i = u_iu_{i+1} : 1 \leq i \leq (n-1), e_{ij} = u_iv_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\}$ are the vertices and edges of the graph $P_n \odot K^c_m$. Define $f : (E(P_n \odot K^c_m)) \to \{\pm 1, \pm 2, \pm 3, ..., \pm(mn+n-1)\}$ as follows:

Case (i) n is even.

Subcase (a). $n = 2$.

Define $f(e_1) = 2$, $f(e_{11}) = 1$, $f(e_{21}) = -3$, for $1 \leq i \leq 2$ and $2 \leq j \leq \frac{m+1}{2}$ $f(e_{ij}) = 2 + \frac{m-1}{2}(i-1) + j = -f(e_{i\frac{m-1+2j}{2}})$. For each edge label f, the induced vertex label f^* is calculated as follows: $f^*(u_1) = 3 = -f^*(v_{21})$, $f^*(u_2) = -1 = -f^*(v_{11})$, for $1 \leq i \leq 2$ and $2 \leq j \leq \frac{m+1}{2}$ $f^*(v_{ij}) = 2 + \frac{m-1}{2}(i-1) + j = -f^*(v_{i\frac{m-1+2j}{2}})$. $f^*(V(P_n \odot K^c_m)) = \{\pm 1, \pm 3\} \bigcup \{\pm(2 + \frac{m-1}{2}(i-1) + j) | 1 \leq i \leq 2, 2 \leq j \leq \frac{m+1}{2}\}$. Hence f is an edge pair sum labeling.

Subcase(b). $n = 4$.

Define $f(e_1) = -2$, $f(e_2) = -1$, $f(e_3) = 3$, $f(e_{11}) = 4 = -f(e_{31})$, $f(e_{21}) = 6 = -f(e_{41})$ and for $1 \leq i \leq n$ and $2 \leq j \leq \frac{m+1}{2}$ $f(e_{ij}) = 5 + \frac{m-1}{2}(i-1) + j = -f(e_{i\frac{m-1+2j}{2}})$. For each edge label f, the induced vertex label f^* is calculated as follows: $f^*(u_1) = 2 = -f^*(u_3)$, $f^*(u_2) = 3 = -f^*(u_4)$, $f^*(v_{11}) = 4 = -f^*(v_{31})$, $f^*(v_{21}) = 6 = -f^*(v_{41})$ and for $1 \le i \le n$ and $2 \le j \le \frac{m+1}{2}$ $f^*(v_{ij}) = 5 + \frac{m-1}{2}(i-1) + j = -f^*(v_{i\frac{m-1+2j}{2}})$. Then we get $f^*(V(P_n \odot K_m^c)) = \{\pm 2, \pm 3, \pm 4, \pm 6\} \bigcup \{\pm(5 + \frac{m-1}{2}(i-1) + j) | 1 \le i \le n, 2 \le j \le \frac{m+1}{2}\}$. Hence f is an edge pair sum labeling. The example for the edge pair sum graph labeling of $P_n \odot K_m^c$ for $n = 4$ and $m = 3$ is shown in Figure 9.

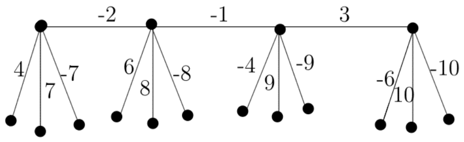

Figure 9. **Edge pair sum labeling of** $P_4 \odot K_3^c$

Subcase(c). $n \ge 6$.

Define $f(e_{\frac{n}{2}-1}) = -2$, $f(e_{\frac{n}{2}}) = -1$, $f(e_{\frac{n}{2}+1}) = 3$, for $1 \le i \le \frac{n}{2} - 2$ $f(e_i) = n+1-2i$, for $\frac{n}{2}+2 \le i \le n-1$ $f(e_i) = n-1-2i$, for $1 \le i \le \frac{n}{2} - 2$ $f(e_{i1}) = n + 2i - 1$, $f(e_{\frac{n}{2}-1,1}) = 2n - 2 = -f(e_{\frac{n}{2}1})$, $f(e_{\frac{n}{2}+1,1}) = 4 = -f(e_{\frac{n}{2}+2,1})$, for $1 \le i \le \frac{n}{2} - 2$ $f(e_{\frac{n}{2}+2+i,1}) = -(2n-2i-3)$ and for $1 \le i \le n$ and $1 \le j \le \frac{m-1}{2}$ $f(e_{i,j+1}) = (3n-1) + \frac{m-1}{2}(i-1) + j = -f(e_{i\frac{m+1}{2}+j})$. For each edge label f, the induced vertex label f^* is calculated as follows: $f^*(u_1) = 2n = -f^*(u_n)$, $f^*(u_{\frac{n}{2}-1}) = 2n + 1 = -f^*(u_{\frac{n}{2}})$, $f^*(u_{\frac{n}{2}+1}) = 6 = -f^*(u_{\frac{n}{2}+2})$, for $2 \le i \le \frac{n}{2} - 2$ $f^*(u_i) = 3n - (2i - 3)$, for $1 \le i \le \frac{n}{2} - 3$ $f^*(u_{\frac{n}{2}+2+i}) = -(2n + 5 + 2i)$, for $1 \le i \le \frac{n}{2} - 2$ $f^*(v_{i1}) = n + 2i - 1$, $f^*(v_{\frac{n}{2}+1,1}) = 4 = -f^*(v_{\frac{n}{2}+2,1})$, $f^*(v_{\frac{n}{2}-1,1}) = 2n - 2 = -f^*(v_{\frac{n}{2},1})$, for $1 \le i \le \frac{n}{2} - 2$ $f^*(v_{\frac{n}{2}+2+i,1}) = -(2n-2i-3)$ and for $1 \le i \le n$ and $1 \le j \le \frac{m-1}{2}$ $f^*(v_{i,j+1}) = (3n-1) + (\frac{m-1}{2})(i-1) + j = -f^*(v_{i,\frac{m+1}{2}+j})$. From the above labeling we get $f^*(V((P_n \odot K_m^c)) = \{\pm 4, \pm 6, \pm(2n-2), \pm 2n, \pm(2n+1), \pm(n+1), \pm(n+3), \pm(n+5), ..., \pm(2n-5), \pm(3n-1), \pm(3n-3), \pm(3n-5), ..., \pm(2n+7)\} \bigcup \{\pm((3n-1) + (\frac{m-1}{2})(i-1) + j) | 1 \le i \le n, 1 \le j \le \frac{m-1}{2}\}$. Hence $P_n \odot K_m^c$ is an edge pair sum graph.

Case (ii) n is odd.

Subcase (a). $n = 3$.

Define $f(e_1) = -4$, $f(e_2) = -2 = -f(e_{11})$, $f(e_{21}) = 3$ and $f(e_{31}) = 1$, for $1 \le i \le n$ and $2 \le j \le \frac{m+1}{2}$ $f(e_{ij}) = 3 + \frac{m-1}{2}(i-1) + j = -f(e_{i,\frac{m-1}{2}+j})$. For each edge label f, the induced vertex label f^* is calculated as follows: $f^*(u_1) = -2 = -f^*(v_{11})$, $f^*(u_2) = -3 = -f^*(v_{21})$, $f^*(u_3) = -1 = -f^*(v_{31})$, for $1 \le i \le n$ and $2 \le j \le \frac{m+1}{2}$ $f^*(v_{ij}) = 3 + \frac{m-1}{2}(i-1) + j = -f^*(v_{i,\frac{m-1}{2}+j})$. $f^*(V(P_n \odot K_m^c)) = \{\pm 1, \pm 2, \pm 3\} \bigcup \{\pm(3 + \frac{m-1}{2}(i-1) + j) | 1 \le i \le n, 2 \le j \le \frac{m+1}{2}\}$. Hence f is an edge pair sum labeling. The example for the edge pair sum graph labeling of $P_n \odot K_m^c$ for $n = 3$ and $m = 3$ is shown in Figure 10.

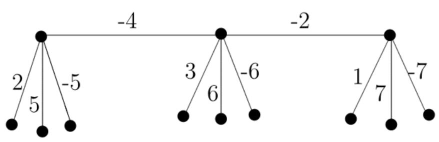

Figure 10. **Edge pair sum labeling of** $P_3 \odot K_3^c$

Subcase(b). $n \ge 5$.

Define $f(e_{\frac{n-1}{2}}) = 2$, $f(e_{\frac{n+1}{2}}) = 1$, $f(e_{\frac{n-1}{2},1}) = -4 = -f(e_{\frac{n+1}{2},1})$, $f(e_{\frac{n+3}{2},1}) = -3$, $f(e_{11}) = -(n+2) = f(e_{n1})$, for $1 \le i \le \frac{n-3}{2}$ $f(e_i) = -(n+2-2i)$, for $\frac{n+3}{2} \le i \le n-1$ $f(e_i) = (-n+2+2i)$, for $2 \le i \le \frac{n-3}{2}$ $f(e_{i1}) = -(n+2i-1)$, for $1 \le i \le \frac{n-5}{2}$ $f(e_{\frac{n+3}{2}+i,1}) = 2(n-1-i)$ and for $1 \le i \le n$ and $1 \le j \le \frac{m-1}{2} f(e_{i,1+j}) = n + 2 + (m-1)(i-1) + 2j = -f(e_{i,\frac{m+1}{2}+j})$. For each edge label f, the induced vertex label f^* is calculated as follows: $f^*(u_1) = -(2n+2) = -f^*(u_n)$, $f^*(v_{11}) = -(n+2)$, $f^*(u_{\frac{n-1}{2}}) = -7 = -f^*(u_{\frac{n+1}{2}})$, $f^*(u_{\frac{n+3}{2}}) = 3 = -f^*(v_{\frac{n+3}{2}1})$, for $1 \leq i \leq \frac{n-5}{2}$ $f^*(u_{1+i}) = -(3n+3-2i)$, for $1 \leq i \leq \frac{n-5}{2}$ $f^*(u_{\frac{n+3}{2}+i}) = (3n+3-2i)$, for $2 \leq i \leq \frac{n-3}{2}$ $f^*(v_{i1}) = -(n+2i-1)$, $f^*(v_{\frac{n-1}{2},1}) = -4 = -f^*(v_{\frac{n+1}{2},1})$, $f^*(v_{n1}) = n+2$, for $1 \leq i \leq \frac{n-5}{2}$ $f^*(v_{\frac{n+3}{2}+i,1}) = 2(n-1-i)$, for $1 \leq i \leq n$ and $1 \leq j \leq \frac{m-1}{2}$ $f^*(v_{i,1+j}) = n+2+(m-1)(i-1)+2j = -f^*(v_{i,\frac{m+1}{2}+j})$. From the above labeling we get $f^*(V((P_n \odot K_m^c)) = \{\pm3, \pm4, \pm7, \pm(2n+2), \pm2n, \pm(3n+1), \pm(3n-1), \pm(3n-3), ..., \pm(2n+8), \pm(2n-4), \pm(2n-6), \pm(2n-8), ..., \pm(n+3)\} \bigcup \{\pm((n+2)+(m-1)(i-1)+2j) | 1 \leq i \leq n, 1 \leq j \leq \frac{m-1}{2}\}$. Hence $P_n \odot K_m^c$ is an edge pair sum graph. □

Remark 2.8. *Let $G(p,q)$ is an edge pair sum graph. Then $G \odot K_n^c$ is also an edge pair sum graph if n is even. This is already proved in [5].*

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