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Fibonacci Sequence Generated From Two Dimensional q-difference Equation

Research Article

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Abstract: In this paper, we define generalized Fibona-sequence using two-dimensional q-difference operator and we derive some algebraic identities as it includes its relationship with Fibonacci numbers. Also we derive theorems using inverse two-dimensional q-difference operator.

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Keywords: Fibonacci numbers, Two-dimensional q-difference operator and Summation solution.

1. Introduction

The study of q-difference equations, initiated at the beginning of the twentieth century in intensive works especially by Jackson [1], Carmichael [2] and other authors such as Poincare, Picard, Ramanujan ([3], [4]), is a very interesting field in difference equations. In the last few decades, it has evolved into a multidisciplinary subject and plays an important role in several fields of physics such as cosmic strings and black holes [5], conformal quantum mechanics [6], nuclear and high energy physics [7].

In 1984, Jerzy Popenda [8] introduced a particular type of difference operator Δ_α defined on $u(k)$ as $\Delta_\alpha u(k) = u(k+1) - \alpha u(k)$. In 1989, K.S.Miller and Ross [9] introduced the discrete analogue of the Riemann-Liouville fractional derivative and proved some properties of the fractional derivative operator. Recently, G.Britto Antony Xavier et al. [10] have got the solution of the generalized q-difference equation $\Delta_q^t v(k) = u(k),\ k \in (-\infty, \infty)$ and $q \neq 1$, in the form

$$\Delta_q^{-t} u(k)\Big|\Big|_{\frac{k}{q^m}}^{k} = \sum_{(r)_{1\to t}}^{m} u\Big(k \prod_{i=1}^{t} q^{-r_i}\Big).$$

In [11], the authors introduced q-alpha difference operator, which is defined as

$$\underset{(q)\alpha}{\Delta} u(k) = u(qk) - \alpha u(k), \tag{1}$$

and then extended to generalized higher oredr q-alpha difference equation

$$\underset{(q_1)\alpha_1}{\Delta}\Big(\underset{(q_2)\alpha_2}{\Delta}\Big(\cdots \underset{(q_t)\alpha_t}{\Delta}\Big(v(k)\Big)\cdots\Big)\Big) = u(k), k \in (-\infty, \infty), \tag{2}$$

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and obtained formula for finite q-alpha multi-series and finite higher order q-alpha series. However, finding the solution of two-dimensional q-difference equation is still in the initial stage and many aspects of this theory need to be explored. The main aim of this paper is to generate generalized Fibonacci sequence using two-dimensional q-difference operator.

The article proceeds as follows: Section 2 presents basic definitions and preliminary results. In Section 3, we show how to find finite solution of two-dimensional q-difference equation and how to generate Fibonacci sequence from that solution, In Section 4, we derive multi-series solution and finally in Section 5, we derive generalized product formula.

2. Preliminaries

Before stating and proving our results, we present some notations, basic definitions and preliminary results which will be used for the subsequent discussions. Let $u(k)$ be a real valued function on $(-\infty,\infty)$, α and q are non-zero reals and m is a positive integer. For simplicity, we use the following notations:

(i) $\sum\limits_{(r)_{1\to i}}^{m} = \sum\limits_{r_1=0}^{m_1}\sum\limits_{r_2=0}^{m_2}\cdots\sum\limits_{r_i=0}^{m_i}$; (ii) $\Delta^{-1}_{q_1\to t} = \Delta^{-1}_{q_1}\Delta^{-1}_{q_2}\Delta^{-1}_{q_3}\cdots\Delta^{-1}_{q_t}$ and

(iii) $\underset{(a_1,\,a_2)}{\Delta^{-1}_{q_1\to t}} = \underset{(a_1,\,a_2)}{\Delta^{-1}_{q_1}}\ \underset{(a_1,\,a_2)}{\Delta^{-1}_{q_2}}\cdots\underset{(a_1,\,a_2)}{\Delta^{-1}_{q_t}}$.

Definition 2.1. *Let a_1 and a_2 be fixed reals, $k\in(-\infty,\infty)$. Then the two-dimensional q-difference operator $\underset{(a_1,a_2)}{\Delta_{\rm q}}$ is defined as*

$$\underset{(a_1,a_2)}{\Delta_{\rm q}} u(k) = u(q^2k) - a_1u(qk) - a_2u(k) \tag{3}$$

and its inverse, denoted by $\underset{(a_1,a_2)}{\Delta^{-1}_{\rm q}}$, is defined as below:

$$\text{if } \underset{(a_1,a_2)}{\Delta_{\rm q}} v(k) = u(k),\ \text{then } v(k) = \underset{(a_1,a_2)}{\Delta^{-1}_{\rm q}} u(k). \tag{4}$$

Remark 2.2. *When $a_1=\alpha$ and $a_2=0$, replacing k by k/q in (3), we get (1).*

Lemma 2.3. *If $q^{2n}-a_1q^n-a_2\neq 0$ for $n=0,1,2,\cdots$, then*

$$\underset{(a_1,a_2)}{\Delta^{-1}_{q}} k^n = \frac{k^n}{q^{2n}-a_1q^n-a_2} \ \textit{and}\ \underset{(a_1,a_2)}{\Delta^{-1}_{\rm q}} (1) = \frac{1}{1-a_1-a_2}. \tag{5}$$

Proof. The proof follows by replacing $u(k)$ by k^n and k^0 in (3) and using (4). □

Lemma 2.4. *Let $k\in(0,\infty)$ and $1-a_1-a_2\neq 0$. Then we have*

$$\underset{(a_1,a_2)}{\Delta^{-1}_{\rm q}} \log k = \frac{\log k}{1-a_1-a_2} - \frac{(2-a_1)\log q}{(1-a_1-a_2)^2}. \tag{6}$$

Proof. From (3), replacing $u(k)$ by $\log k$, we get

$$\underset{(a_1,a_2)}{\Delta_{\rm q}} \log k = (2-a_1)\log q + (1-a_1-a_2)\log k, \tag{7}$$

which yields (6) by using the Lemma 2.3. □

Lemma 2.5. *Let $k\in(-\infty,\infty)$ and $q\neq 0$. Then we have*

$$\underset{(q)\alpha}{\Delta^2} u(k) = \underset{(2\alpha,-\alpha^2)}{\Delta_{\rm q}} u(k). \tag{8}$$

Proof. From (1), $\underset{(q)\alpha}{\Delta^2} u(k) = u(q^2k) - 2\alpha u(qk) + \alpha^2u(k)$.

Hence the proof completes from the above equation and by putting $a_1=2\alpha$ and $a_2=-\alpha^2$ in (3). □

3. Fibonacci Sequence Using Two-dimensional q-difference Operator

In this section, we introduce two dimensional Fibonacci sequence and its sum.

Definition 3.1. *For each pair $(a_1, a_2) \in \mathbb{R}^2$, the two-dimensional Fibonacci sequence is defined as*

$$F_{(a_1,a_2)} = \{F_n\}_{n=0}^{\infty}, \tag{9}$$

where $F_0 = 1, F_1 = a_1$ and $F_n = a_1F_{n-1} + a_2F_{n-2}$ for $n \geq 2$.

When $a_1 = a_2 = 1$, (9) becomes the Fibonacci sequence.

Example 3.2. $F_{(2,-3)} = \{1, 2, 1, -4, -11, \cdots\}$

Theorem 3.3 (Two-Dimensional Finite q-Series)**.** *Let $F_n \in F_{(a_1,a_2)}$ and $k \in (-\infty, \infty)$. Then we have*

$$\sum_{r=0}^{m} F_r u\Big(\frac{k}{q^{r+2}}\Big) = \underset{(a_1,a_2)}{\Delta_q^{-1}} u(k) - F_{m+1} \underset{(a_1,a_2)}{\Delta_q^{-1}} u\Big(\frac{k}{q^{m+1}}\Big) - a_2F_m \underset{(a_1,a_2)}{\Delta_q^{-1}} u\Big(\frac{k}{q^{m+2}}\Big), \tag{10}$$

Proof. Taking $\underset{(a_1,a_2)}{\Delta_q^{-1}} u(k) = v(k)$, $\underset{(a_1,a_2)}{\Delta_q} v(k) = u(k)$ and by (3), we write

$$v(q^2k) = u(k) + a_1v(qk) + a_2v(k). \tag{11}$$

Substituting the value of $v(qk)$ in (11), we get

$$v(q^2k) = u(k) + a_1u\Big(\frac{k}{q}\Big) + (a_1^2 + a_2)v(k) + a_1a_2v\Big(\frac{k}{q}\Big). \tag{12}$$

Again putting the value of $v(k)$ in (12), we obtain

$$v(q^2k) = u(k) + a_1u\Big(\frac{k}{q}\Big) + (a_1^2 + a_2)u\Big(\frac{k}{q^2}\Big) + \big\{a_1(a_1^2 + a_2) + a_1a_2\big\}\, v\Big(\frac{k}{q}\Big) + a_2(a_1^2 + a_2)v\Big(\frac{k}{q^2}\Big). \tag{13}$$

Since $F_n \in F_{(a_1,a_2)}$, we get

$$v(q^2k) = F_0u(k) + F_1u\Big(\frac{k}{q}\Big) + F_2u\Big(\frac{k}{q^2}\Big) + F_3v\Big(\frac{k}{q}\Big) + a_2F_2v\Big(\frac{k}{q^2}\Big). \tag{14}$$

Proceeding like this, we arrive

$$v(q^2k) = F_0u(k) + F_1u\Big(\frac{k}{q}\Big) + \cdots + F_mu\Big(\frac{k}{q^m}\Big) + F_{m+1}v\Big(\frac{k}{q^{m-1}}\Big) + a_2F_mv\Big(\frac{k}{q^m}\Big), \tag{15}$$

which completes the proof of the theorem. □

Corollary 3.4. *Assume that $a_1 + a_2 \neq 1$ and $F_n \in F_{(a_1,a_2)}$. Then we have*

$$\sum_{r=0}^{m} F_r = \frac{1 - F_{m+1} - a_2F_m}{1 - a_1 - a_2}.$$

Proof. The proof is trivial by replacing $u(k)$ by k^0 in (10). □

4. Two-Dimensional q Multi-Series

In this section, we obtain formula for sum of q-multi series.

Theorem 4.1. *Let* $0 \neq q_i, k \in (-\infty, \infty)$ *and* $F_n \in F_{(a_1,a_2)}$*. Then*

$$\sum_{i=1}^{t-1} \sum_{(r)_{1\to i}}^{m} \prod_{j=1}^{i} F_{r_j} \underset{(a_1,\ a_2\ )}{\underset{q_{i+1\to t}}{\Delta^{-1}}} \Bigg\{ F_{m_{i+1}+1}\, u\bigg(\frac{\prod_{p=i+1}^{t-1} q_p^2 k}{\prod_{p=1}^{i} q_p^{r_p} q_{i+1}^{m_{i+1}+1}} \bigg)$$

$$+ a_2 F_{m_{i+1}} u\bigg(\frac{\prod_{p=i+1}^{t-1} q_p^2 k}{\prod_{p=1}^{i} q_p^{r_p} q_{i+1}^{m_{i+1}+2}} \bigg) \Bigg\} + \sum_{(r)_{1\to t}}^{m} \prod_{i=1}^{t} F_{r_i} u\bigg(\frac{k}{\prod_{i=1}^{t} q_i^{r_i} q_t^2} \bigg)$$

$$= \underset{(a_1,\ a_2)}{\underset{q_{1\to t}}{\Delta^{-1}}} \Bigg\{ u\bigg(\prod_{p=1}^{t-1} q_p^2 k \bigg) - F_{m_1+1} u\bigg(\frac{\prod_{p=1}^{t-1} q_p^2 k}{q_1^{m_1+1}} \bigg) - a_2 F_{m_1} u\bigg(\frac{\prod_{p=1}^{t-1} q_p^2 k}{q_1^{m_1+2}} \bigg) \Bigg\}. \tag{16}$$

Proof. Replacing q, m, r by q_2, m_2, r_2 in (10), we get

$$\sum_{r_2=0}^{m_2} F_{r_2} u\Big(\frac{k}{q_2^{r_2+2}}\Big) = \underset{(a_1,a_2)}{\Delta_{q_2}^{-1}} u(k) - F_{m_2+1} \underset{(a_1,a_2)}{\Delta_{q_2}^{-1}} u\Big(\frac{k}{q_2^{m_2+1}}\Big) - a_2 F_{m_2} \underset{(a_1,a_2)}{\Delta_{q_2}^{-1}} u\Big(\frac{k}{q_2^{m_2+2}}\Big). \tag{17}$$

Replacing k by $k/q_1^{r_1}$ and multiplying by F_{r_1} for $r_1 = 1, 2, \cdots, m_1$ in (17), and using (10) after summing the resultant expressions with (17), we arrive

$$\sum_{r_1=0}^{m_1} F_{r_1} \sum_{r_2=0}^{m_2} F_{r_2} u\Big(\frac{k}{q_1^{r_1} q_2^{r_2+2}}\Big) = \underset{(a_1,a_2)}{\Delta_{q_1}^{-1}} \underset{(a_1,a_2)}{\Delta_{q_2}^{-1}} u(q_1^2 k) - F_{m_1+1} \underset{(a_1,a_2)}{\Delta_{q_1}^{-1}} \underset{(a_1,a_2)}{\Delta_{q_2}^{-1}} u\Big(\frac{q_1^2 k}{q_1^{m_1+1}}\Big)$$

$$-a_2 F_{m_1} \underset{(a_1,a_2)}{\Delta_{q_1}^{-1}} \underset{(a_1,a_2)}{\Delta_{q_2}^{-1}} u\Big(\frac{q_1^2 k}{q_1^{m_1+2}}\Big) - \sum_{r_1=0}^{m_1} F_{r_1} F_{m_2+1} \underset{(a_1,a_2)}{\Delta_{q_2}^{-1}} u\Big(\frac{k}{q_1^{r_1} q_2^{m_2+1}}\Big)$$

$$- \sum_{r_1=0}^{m_1} a_2 F_{r_1} F_{m_2} \underset{(a_1,a_2)}{\Delta_{q_2}^{-1}} u\Big(\frac{k}{q_1^{r_1} q_2^{m_2+2}}\Big). \tag{18}$$

Again replacing $q_1, q_2, r_1, r_2, m_1, m_2$ by $q_2, q_3, r_2, r_3, m_2, m_3$ in (18), then k by $k/q_1^{r_1}$ and multiplying by F_{r_1} for $r_1 = 1, 2, \cdots, m_1$ and then summing all the resultant expressions, we arrive

$$\sum_{r_1=0}^{m_1} F_{r_1} \sum_{r_2=0}^{m_2} F_{r_2} \sum_{r_3=0}^{m_3} F_{r_3} u\Big(\frac{k}{q_1^{r_1} q_2^{r_2} q_3^{r_3+2}}\Big) = \sum_{r_1=0}^{m_1} F_{r_1} \Bigg\{ \underset{(a_1,a_2)}{\Delta_{q_2}^{-1}} \underset{(a_1,a_2)}{\Delta_{q_3}^{-1}} u(q_2^2 k)$$

$$-F_{m_2+1} \underset{(a_1,a_2)}{\Delta_{q_2}^{-1}} \underset{(a_1,a_2)}{\Delta_{q_3}^{-1}} u\Big(\frac{q_2^2 k}{q_2^{m_2+1}}\Big) - a_2 F_{m_2} \underset{(a_1,a_2)}{\Delta_{q_2}^{-1}} \underset{(a_1,a_2)}{\Delta_{q_3}^{-1}} u\Big(\frac{q_2^2 k}{q_2^{m_2+2}}\Big)$$

$$- \sum_{r_2=0}^{m_2} F_{r_2} F_{m_3+1} \underset{(a_1,a_2)}{\Delta_{q_3}^{-1}} u\Big(\frac{k}{q_2^{r_2} q_3^{m_3+1}}\Big) - \sum_{r_2=0}^{m_2} a_2 F_{r_2} F_{m_3} \underset{(a_1,a_2)}{\Delta_{q_3}^{-1}} u\Big(\frac{k}{q_2^{r_2} q_3^{m_3+2}}\Big) \Bigg\}.$$

By applying (10) on the above equation and repeating the above procedure complete the proof of this theorem. □

Corollary 4.2. *Let* $k \in (0, \infty)$*,* $q \neq 0$ *and* $F_n \in F_{(a_1,a_2)}$*. Then we obtain*

$$\sum_{r_1=0}^{m_1} \frac{F_{r_1} q_3^4}{q_1^{2r_1} q_2^{2m_2-2}} \Big(F_{m_2+1} + \frac{a_2 F_{m_2}}{q_2^2} \Big) \underset{(a_1,\ a_2)}{\underset{q_{2\to 4}}{\Delta^{-1}}} k^2 + \sum_{(r)_{1\to 2}}^{m} \frac{F_{r_1} F_{r_2}}{q_1^{2r_1} q_2^{2r_2} q_3^{2m_3-2}}$$

$$\Big(F_{m_3+1} + \frac{a_2 F_{m_3}}{q_3^2} \Big) \underset{(a_1,\ a_2)}{\underset{q_{3\to 4}}{\Delta^{-1}}} k^2 + \sum_{(r)_{1\to 3}}^{m} \prod_{i=1}^{3} \Big(\frac{F_{r_i}}{q_i^{2r_i} q_4^{2m_4+2}} \Big) \Big(F_{m_4+1} + \frac{a_2 F_{m_4}}{q_4^2} \Big)$$

$$\underset{(a_1,a_2)}{\Delta_{q_4}^{-1}} k^2 + \sum_{(r)_{1\to 4}}^{m} \prod_{i=1}^{4} \frac{F_{r_i}}{q_i^{2r_i} q_4^4} k^2 = q_1^4 q_2^4 q_3^4 \Big(1 - \frac{F_{m_1+1}}{q_1^{2m_1+2}} - \frac{a_2 F_{m_1}}{q_1^{2m_1+4}} \Big) \underset{(a_1,\ a_2)}{\underset{q_{1\to 4}}{\Delta^{-1}}} k^2. \tag{19}$$

Proof. The proof is trivial by taking $t = 4$ and $u(k) = k^2$ in (16). ☐

The following example illustrates (19).

Example 4.3. *Taking $m_1 = m_2 = 1, m_3 = 2$ and $m_4 = 2$ in (19), we get*

$$q_3^4\Big(F_2 + \frac{a_2F_1}{q_2^2}\Big)\sum_{r_1=0}^{1}\frac{F_{r_1}}{q_1^{2r_1}}\underset{(a_1,\,a_2)}{\Delta^{-1}_{q_2\to 4}}\,k^2 + \frac{1}{q_3^2}\Big(F_3 + \frac{a_2F_2}{q_3^2}\Big)\sum_{r_1=0}^{1}\sum_{r_2=0}^{1}\frac{F_{r_1}F_{r_2}}{q_1^{2r_1}q_2^{2r_2}}\underset{(a_1,\,a_2)}{\Delta^{-1}_{q_3\to 4}}\,k^2$$
$$+\frac{1}{q_4^6}\Big(F_3 + \frac{a_2F_2}{q_4^2}\Big)\sum_{r_1=0}^{1}\sum_{r_2=0}^{1}\sum_{r_3=0}^{2}\prod_{i=1}^{3}\frac{F_{r_i}}{q_i^{2r_i}}\underset{(a_1,a_2)}{\Delta^{-1}_{q_4}}\,k^2 + \sum_{r_1=0}^{1}\sum_{r_2=0}^{1}\sum_{r_3=0}^{2}\sum_{r_4=0}^{2}\prod_{i=1}^{4}\frac{F_{r_i}k^2}{q_i^{2r_i}q_4^4}$$
$$= q_1^4q_2^4q_3^4\left\{1 - \frac{F_2}{q_1^4} - \frac{a_2F_1}{q_1^6}\right\}\underset{(a_1,\,a_2)}{\Delta^{-1}_{q_1\to 4}}\,k^2. \tag{20}$$

From (5), we have
$\underset{(a_1,a_2)}{\Delta^{-1}_{q_4}}\,k^2 = \dfrac{k^2}{q_4^4 - a_1q_4^2 - a_2}$ and so $\underset{(a_1,\,a_2)}{\Delta^{-1}_{q_3\to 4}}\,k^2 = \dfrac{k^2}{(q_3^4 - a_1q_3^2 - a_2)(q_4^4 - a_1q_4^2 - a_2)}$.
Similarly, we can find $\underset{(a_1,\,a_2)}{\Delta^{-1}_{q_2\to 4}}\,k^2$ and $\underset{(a_1,\,a_2)}{\Delta^{-1}_{q_1\to 4}}\,k^2$. Hence (20) bocomes

$$\frac{q_3^4\Big(F_2 + \frac{a_2F_1}{q_2^2}\Big)\Big(1 + \frac{F_1}{q_1^2}\Big)k^2}{\prod_{i=2}^{4}(q_i^4 - a_1q_i^2 - a_2)} + \frac{\Big(F_3 + \frac{a_2F_2}{q_3^2}\Big)\Big(1 + \frac{F_1}{q_1^2}\Big)\Big(1 + \frac{F_1}{q_2^2}\Big)k^2}{q_3^2\prod_{i=3}^{4}(q_i^4 - a_1q_i^2 - a_2)}$$
$$+\frac{\Big(1 + \frac{F_1}{q_1^2}\Big)\Big(1 + \frac{F_1}{q_2^2}\Big)\Big(1 + \frac{F_1}{q_3^2} + \frac{F_2}{q_3^4}\Big)\Big(F_3 + \frac{a_2F_2}{q_4^2}\Big)k^2}{q_4^6(q_4^4 - a_1q_4^2 - a_2)}$$
$$+\Big(1 + \frac{F_1}{q_1^2}\Big)\Big(1 + \frac{F_1}{q_2^2}\Big)\Big(1 + \frac{F_1}{q_3^2} + \frac{F_2}{q_3^4}\Big)\Big(1 + \frac{F_1}{q_4^2} + \frac{F_2}{q_4^4}\Big)\frac{k^2}{q_4^4}$$
$$= q_1^4q_2^4q_3^4\Big(1 - \frac{F_2}{q_1^4} - \frac{a_2F_1}{q_1^6}\Big)\frac{k^2}{\prod_{i=1}^{4}(q_i^4 - a_1q_i^2 - a_2)}. \tag{21}$$

5. Discrete Version of Generalized Product Formula

Here, we obtain inverse for product of two functions with respect to $\underset{(a_1,a_2)}{\Delta_q}$.

Theorem 5.1. *For the real valued functions $u(k)$ and $v(k)$, we have*

$$\underset{(a_1,a_2)}{\Delta_q^{-1}}\big(u(k)v(k)\big) = \frac{1}{a_2}\Bigg\{u(k)\underset{(0,1)}{\Delta_q^{-1}}v(k) - \underset{(a_1,a_2)}{\Delta_q^{-1}}\Big(\underset{(a_1,a_2)}{\Delta_q}u(k)\underset{(0,1)}{\Delta_q^{-1}}v(q^2k)\Big)$$
$$-a_1\underset{(a_1,a_2)}{\Delta_q^{-1}}\Big(u(qk)\underset{(1,0)}{\Delta_q}\big(\underset{(0,1)}{\Delta_q^{-1}}v(k)\big)\Big)\Bigg\}. \tag{22}$$

Proof. From (1), we find that

$$\underset{(a_1,a_2)}{\Delta_q}\big(u(k)w(k)\big) = \underset{(a_1,a_2)}{\Delta_q}u(k)w(q^2k) + a_1u(qk)\left\{w(q^2k) - w(qk)\right\}$$
$$+a_2u(k)\left\{w(q^2k) - w(k)\right\},$$

which gives $\underset{(a_1,a_2)}{\Delta_q}\big(u(k)w(k)\big) = \underset{(a_1,a_2)}{\Delta_q}u(k)w(q^2k) + a_1u(qk)\underset{(1,0)}{\Delta_q}w(k) + a_2u(k)\underset{(0,1)}{\Delta_q}w(k)$. The proof follows by applying equation (4) in the above equation and using the relation $v(k) = \underset{(0,1)}{\Delta_q}w(k)$. ☐

Corollary 5.2. *For real valued function $v(k)$ and for $k>0$, we have*

$$\underset{(a_1,a_2)}{\Delta_q^{-1}}\big(v(k)\log(k)\big)=\frac{1}{a_2}\Bigg\{\log(k)\underset{(0,1)}{\Delta_q^{-1}}v(k)-\underset{(a_1,a_2)}{\Delta_q^{-1}}\Big(\underset{(a_1,a_2)}{\Delta_q}\log(k)\underset{(0,1)}{\Delta_q^{-1}}v(q^2k)\Big)$$

$$-a_1\underset{(a_1,a_2)}{\Delta_q^{-1}}\Big(\log(qk)\underset{(1,0)}{\Delta_q}\big(\underset{(0,1)}{\Delta_q^{-1}}v(k)\big)\Big)\Bigg\} \tag{23}$$

Proof. The proof follows by replacing $u(k)$ by $\log(k)$ in (22). ☐

Corollary 5.3. *Let $q,k>0$, $1-a_1q-a_2q^2\neq 0$ and $F_n\in F_{(a_1,a_2)}$. Then we have*

$$\underset{(a_1,a_2)}{\Delta_q^{-1}}\big(\frac{1}{k}\log(k)\big)=\frac{q^2}{(1-a_1q-a_2q^2)k}\left\{\log(k)-\frac{(2-a_1q)\log(q)}{(1-a_1q-a_2q^2)}\right\} \tag{24}$$

and hence

$$\big(1-F_{m+1}q^{m+1}-a_2F_mq^{m+2}\big)\underset{(a_1,a_2)}{\Delta_q^{-1}}\big(\frac{1}{k}\log(k)\big)+\big((m+1)F_{m+1}q^{m+1}+a_2(m+2)F_mq^{m+2}\big)$$

$$\times\frac{q^2\log(q)}{(1-a_1q-a_2q^2)k}=\sum_{r=0}^{m}F_r\frac{q^{r+2}}{k}\log\Big(\frac{k}{q^{r+2}}\Big). \tag{25}$$

Proof. Taking $v(k)=1/k$ in (23) results (24). The proof of (25) is obvious from (24) and replacing $u(k)$ by $\frac{1}{k}\log(k)$ in (10). ☐

6. Conclusion

In this paper, we have introduced two-dimensional q-difference operator and its equation. The closed form solution found in this paper agreed very well with the numerical solution of the generalized two-dimensional q-difference equation which generates various summation formulae on Fibonacci series

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