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Some Finite Summations Involving Srivastava-Daoust Double Hypergeometric Functions

Research Article

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Abstract: The main object of the present paper is to obtain seven finite summation formulae (not recorded earlier) involving Srivastava-Daoust double hypergeometric functions using series rearrangement technique.

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Keywords: Pochhammer's symbol; Double hypergeometric functions of Exton and Srivastava-Daoust; Ordinary Gaussian hypergeometric function.

1. Introduction

In 1979, Exton [4] defined the double hypergeometric function $\mathscr{H}$ in the following form

$$\mathscr{H}^{A:B;C;D}_{E:G;M;N}\left[\begin{array}{l}(a_A):(b_B);\ (c_C);\ (d_D)\ ;\\ \\ (e_E):(g_G);(m_M);(n_N)\ ;\end{array}\ x,\ y\right]$$

$$= F^{A+B;C;D}_{E+G;M;N}\left(\begin{array}{l}[(a_A):2,1],\ [(b_B):1,1]:\ [(c_C):1];\ [(d_D):1]\ ;\\ \\ [(e_E):2,1],\ [(g_G):1,1]:[(m_M):1];[(n_N):1]\ ;\end{array}\ x,\ y\right)$$

$$= \sum_{i,j=0}^{\infty}\frac{[(a_A)]_{2i+j}\ [(b_B)]_{i+j}\ [(c_C)]_i\ [(d_D)]_j\ x^i\ y^j}{[(e_E)]_{2i+j}\ [(g_G)]_{i+j}\ [(m_M)]_i\ [(n_N)]_j\ i!\ j!} \tag{1}$$

Making suitable adjustment in numbers of numerator and denominator parameters of (1), we obtain Kampé de Fériet's double hypergeometric function [15, 16] given by $F^{B:C;D}_{G:M;N}\equiv\mathscr{H}^{0:B;C;D}_{0:G;M;N}$, another additional double hypergeometric function of Exton [5] given by $X^{E:M;N}_{A:C;D}\equiv\mathscr{H}^{E:0;M;N}_{A:0;C;D}$, Appell's four double hypergeometric functions $F_1,\ F_2,\ F_3,\ F_4$ [2], Humbert's seven double hypergeometric functions $\Phi_1,\ \Phi_2,\ \Phi_3,\ \Psi_1,\ \Psi_2,\ \Xi_1,\ \Xi_2$ [2], Horn's non confluent double hypergeometric

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functions H_3 and H_4 [2] and its confluent forms H_6 and H_7 [2] respectively. In 1984, Exton [1, 6] defined the double hypergeometric function G in the following form

$$G^{A:B;D}_{E:H;M}\left[\begin{array}{lr}(a_A):(b_B);(d_D)\ ; & \\ & x,y\\ (e_E):(h_H);(m_M)\ ; & \end{array}\right]=F^{A:B;D}_{E:H;M}\left(\begin{array}{lr}[(a_A):1,-1]:[(b_B):1]\ ;\ [(d_D):1]\ ; & \\ & x,\ y\\ [(e_E):1,-1]:[(h_H):1]\ ;[(m_M):1]\ ; & \end{array}\right)$$

$$=\sum_{i,j=0}^{\infty}\frac{[(a_A)]_{i-j}\ [(b_B)]_i\ [(d_D)]_j\ x^i\ y^j}{[(e_E)]_{i-j}\ [(h_H)]_i\ [(m_M)]_j\ i!\ j!} \tag{2}$$

It is the generalization and unification of Horn's non confluent double hypergeometric function G_2 [2], H_2 [2] and Horn's confluent double hypergeometric functions $\Gamma_1,\Gamma_2,\mathrm{H}_2,\mathrm{H}_3,\mathrm{H}_4,\mathrm{H}_5,\mathrm{H}_{11}$ [2, 7, 8]. The notations $F^{A+B;C;D}_{E+G;M;N}$ and $F^{A:B;D}_{E:H;M}(\cdot)$ in (1) and (2) are due to Srivastava-Daoust [17]. The symbol (a_A) denotes the array of A parameters in Slater's contracted notation [11, 12] given by $a_1,a_2,\cdots,a_A$ with similar interpretations for others. The Pochhammer's symbol $[(b_B)]_u$ is defined by

$$\begin{aligned}[(b_B)]_u&=(b_1)_u(b_2)_u\cdots(b_B)_u=\prod_{m=1}^{B}\{(b_m)_u\}\\ &=\prod_{m=1}^{B}\left\{\frac{\Gamma(b_m+u)}{\Gamma(b_m)}\right\},\text{if } b_m\neq 0,-1,-2,-3,\dots\\ &=\prod_{m=1}^{B}\{(b_m)(b_m+1)(b_m+2)\cdots(b_m+u-1)\},\text{if } u=1,2,3,\dots\end{aligned} \tag{3}$$

and $(b_m)_0=1$. The notation Γ is used for Gamma function. The denominator parameters in (1) and (2) are neither zero nor negative integers, numerator parameters may be zero or negative integers. In our investigation, we shall use the following results:

$$\sum_{n=0}^{m}\sum_{r=0}^{n}\sum_{s=0}^{m-n}\Phi(m,n,r,s)=\sum_{s=0}^{m}\sum_{r=0}^{m-s}\sum_{n=0}^{m-s-r}\Phi(m,n+r,r,s) \tag{4}$$

$$\sum_{n=0}^{k}(-1)^n\begin{pmatrix}k\\ n\end{pmatrix}=\sum_{n=0}^{k}\frac{(-k)_n}{n!}=\begin{cases}1; & \text{if } k=0\\ 0; & \text{if } k=1,2,3,\dots\end{cases} \tag{5}$$

where $\begin{pmatrix}k\\ n\end{pmatrix}$ is binomial coefficient.

$$\sum_{s=0}^{m}\sum_{r=0}^{m-s}\frac{\Psi(r+s)\ x^s\ y^r}{s!\ r!}=\sum_{p=0}^{m}\frac{\Psi(p)\ (x+y)^p}{p!} \tag{6}$$

The finite triple series identity (4) and combinatorial identity (5) are due to Srivastava [14]. The finite double series identity (6) is also due to Srivastava [10, 13]. The following Pochhammer's symbols identities used in the derivation (13)-(19), can be proved in view of the definition (3) of Pochhammer's symbol:

$$[1-(b_B)-n-r]_r=\frac{(-1)^{Br}[(b_B)]_{n+r}}{[(b_B)]_n} \tag{7}$$

$$[1-(e_E)-n-r]_{i+r}=\frac{[(e_E)]_{n+r}\ [1-(e_E)]_{i-n}}{(-1)^{E(n+r)}} \tag{8}$$

$$(-n-r)_r=\frac{(-1)^r(n+r)!}{n!} \tag{9}$$

$$[(k_K)+m]_{-s}=\frac{(-1)^{Ks}}{[1-(k_K)-m]_s} \tag{10}$$

$$[(a_A)]_{m+n+p}=[(a_A)]_m\ [(a_A)+m]_n[(a_A)+m+n]_p \tag{11}$$

$$[(e_E)]_{m-n}=\frac{[(e_E)]_m(-1)^{nE}}{[1-(e_E)-m]_n} \tag{12}$$

2. Main Finite Summation Formulae

Since Pochhammer symbols are associated with Gamma function therefore numerator, denominator parameters and arguments are adjusted in such a manner that each side is completely meaningful and well defined, then without any loss of convergence, we have

$$\sum_{n=0}^{m}\frac{(-m)_n}{n!}G^{A:B+1;D+1}_{E:H;T}\left[\begin{array}{l}(a_A):-m+n,\,(b_B);\,-n,\,(d_D)\;;\\ \qquad\qquad\qquad\qquad\qquad\qquad x,y\\ (e_E)\;:\;(h_H)\;;\;(t_T)\;;\end{array}\right]=\frac{(-1)^{m(A-E)}[(d_D)]_m\,[1-(e_E)]_m\,y^m}{[(t_T)]_m\,[1-(a_A)]_m}$$
$$\times\,{}_{U_1}F_{V_1}\left[\begin{array}{l}-m,\,\frac{(a_A)-m}{2},\,\frac{(a_A)-m+1}{2},\,(b_B),\,1-(t_T)-m\;;\\ \qquad\qquad\qquad\qquad\qquad 4^{(A-E)}(-1)^{(D-T)}\left(\frac{x}{y}\right)\\ \frac{(e_E)-m}{2},\,\frac{(e_E)-m+1}{2},\,(h_H),\,1-(d_D)-m\;;\end{array}\right]\tag{13}$$

$$\sum_{n=0}^{m}\frac{(-m)_n}{n!}G^{A:B+1;D+1}_{E:H;T}\left[\begin{array}{l}(a_A):\,-n,\,(b_B);\,-m+n,\,(d_D)\;;\\ \qquad\qquad\qquad\qquad\qquad\qquad x,\,y\\ (e_E)\;:\;(h_H)\;;\;(t_T)\;;\end{array}\right]=\frac{[(a_A)]_m\,[(b_B)]_m\,x^m}{[(e_E)]_m\,[(h_H)]_m}$$
$$\times\,{}_{U_2}F_{V_2}\left[\begin{array}{l}-m,\,\frac{1-(e_E)-m}{2},\,\frac{2-(e_E)-m}{2},\,(d_D),\,1-(h_H)-m\;;\\ \qquad\qquad\qquad\qquad\qquad 4^{(E-A)}(-1)^{(B-H)}\left(\frac{y}{x}\right)\\ \frac{1-(a_A)-m}{2},\,\frac{2-(a_A)-m}{2},\,(t_T),\,1-(b_B)-m\;;\end{array}\right]\tag{14}$$

$$\sum_{n=0}^{m}\frac{(-m)_n\,[(a_A)]_{2n}\,[(b_B)]_n\,[(d_D)]_{m-n}}{n!\,[(e_E)]_{2n}[(h_H)]_n[(t_T)]_{m-n}}\left(\frac{x}{y}\right)^n$$
$$\times\,G^{A:B+1;D+1}_{E:H;T}\left[\begin{array}{l}(a_A)+2n:\,-m+n,\,(b_B)+n;\,-n,\,(d_D)+m-n\;;\\ \qquad\qquad\qquad\qquad\qquad\qquad x,\,y\\ (e_E)+2n:\,(h_H)+n;\,(t_T)+m-n\;;\end{array}\right]$$
$$=\frac{[(d_D)]_m}{[(t_T)]_m}\,{}_{U_1}F_{V_1}\left[\begin{array}{l}-m,\,\frac{(a_A)}{2},\,\frac{(a_A)+1}{2},\,(b_B),\,1-(t_T)-m\;;\\ \qquad\qquad\qquad\qquad\qquad 4^{(A-E)}(-1)^{(D-T)}\left(\frac{x}{y}\right)\\ \frac{(e_E)}{2},\,\frac{(e_E)+1}{2},\,(h_H),\,1-(d_D)-m\;;\end{array}\right]\tag{15}$$

$$\sum_{n=0}^{m}\frac{(-m)_n\,[(a_A)]_{-2n}\,[(b_B)]_{m-n}\,[(d_D)]_n}{n!\,[(e_E)]_{-2n}[(h_H)]_{m-n}[(t_T)]_n}\left(\frac{y}{x}\right)^n$$
$$\times\,G^{A:B+1;D+1}_{E:H;T}\left[\begin{array}{l}(a_A)-2n:\,-n,\,(b_B)+m-n;\,-m+n,\,(d_D)+n\;;\\ \qquad\qquad\qquad\qquad\qquad\qquad x,\,y\\ (e_E)-2n:\,(h_H)+m-n;\,(t_T)+n\;;\end{array}\right]$$
$$=\frac{[(b_B)]_m}{[(h_H)]_m}\,{}_{U_2}F_{V_2}\left[\begin{array}{l}-m,\,\frac{1-(e_E)}{2},\,\frac{2-(e_E)}{2},\,(d_D),\,1-(h_H)-m\;;\\ \qquad\qquad\qquad\qquad\qquad 4^{(E-A)}(-1)^{(B-H)}\left(\frac{y}{x}\right)\\ \frac{1-(a_A)}{2},\,\frac{2-(a_A)}{2},\,(t_T),\,1-(b_B)-m\;;\end{array}\right]\tag{16}$$

$$\sum_{n=0}^{m}\frac{(-m)_n[(a_A)]_n[(d_D)]_n[(e_E)]_{m-n}(\frac{z}{y})^n}{[(g_G)]_n[(k_K)]_n[(t_T)]_{m-n}\ n!}$$

$$\times\mathscr{H}^{A:B;D+1;E+1}_{G:H;K;T}\left[\begin{array}{l}(a_A)+n:(b_B);-m+n,(d_D)+n;-n,(e_E)+m-n\ ;\\ \\ (g_G)+n:(h_H);\qquad (k_K)+n;\qquad (t_T)+m-n\ ;\end{array} x,\ y\right]$$

$$=\frac{[(e_E)]_m}{[(t_T)]_m}\mathscr{H}^{A:D+1;B;T}_{G:K;H;E}\left[\begin{array}{l}(a_A):-m,(d_D);\ (b_B);\ 1-(t_t)-m\ \ ;\\ \\ (g_G):\qquad (k_K);\ (h_H);\ 1-(e_E)-m\ ;\end{array} (x-z),\ (-1)^{E-T}(\tfrac{z}{y})\right] \tag{17}$$

$$\sum_{n=0}^{m}\frac{(-m)_n}{n!}\mathscr{H}^{A:B;D+1;E+1}_{G:H;K;T}\left[\begin{array}{l}(a_A):(b_B);-n,(d_D);-m+n,(e_E)\ ;\\ \\ (g_G):(h_H);\qquad (k_K);\qquad (t_T)\qquad ;\end{array} x,\ y\right]=\frac{[(a_A)]_{2m}[(b_B)]_m[(d_D)]_m\ x^m}{[(g_G)]_{2m}[(h_H)]_m[(k_K)]_m}$$

$$\times\ {}_{U_3}F_{V_3}\left[\begin{array}{l}-m,1-(g_G)-2m,1-(k_K)-m,(e_E)\ ;\\ \\ 1-(a_A)-2m,\ \ 1-(d_D)-m,\ \ (t_T)\ ;\end{array} (-1)^{A+D-G-K}(\tfrac{y}{x})\right] \tag{18}$$

$$\sum_{n=0}^{m}\frac{(-m)_n}{n!}\mathscr{H}^{A:B;D+1;E+1}_{G:H;K;T}\left[\begin{array}{l}(a_A):(b_B);-m+n,(d_D);-n,(e_E)\ ;\\ \\ (g_G):(h_H);\qquad (k_K);\qquad (t_T)\qquad ;\end{array} x,\ y\right]=\frac{[(a_A)]_m[(b_B)]_m[(e_E)]_m\ y^m}{[(g_G)]_m[(h_H)]_m[(t_T)]_m}$$

$$\times\ {}_{U_4}F_{V_4}\left[\begin{array}{l}-m,(a_A)+m,(d_D),1-(t_T)-m\ ;\\ \\ (g_G)+m,\ \ (k_K),\ \ \ 1-(e_E)-m\ ;\end{array} (-1)^{E-T}(\tfrac{x}{y})\right] \tag{19}$$

where $U_1=1+2A+B+T,\ U_2=1+2E+D+H,\ U_3=1+G+K+E,\ U_4=1+A+D+T,\ V_1=2E+H+D,\ V_2=2A+T+B,\ V_3=A+D+T,\ V_4=G+K+E$ and ${}_{U_1}F_{V_1},{}_{U_2}F_{V_2},{}_{U_3}F_{V_3},{}_{U_4}F_{V_4}$ are generalized hypergeometric polynomials of one variable[8, p.41(2.1.1.3)].

3. Derivations of (13)-(19)

Expressing the left hand side of (13) in power series form, we get:

$$W_1=\sum_{n=0}^{m}\frac{(-m)_n}{n!}\sum_{s=0}^{m-n}\sum_{r=0}^{n}\frac{[(a_A)]_{s-r}(-m+n)_s[(b_B)]_s(-n)_r[(d_D)]_r x^s y^r}{[(e_E)]_{s-r}[(h_H)]_s[(t_T)]_r\ s!\ r!} \tag{20}$$

Now applying finite triple series identity (4) of Srivastava, we get:

$$W_1=\sum_{s=0}^{m}\left[\sum_{r=0}^{m-s}\frac{[(a_A)]_{s-r}(-m)_{r+s}[(b_B)]_s[(d_D)]_r\ x^s(-y)^r}{[(e_E)]_{s-r}[(h_H)]_s[(t_T)]_r\ s!\ r!}\left\{\sum_{n=0}^{m-r-s}\frac{(-m+r+s)_n}{n!}\right\}\right] \tag{21}$$

Now apply combinatorial identity (5) in the square bracket of (21), When r varies from 0 to $m-s-1$, corresponding terms in square bracket vanish, therefore when $r=m-s$, we get:

$$W_1=\frac{[(a_A)]_{-m}\ (-m)_m\ [(d_D)]_m\ (-y)^m}{[(e_E)]_{-m}\ [(t_T)]_m\ m!}\sum_{s=0}^{m}\frac{[(a_A)-m]_{2s}\ [(b_B)]_s[(d_D)+m]_{-s}\ (-m)_s}{[(e_E)-m]_{2s}[(h_H)]_s[(t_T)+m]_{-s}\ s!}\left(\frac{x}{y}\right)^s$$

$$= \frac{(-1)^{m(A-E)}[1-(e_E)]_m[(d_D)]_m y^m}{[1-(a_A)]_m[(t_T)]_m}\sum_{s=0}^{m}\frac{(-m)_s\ [(b_B)]_s\ [1-(t_T)-m]_s}{[(h_H)]_s\ [1-(d_D)-m]_s}\times\frac{\prod_{u=1}^{2}\left\{\left[\frac{(a_A)-m+u-1}{2}\right]_s\right\}2^{2s(A-E)}\ (-1)^{s(D-T)}}{\prod_{u=1}^{2}\left\{\left[\frac{(e_E)-m+u-1}{2}\right]_s\right\}\ s!}\left(\frac{x}{y}\right)^s$$

Now express above power series into its hypergeometric form, we get right hand side of (13). Similarly we can obtain (14). Now expressing left hand side of (15) in power series form, we get:

$$W_2=\sum_{n=0}^{m}\sum_{r=0}^{n}\sum_{s=0}^{m-n}\frac{(-n)_r\ (-m)_{n+s}\ [(a_A)]_{2n+s-r}[(b_B)]_{n+s}[(d_D)]_{m-n+r}\ x^s y^r}{[(e_E)]_{2n+s-r}[(h_H)]_{n+s}[(t_T)]_{m-n+r}\ n!\ s!\ r!}\left(\frac{x}{y}\right)^n$$

Now applying finite series identities (4) and (6), we get:

$$W_2=\frac{[(d_D)]_m}{[(t_T)]_m}\sum_{s=0}^{m}\sum_{r=0}^{m-s}\frac{(-m)_{r+s}\ [(a_A)]_{r+s}[(b_B)]_{r+s}x^s(-x)^r}{[(e_E)]_{r+s}\ [(h_H)]_{r+s}\ s!\ r!}$$
$$\times\sum_{n=0}^{m-r-s}\frac{(-m+r+s)_n\ [(a_A)+s+r]_{2n}\ [(b_B)+s+r]_n\ [(d_D)+m]_{-n}}{n!\ [(e_E)+s+r]_{2n}\ [(h_H)+s+r]_n\ [(t_T)+m]_{-n}}\left(\frac{x}{y}\right)^n$$
$$=\frac{[(d_D)]_m}{[(t_T)]_m}\sum_{p=0}^{m}\frac{(-m)_p\ [(a_A)]_p[(b_B)]_p(x-x)^p}{[(e_E)]_p\ [(h_H)]_p\ p!}$$
$$\times\sum_{n=0}^{m-p}\frac{(-m+p)_n\ [(a_A)+p]_{2n}\ [(b_B)+p]_n\ [(d_D)+m]_{-n}}{n!\ [(e_E)+p]_{2n}\ [(h_H)+p]_n\ [(t_T)+m]_{-n}}\left(\frac{x}{y}\right)^n$$
$$=\frac{[(d_D)]_m}{[(t_T)]_m}\sum_{n=0}^{m}\frac{(-m)_n\ 2^{2n(A-E)}\ \left[\frac{(a_A)}{2}\right]_n\ \left[\frac{(a_A)+1}{2}\right]_n\ [(b_B)]_n}{\left[\frac{(e_E)}{2}\right]_n\ \left[\frac{(e_E)+1}{2}\right]_n\ [(h_H)]_n\ n!}\frac{(-1)^{n(D-T)}\ [1-(t_T)-m]_n}{[1-(d_D)-m]_n}\left(\frac{x}{y}\right)^n$$

Now express above power series into its corresponding hypergeometric form, we get the right hand side of (15). Similarly we can obtain (16). The left hand side of (17) can be written in the following form

$$W_3=\sum_{n=0}^{m}\frac{(-m)_n[(a_A)]_n[(d_D)]_n[(e_E)]_{m-n}(\frac{z}{y})^n}{[(g_G)]_n[(k_K)]_n[(t_T)]_{m-n}\ n!}$$
$$\times\sum_{s=0}^{m-n}\sum_{r=0}^{n}\frac{[(a_A)+n]_{2s+r}[(b_B)]_{s+r}(-m+n)_s[(d_D)+n]_s(-n)_r[(e_E)+m-n]_r x^s\ y^r}{[(g_G)+n]_{2s+r}[(h_H)]_{s+r}[(k_K)+n]_s[(t_T)+m-n]_r s!r!}$$
$$=\sum_{n=0}^{m}\sum_{r=0}^{n}\sum_{s=0}^{m-n}\frac{(-m)_{n+s}[(a_A)]_{n+2s+r}[(d_D)]_{n+s}[(e_E)]_{m-n+r}(-n)_r[(b_B)]_{s+r}x^s\ z^n\ y^{r-n}}{[(g_G)]_{n+2s+r}[(k_K)]_{n+s}[(t_T)]_{m-n+r}[(h_H)]_{s+r}\ n!r!s!}\tag{22}$$

Now using finite triple series identity (4), after simplification we get

$$W_3=\frac{[(e_E)]_m}{[(t_T)]_m}\sum_{s=0}^{m}\sum_{r=0}^{m-s}\frac{(-m)_{s+r}[(a_A)]_{2s+2r}[(d_D)]_{r+s}[(b_B)]_{r+s}(-z)^r\ x^s}{[(g_G)]_{2s+2r}[(k_K)]_{r+s}[(h_H)]_{r+s}\ r!s!}$$
$$\times\sum_{n=0}^{m-r-s}\frac{(-m+s+r)_n[(a_A)+2s+2r]_n[(d_D)+r+s]_n(-1)^{n(E-T)}[1-(t_T)-m]_n(\frac{z}{y})^n}{[(g_G)+2s+2r]_n[(k_K)+r+s]_n[1-(e_E)-m]_n\ n!}\tag{23}$$

Now using finite double series identity (6), we get

$$W_3=\frac{[(e_E)]_m}{[(t_T)]_m}\sum_{p=0}^{m}\sum_{n=0}^{m-p}\frac{(-m)_{p+n}[(a_A)]_{2p+n}[(d_D)]_{p+n}[(b_B)]_p[1-(t_T)-m]_n(x-z)^p}{[(g_G)]_{2p+n}[(k_K)]_{p+n}[(h_H)]_p[1-(e_E)-m]_n\ p!n!}\left[(-1)^{(E-T)}\left(\frac{z}{y}\right)\right]^n\tag{24}$$

which is the right hand side of (17). The left hand side of (18) can be written in the following form

$$W_4=\sum_{n=0}^{m}\sum_{r=0}^{n}\sum_{s=0}^{m-n}\frac{(-m)_{n+s}[(a_A)]_{2r+s}[(b_B)]_{r+s}(-n)_r[(d_D)]_r[(e_E)]_s\ x^r\ y^s}{[(g_G)]_{2r+s}[(h_H)]_{r+s}[(k_K)]_r[(t_T)]_s\ n!r!s!}\tag{25}$$

Now using identity (4), we get

$$W_4 = \sum_{s=0}^{m}\left[\sum_{r=0}^{m-s}\frac{(-m)_{r+s}[(a_A)]_{2r+s}[(b_B)]_{r+s}[(d_D)]_r[(e_E)]_s(-x)^r\ y^s}{[(g_G)]_{2r+s}[(h_H)]_{r+s}[(k_K)]_r[(t_T)]_s\ r!s!}\left\{\sum_{n=0}^{m-r-s}\frac{(-m+r+s)_n}{n!}\right\}\right] \tag{26}$$

Now apply combinatorial identity (5) in the square bracket of (26), when r varies from 0 to $m-s-1$, corresponding terms in square bracket vanish, therefore when $r=m-s$, we get

$$W_4 = \frac{[(a_A)]_{2m}[(b_B)]_m[(d_D)]_m\ x^m}{[(g_G)]_{2m}[(h_H)]_m[(k_K)]_m}\times\sum_{s=0}^{m}\frac{(-m)_s[(e_E)]_s[1-(g_G)-2m]_s[1-(k_K)-m]_s(-1)^{(A+D-G-K)s}(\frac{y}{x})^s}{[(t_T)]_s[1-(a_A)-2m]_s[1-(d_D)-m]_s\ s!}$$

which is the right hand side of (18). Similarly we can derive (19). By making suitable adjustment of parameters and variables in (2.1)-(2.7), applying the definitions of different types of hypergeometric polynomials and multiple hypergeometric functions scattered in the literature [3, 9, 17–20], we can obtain a number of results.

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