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A Hendecagonal Fuzzy Number and Its Vertex Method

Research Article

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Abstract: The decision data of human judgements with preferences are often vague so that the usual ways of using crisp values are inadequate in many real life situation. Also using fuzzy numbers such as triangular, trapezoidal are not suitable in few cases where the uncertainties arises in eleven different points. Therefore, Hendecagonal Fuzzy Number (H_DFN) and its arithmetic operations based on $\alpha-$cut is introduced. Also, the linguistic terms can be expressed in hendecagonal fuzzy numbers and a vertex method is defined to calculate the distance between two hendecagonal fuzzy numbers.

Keywords: Fuzzy Arithmetic, Linguistic values, Hendecagonal Fuzzy Number (H_DFN), $\alpha-$cut.

1. Introduction

In 1965, fuzzy sets were introduced by Zadeh, L.A [17] to provide natural way of dealing with problems in which the source of imprecision and vagueness occurs. It can be applied in many fields such as artificial intelligence, control system, decision making, expert system etc. Fuzzy numbers and their fuzzy operations [17–19] are seeds of fuzzy number theory. Fuzzy number and their arithmetic operation for modelling expert system, Cognitive computational models, measurement, knowledge, intelligence [14–16, 19]. The concept of fuzzy number has been defined as a fuzzy subset of real line by Dubois, D and Prade, H [2]. A fuzzy number is a quantity whose values are precise, rather than exact as in the case with single valued numbers [3, 7]. To deal imprecise in real life situation, many researchers used triangular and trapezoidal fuzzy number [10–12, 21]. Also hexagonal, nonagonal, decagonal fuzzy numbers have been introduced to clear the vagueness [6, 7, 9, 11]. Most of the researcher has focused on uncertain linguistic term in group decision making processes [4, 13, 18].

In decision making problem experts may provide uncertain linguistic term to express their opinion when they have no clear idea and lack of information. The uncertain linguistic term is frequently used as input in decision analysis activities. So far linguistic values are usually represented as fuzzy numbers such as triangular, trapezoidal. But it is complex to restrict the membership functions to take triangular, trapezoidal when vagueness arises in eleven different points. Therefore, in this paper a new form of Hendecagonal Fuzzy Number(H_DFN) is proposed under uncertain linguistic environment. This paper is organized as follows: section one presents introduction.

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Basic definition of fuzzy number and linguistic terms are given in section two, section three provides hendecagonal fuzzy number and it linguistic values. New operation for addition, multiplication, division of hendecagonal fuzzy number $\alpha-$cut is proposed in section four. Section five proposes vertex method to calculate the distance between two hendecagonal fuzzy numbers. Finally, conclusion and future directions are given.

2. Basic Definitions and Notations

In this section, some basic definitions of fuzzy set theory and fuzzy numbers are reviewed.

Definition 2.1. *A fuzzy set $\widetilde{A}$ in X is characterized by a membership function $\mu_{\widetilde{A}}(x)$ which associates each point in X, to a real number in the interval $[0,1]$. The value of $\mu_{\widetilde{A}}(x)$ represents "grade of membership"of $x \in \mu_{\widetilde{A}(x)}$. More general representation for a fuzzy set is $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x))\}$*

Definition 2.2. *The $\alpha-$cut of the fuzzy set $\widetilde{A}$ of the universe of discourse X is defined as $\widetilde{A}_\alpha = \{x \in X/\mu_{\widetilde{A}}(x) \geq \alpha\}$, where $\alpha \in [0,1]$.*

Definition 2.3. *A fuzzy set $\widetilde{A}$ defined on the set of real numbers $\mathbb{R}$ is said to be a fuzzy number if its membership function $\widetilde{A} : \mathbb{R} \to [0,1]$ has the following characteristics.*

(1). $\widetilde{A}$ is convex $\mu_{\widetilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_{\widetilde{A}}(x_1),\, \mu_{\widetilde{A}}(x_2))$, $\forall x \in [0,1]$, $\lambda \in [0,1]$

(2). $\widetilde{A}$ is normal. i.e. there exists an $x \in \mathbb{R}$ such that if $\max \mu_{\widetilde{A}}(x) = 1$.

(3). $\widetilde{A}$ is piecewise continuous.

Definition 2.4. *A triangular fuzzy number $\widetilde{A}$ denoted by (a_1, a_2, a_3), and the membership function is defined as*

$$\mu_{\widetilde{A}}(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1}, & \text{for } a_1 \leq x \leq a_2 \\ \dfrac{a_3-x}{a_3-a_2}, & \text{for } a_2 \leq x \leq a_3 \\ 0, & \text{elsewhere.} \end{cases}$$

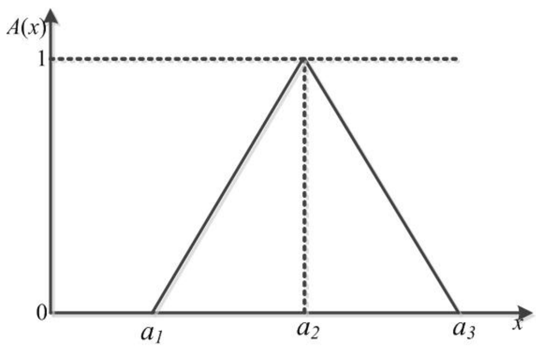

Figure 1. **Triangular fuzzy number**

Definition 2.5. *A trapezoidal fuzzy number $\widetilde{A}$ can be defined as (a_1, a_2, a_3, a_4), and the membership function is defined as*

$$\mu_{\widetilde{A}}(x) = \begin{cases} \dfrac{(x-a_1)}{(a_2-a_1)}, & a_1 \leq x \leq a_2 \\ 1, & a_2 \leq x \leq a_3 \\ \dfrac{(a_4-x)}{(a_4-a_3)}, & a_3 \leq x \leq a_4 \\ 0, & \text{otherwise.} \end{cases}$$

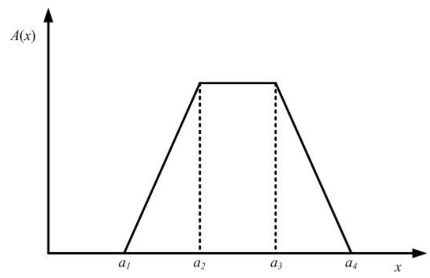

Figure 2. **Trapezoidal fuzzy number**

Definition 2.6. *A linguistic variable/term is a variable whose value is not crisp number but word or sentence in a natural language.*

3. Hendecagonal Fuzzy Number (H_DFN)

Definition 3.1. *A Hendecagonal fuzzy number* $\widetilde{H}_D$ *denoted as* $(a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,\ a_7,\ a_8,\ a_9,\ a_{10},\ a_{11})$, *and the membership function is defined as*

$$\mu_{\widetilde{H}_D}(x) = \begin{cases} \frac{1}{5}\frac{(x-a_1)}{(a_2-a_1)}, & a_1 \le x \le a_2 \\ \frac{1}{5}+\frac{1}{5}\frac{(x-a_2)}{(a_3-a_2)}, & a_2 \le x \le a_3 \\ \frac{2}{5}+\frac{1}{5}\frac{(x-a_3)}{(a_4-a_3)}, & a_3 \le x \le a_4 \\ \frac{3}{5}+\frac{1}{5}\frac{(x-a_4)}{(a_5-a_4)}, & a_4 \le x \le a_5 \\ \frac{4}{5}+\frac{1}{5}\frac{(x-a_5)}{(a_6-a_5)}, & a_5 \le x \le a_6 \\ 1-\frac{1}{5}\frac{(x-a_6)}{(a_7-a_6)}, & a_6 \le x \le a_7 \\ \frac{4}{5}-\frac{1}{5}\frac{(x-a_7)}{(a_8-a_7)}, & a_7 \le x \le a_8 \\ \frac{3}{5}-\frac{1}{5}\frac{(x-a_8)}{(a_9-a_8)}, & a_8 \le x \le a_9 \\ \frac{2}{5}-\frac{1}{5}\frac{(x-a_9)}{(a_{10}-a_9)}, & a_9 \le x \le a_{10} \\ \frac{1}{5}\frac{(a_{11}-x)}{(a_{11}-a_{10})}, & a_{10} \le x \le a_{11} \\ 0, & otherwise. \end{cases}$$

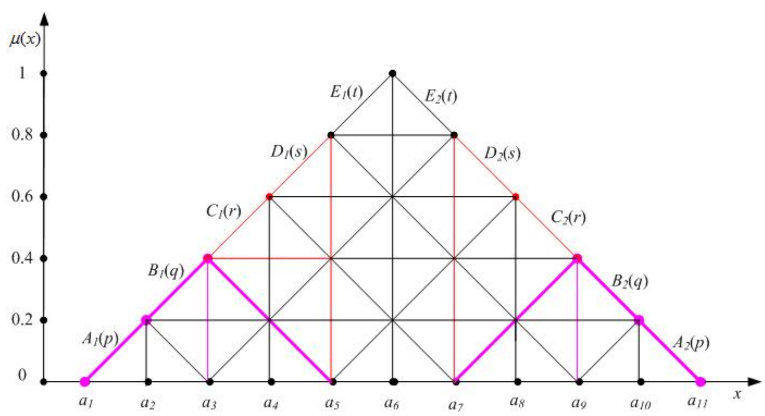

Figure 3. **The Hendecagonal fuzzy number**

3.1. Generalized Hendecagonal Fuzzy Number (H_DFN)

Definition 3.2. *A Hendecagonal fuzzy number* $\widetilde{H}_D$ *denoted as* $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, w)$*, and the membership function is defined as*

$$\mu_{\widetilde{H}_D}(x) = \begin{cases} \frac{1}{5}\frac{(x-a_1)}{(a_2-a_1)}, & a_1 \le x \le a_2 \\ \frac{1}{5} + \frac{1}{5}w\frac{(x-a_2)}{(a_3-a_2)}, & a_2 \le x \le a_3 \\ \frac{2}{5} + \frac{1}{5}w\frac{(x-a_3)}{(a_4-a_3)}, & a_3 \le x \le a_4 \\ \frac{3}{5} + \frac{1}{5}w\frac{(x-a_4)}{(a_5-a_4)}, & a_4 \le x \le a_5 \\ \frac{4}{5} + \frac{1}{5}w\frac{(x-a_5)}{(a_6-a_5)}, & a_5 \le x \le a_6 \\ 1 - \frac{1}{5}w\frac{(x-a_6)}{(a_7-a_6)}, & a_6 \le x \le a_7 \\ \frac{4}{5} - \frac{1}{5}w\frac{(x-a_7)}{(a_8-a_7)}, & a_7 \le x \le a_8 \\ \frac{3}{5} - \frac{1}{5}w\frac{(x-a_8)}{(a_9-a_8)}, & a_8 \le x \le a_9 \\ \frac{2}{5} - \frac{1}{5}w\frac{(x-a_9)}{(a_{10}-a_9)}, & a_9 \le x \le a_{10} \\ \frac{1}{5}\frac{(a_{11}-x)}{(a_{11}-a_{10})}, & a_{10} \le x \le a_{11} \\ 0, & otherwise. \end{cases}$$

Table 1. **Hendecagonal Fuzzy linguistic scale**

Linguistic terms	Linguistic values
No influence	(0 0 0 0 0 0 0.03 0.06 0.09 0.12 0.15)
Very Low influence	(0 0.03 0.06 0.09 0.12 0.15 0.18 0.21 0.24 0.27 0.3)
Low influence	(0.15 0.18 0.21 0.24 0.27 0.3 0.33 0.39 0.42 0.45 0.48)
Medium	(0.3 0.33 0.39 0.42 0.45 0.48 0.51 0.54 0.57 0.6 0.63)
High influence	(0.48 0.51 0.54 0.57 0.6 0.63 0.66 0.69 0.72 0.75 0.78)
Very High influence	(0.63 0.66 0.69 0.72 0.75 0.78 0.81 0.84 0.87 0.9 0.93)
Very Very High influence	(0.78 0.81 0.84 0.87 0.9 1 1 1 1 1 1)

Furthermore, in light of the research work by Kaufman, A [8] and Zadeh, LA [18, 19], the arithmetic operations of hendecagonal fuzzy number and vertex method are provided in the following section.

4. Arithmetic Operations of Hendecagonal Fuzzy Numbers (H_DFN)

In this section, arithmetic operations of (H_DFN) based on α-cut method is reviewed.

4.1. Arithmetic Operation on α-cut

Definition 4.1. *A Hendecagonal fuzzy number* $\widetilde{H}_D$ *can also be defined as* $\widetilde{H}_D = A_1(p), B_1(q), C_1(r), D_1(s), E_1(t), A_2(p), B_2(q), C_2(r), D_2(s), E_2(t)$ $p \in [0, 0.2]$, $q \in [0.2, 0.4]$, $r \in [0.4, 0.6]$ $s \in [0.6, 0.8]$ *and* $t \in [0.8, 1]$, *where*

$A_1(p) = \frac{1}{5}\frac{(x-a_1)}{(a_2-a_1)}$, $B_1(q) = \frac{1}{5} + \frac{1}{5}\frac{(x-a_2)}{(a_3-a_2)}$, $C_1(r) = \frac{2}{5} + \frac{1}{5}\frac{(x-a_3)}{(a_4-a_3)}$, $D_1(s) = \frac{3}{5} + \frac{1}{5}\frac{(x-a_4)}{(a_5-a_4)}$, $E_1(t) = \frac{4}{5} + \frac{1}{5}\frac{(x-a_5)}{(a_6-a_5)}$,

$E_2(t) = 1 - \frac{1}{5}\frac{(x-a_6)}{(a_7-a_6)}$, $D_2(s) = \frac{4}{5} - \frac{1}{5}\frac{(x-a_7)}{(a_8-a_7)}$, $C_2(r) = \frac{3}{5} - \frac{1}{5}\frac{(x-a_8)}{(a_9-a_8)}$, $B_2(q) = \frac{2}{5} - \frac{1}{5}\frac{(x-a_9)}{(a_{10}-a_9)}$, $A_2(p) = \frac{1}{5}\frac{(a_{11}-x)}{(a_{11}-a_{10})}$

Here,

- $A_1(p), B_1(q), C_1(r), D_1(s), E_1(t)$, *is bounded and continuous incresing function over* $[0, 0.2)$, $[0.2, 0.4)$, $[0.4, 0.6)$, $[0.6, 0.8)$ *and* $[0.8, 1]$ *respectively.*

- $A_2(p), B_2(q), C_2(r), D_2(s), E_2(t)$, *is bounded and continuous decresing function over* $[0,0.2)$, $[0.2,0.4)$, $[0.4,0.6)$, $[0.6,0.8)$ *and* $[0.8,1]$ *respectively.*

Definition 4.2. *The α-cut of the fuzzy set of the universe of discourse X is defined as $\widetilde{H_D}_\alpha = \{x \in X/\mu_{\widetilde{A}}(x) \geq \alpha\}$, where $\alpha \in [0,1]$.*

$$\widetilde{H_D}_\alpha = \begin{cases} [A_1(\alpha), A_2(\alpha)], & \textit{for } \alpha \in [0,0.2) \\ [B_1(\alpha), B_2(\alpha)], & \textit{for } \alpha \in [0.2,0.4) \\ [C_1(\alpha), C_2(\alpha)], & \textit{for } \alpha \in [0.4,0.6) \\ [D_1(\alpha), D_2(\alpha)], & \textit{for } \alpha \in [0.6,0.8) \\ [E_1(\alpha), E_2(\alpha)], & \textit{for } \alpha \in [0.8,1] \end{cases}$$

Definition 4.3. *If $A_1(x) = \alpha$ and $A_2(x) = \alpha$, then α cut operations interval $\widetilde{H_D}_\alpha$ is obtained as*

(1). $[A_1(\alpha), A_2(\alpha)] = [5\alpha(a_2 - a_1) + a_1, -5\alpha(a_{11} - a_{10}) + a_{11})]$

Similarly, we can obtain $\alpha-$ cut operations interval $\widetilde{D}_\alpha$ for $[Q_1(\alpha), Q_2(\alpha)]$, $[R_1(\alpha), R_2(\alpha)]$ and $[S_1(\alpha), S_2(\alpha)]$ as follows;

(2). $[B_1(\alpha), B_2(\alpha)] = [5\alpha(a_3 - a_2) + 2a_2 - a_3, -5\alpha(a_{10} - a_9) + 2a_{10} - a_9]$

(3). $[C_1(\alpha), C_2(\alpha)] = [5\alpha(a_4 - a_3) + 3a_3 - 2a_4, -5\alpha(a_9 - a_8) + 3a_9 - 2a_8]$

(4). $[D_1(\alpha), D_2(\alpha)] = [5\alpha(a_5 - a_4) + 4a_4 - 3a_5, -5\alpha(a_8 - a_7) + 4a_8 - 3a_7]$

(5). $[D_1(\alpha), D_2(\alpha)] = [5\alpha(a_6 - a_5) + 5a_5 - 3a_6, -5\alpha(a_7 - a_6) + 5a_7 - 3a_6]$

Hence, α-cut of Hendecagonal Fuzzy Number

$$\widetilde{H_D}_\alpha = \begin{cases} [5\alpha(a_2 - a_1) + a_1, -5\alpha(a_{11} - a_{10}) + a_{11})], & \textit{for } \alpha \in [0,0.2) \\ [5\alpha(a_3 - a_2) + 2a_2 - a_3, -5\alpha(a_{10} - a_9) + 2a_{10} - a_9], & \textit{for } \alpha \in [0.2,0.4) \\ [5\alpha(a_4 - a_3) + 3a_3 - 2a_4, -5\alpha(a_9 - a_8) + 3a_9 - 2a_8], & \textit{for } \alpha \in [0.4,0.6) \\ [5\alpha(a_5 - a_4) + 4a_4 - 3a_5, -5\alpha(a_8 - a_7) + 4a_8 - 3a_7], & \textit{for } \alpha \in [0.6,0.8) \\ [5\alpha(a_6 - a_5) + 5a_5 - 3a_6, -5\alpha(a_7 - a_6) + 5a_7 - 3a_6], & \textit{for } \alpha \in [0.8,1] \end{cases}$$

Theorem 4.4. *If $\widetilde{A}$=(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}) and $\widetilde{B}$=(b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}) are the Hendecagonal fuzzy numbers, then $\widetilde{C}$=$\widetilde{A} \oplus \widetilde{B}$ is also Hendecagonal fuzzy number $\widetilde{A} \oplus \widetilde{B}$=($a_1 + b_1$, $a_2 + b_2$, $a_3 + b_3$, $a_4 + b_4$, $a_5 + b_5$, $a_6 + b_6$, $a_7 + b_7$, $a_8 + b_8$, $a_9 + b_9$, $a_{10} + b_{10}$, $a_{11} + b_{11}$).*

Proof. The membership function of H_DFN $\widetilde{C}$=$\widetilde{A} \oplus \widetilde{B}$ can be found by $\alpha-$cut method with the transform $z = x + y$, $\alpha-$cut membership function of $\widetilde{A}(x)$ is,

$$x \in \begin{cases} [5\alpha(a_2 - a_1) + a_1, -5\alpha(a_{11} - a_{10}) + a_{11})], & \text{for } \alpha \in [0,0.2) \\ [5\alpha(a_3 - a_2) + 2a_2 - a_3, -5\alpha(a_{10} - a_9) + 2a_{10} - a_9], & \text{for } \alpha \in [0.2,0.4) \\ [5\alpha(a_4 - a_3) + 3a_3 - 2a_4, -5\alpha(a_9 - a_8) + 3a_9 - 2a_8], & \text{for } \alpha \in [0.4,0.6) \\ [5\alpha(a_5 - a_4) + 4a_4 - 3a_5, -5\alpha(a_8 - a_7) + 4a_8 - 3a_7], & \text{for } \alpha \in [0.6,0.8) \\ [5\alpha(a_6 - a_5) + 5a_5 - 3a_6, -5\alpha(a_7 - a_6) + 5a_7 - 3a_6], & \text{for } \alpha \in [0.8,1] \end{cases}$$

$\alpha-$cut membership function of $\widetilde{B}(y)$ is,

$$y \in \begin{cases} [5\alpha(b_2-b_1)+b_1, -5\alpha(b_{11}-b_{10})+b_{11})], & \text{for } \alpha \in [0,0.2) \\ [5\alpha(b_3-b_2)+2b_2-b_3, -5\alpha(b_{10}-b_9)+2b_{10}-b_9], & \text{for } \alpha \in [0.2,0.4) \\ [5\alpha(b_4-b_3)+3b_3-2b_4, -5\alpha(b_9-b_8)+3b_9-2b_8], & \text{for } \alpha \in [0.4,0.6) \\ [5\alpha(b_5-b_4)+4b_4-3b_5, -5\alpha(b_8-b_7)+4b_8-3b_7], & \text{for } \alpha \in [0.6,0.8) \\ [5\alpha(b_6-b_5)+5b_5-3b_6, -5\alpha(b_7-b_6)+5b_7-3b_6], & \text{for } \alpha \in [0.8,1] \end{cases}$$

so,

$$z = x+y \in \begin{cases} [5\alpha(a_2-a_1)+a_1, -5\alpha(a_{11}-a_{10})+a_{11})]+ & \\ [5\alpha(b_2-b_1)+b_1, -5\alpha(b_{11}-b_{10})+b_{11})], & \text{for } \alpha \in [0,0.2) \\ [5\alpha(a_3-a_2)+2a_2-a_3, -5\alpha(a_{10}-a_9)+2a_{10}-a_9]+ & \\ [5\alpha(b_3-b_2)+2b_2-b_3, -5\alpha(b_{10}-b_9)+2b_{10}-b_9], & \text{for } \alpha \in [0.2,0.4) \\ [5\alpha(a_4-a_3)+3a_3-2a_4, -5\alpha(a_9-a_8)+3a_9-2a_8]+ & \\ [5\alpha(b_4-b_3)+3b_3-2b_4, -5\alpha(b_9-b_8)+3b_9-2b_8], & \text{for } \alpha \in [0.4,0.6) \\ [5\alpha(a_5-a_4)+4a_4-3a_5, -5\alpha(a_8-a_7)+4a_8-3a_7]+ & \\ [5\alpha(b_5-b_4)+4b_4-3b_5, -5\alpha(b_8-b_7)+4b_8-3b_7], & \text{for } \alpha \in [0.6,0.8) \\ [5\alpha(a_6-a_5)+5a_5-3a_6, -5\alpha(a_7-a_6)+5a_7-3a_6]+ & \\ [5\alpha(b_6-b_5)+5b_5-3b_6, -5\alpha(b_7-b_6)+5b_7-3b_6], & \text{for } \alpha \in [0.8,1] \end{cases}$$

the membership function of $\widetilde{C}$=$\widetilde{A} \oplus \widetilde{B}$ is,

$$\mu_{\widetilde{C}}(x) = \begin{cases} \dfrac{z-(a_1+b_1)}{5[(a_2-a_1)+(b_2-b_1)]} & a_1+b_1 \le z \le a_2+b_2 \\ \dfrac{z-2(a_2+b_2)+(a_3+b_3)}{5[(a_3-a_2)+(b_3-b_2)]} & a_2+b_2 \le z \le a_3+b_3 \\ \dfrac{z-3(a_3+b_3)+2(a_4+b_4)}{5[(a_4-a_3)+(b_4-b_3)]} & a_3+b_3 \le z \le a_4+b_4 \\ \dfrac{z-4(a_4+b_4)+3(a_5+b_5)}{5[(a_5-a_4)+(b_5-b_4)]} & a_4+b_4 \le z \le a_5+b_5 \\ \dfrac{z-5(a_5+b_5)+4(a_6+b_6)}{5[(a_6-a_5)+(b_6-b_5)]} & a_5+b_5 \le z \le a_6+b_6 \\ \dfrac{z-5(a_7+b_7)+4(a_6+b_6)}{5[(a_6-a_7)+(b_6-b_7)]} & a_6+b_6 \le z \le a_7+b_7 \\ \dfrac{z-4(a_8+b_8)+3(a_7+b_7)}{5[(a_7-a_8)+(b_7-b_8)]} & a_7+b_7 \le z \le a_8+b_8 \\ \dfrac{z-3(a_9+b_9)+2(a_8+b_8)}{5[(a_8-a_9)+(b_8-b_9)]} & a_8+b_8 \le z \le a_9+b_9 \\ \dfrac{z-2(a_{10}+b_{10})+(a_9+b_9)}{5[(a_9-a_{10})+(b_9-b_{10})]} & a_9+b_9 \le z \le a_{10}+b_{10} \\ \dfrac{z-(a_{11}+b_{11})}{5[(a_{10}-a_{11})+(b_{10}-b_{11})]} & a_{10}+b_{10} \le z \le a_{11}+b_{11} \\ 0, & \text{otherwise.} \end{cases}$$

Hence, addition rule is proved. Therefore we have $\widetilde{A} \oplus \widetilde{B}$=($a_1+b_1$, a_2+b_2, a_3+b_3, a_4+b_4, a_5+b_5, a_6+b_6, a_7+b_7, a_8+b_8 a_9+b_9, $a_{10}+b_{10}$, $a_{11}+b_{11}$) is a Hendecagonal Fuzzy Number.

□

Example 4.5. *Let us consider two* $H_DFN\tilde{A} = (1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5, 5.5, 6, 6.5)$ *and* $\tilde{B} = (2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)$. *Then, the addition of these two* H_DFN *is defined as* $\tilde{A} \oplus \tilde{B} = (3.5, 5, 6.5, 8, 9.5, 11, 12.5, 14, 15.5, 17, 18.5)$ *with membership function as follows*

$$\mu_{\widetilde{C}}(x) = \begin{cases} \frac{1}{5}\frac{(x-3.5)}{1.5}, & 3.5 \le x \le 5 \\ \frac{1}{5}\frac{(x-3.5)}{1.5}, & 5 \le x \le 6.5 \\ \frac{1}{5}\frac{(x-3.5)}{1.5}, & 6 \le x \le 8 \\ \frac{1}{5}\frac{(x-3.5)}{1.5}, & 8 \le x \le 9.5 \\ \frac{1}{5}\frac{(x-3.5)}{1.5}, & 9.5 \le x \le 11 \\ \frac{-1}{5}\frac{(x-18.5)}{1.5}, & 11 \le x \le 12.5 \\ \frac{-1}{5}\frac{(x-18.5)}{1.5}, & 12.5 \le x \le 14 \\ \frac{-1}{5}\frac{(x-18.5)}{1.5}, & 14 \le x \le 15.5 \\ \frac{-1}{5}\frac{(x-18.5)}{1.5}, & 15.5 \le x \le 17 \\ \frac{-1}{5}\frac{(x-18.5)}{1.5}, & 17 \le x \le 18.5 \\ 0, & \textit{otherwise.} \end{cases}$$

Theorem 4.6. *If* $\widetilde{A}$=(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9) *and* $\widetilde{B}$=(b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9) *are the Hendecagonal fuzzy numbers, then* $\widetilde{P}$=$\widetilde{A} \otimes \widetilde{B}$ *is also Hendecagonal fuzzy number* $\widetilde{A} \otimes \widetilde{B}$=($a_1b_1$, a_2b_2, a_3b_3, a_4b_4, a_5b_5, a_6b_6, a_7b_7, a_8b_8, a_9b_9)

Proof. The membership function of H_DFN $\widetilde{P}$=$\widetilde{A} \otimes \widetilde{B}$ can be found by $\alpha-$cut method with the transform $z = x \times y$,

$$z = x \times y \in \begin{cases} [5\alpha(a_2 - a_1) + a_1, -5\alpha(a_{11} - a_{10}) + a_{11})]\times & \\ [5\alpha(b_2 - b_1) + b_1, -5\alpha(b_{11} - b_{10}) + b_{11})], & \text{for } \alpha \in [0, 0.2) \\ [5\alpha(a_3 - a_2) + 2a_2 - a_3, -5\alpha(a_{10} - a_9) + 2a_{10} - a_9]\times & \\ [5\alpha(b_3 - b_2) + 2b_2 - b_3, -5\alpha(b_{10} - b_9) + 2b_{10} - b_9], & \text{for } \alpha \in [0.2, 0.4) \\ [5\alpha(a_4 - a_3) + 3a_3 - 2a_4, -5\alpha(a_9 - a_8) + 3a_9 - 2a_8]\times & \\ [5\alpha(b_4 - b_3) + 3b_3 - 2b_4, -5\alpha(b_9 - b_8) + 3b_9 - 2b_8], & \text{for } \alpha \in [0.4, 0.6) \\ [5\alpha(a_5 - a_4) + 4a_4 - 3a_5, -5\alpha(a_8 - a_7) + 4a_8 - 3a_7]\times & \\ [5\alpha(b_5 - b_4) + 4b_4 - 3b_5, -5\alpha(b_8 - b_7) + 4b_8 - 3b_7], & \text{for } \alpha \in [0.6, 0.8) \\ [5\alpha(a_6 - a_5) + 5a_5 - 3a_6, -5\alpha(a_7 - a_6) + 5a_7 - 3a_6]\times & \\ [5\alpha(b_6 - b_5) + 5b_5 - 3b_6, -5\alpha(b_7 - b_6) + 5b_7 - 3b_6], & \text{for } \alpha \in [0.8, 1] \end{cases}$$

So, the membership function of $\widetilde{P}$=$\widetilde{A} \otimes \widetilde{B}$ is,

$$\mu_{\widetilde{P}}(x) = \begin{cases} \dfrac{-B_1 + \sqrt{B_1^2 - 4A_1(a_1b_1 - z)}}{2A_1} & a_1b_1 \le z \le a_2b_2 \\ \dfrac{-B_2 + \sqrt{B_2^2 - 4A_2(4a_2b_2 - 2(a_2b_3 + a_3b_2) + a_3b_3 - z)}}{2A_2} & a_2b_2 \le z \le a_3b_3 \\ \dfrac{-B_3 + \sqrt{B_3^2 - 4A_3(9a_3b_3 - 6(a_3b_4 + a_4b_3) + 4a_4b_4 - z)}}{2A_3} & a_3b_3 \le z \le a_4b_4 \\ \dfrac{-B_4 + \sqrt{B_4^2 - 4A_4(16a_4b_4 - 12(a_4b_5 - a_5b_4) + 9a_5b_5 - z)}}{2A_4} & a_4b_4 \le z \le a_5b_5 \\ \dfrac{B_5 - \sqrt{B_5^2 - 4A_5(25a_5b_5 - 20(a_5b_6 - a_6b_5) + 16a_6b_6 - z)}}{2A_5} & a_5b_5 \le z \le a_6b_6 \\ \dfrac{B_6 - \sqrt{B_6^2 - 4A_6(16a_6b_6 - 20(a_6b_7 + a_7b_6) + 25a_7b_7 - z)}}{2A_6} & a_6b_6 \le z \le a_7b_7 \\ \dfrac{B_7 - \sqrt{B_7^2 - 4A_7(9a_7b_7 - 12(a_7b_8 + a_8b_7) + 16a_8b_8 - z)}}{2A_7} & a_7b_7 \le z \le a_8b_8 \\ \dfrac{B_8 - \sqrt{B_8^2 - 4A_8(4a_8b_8 - 6(a_9b_8 + a_8b_9) + 9a_9b_9 - z)}}{2A_8} & a_8b_8 \le z \le a_9b_9 \\ \dfrac{B_9 - \sqrt{B_9^2 - 4A_9(4a_9b_9 - 2(a_9b_{10} + a_{10}b_9) + a_9b_9 - z)}}{2A_9} & a_9b_9 \le z \le a_{10}b_{10} \\ \dfrac{B_{10} - \sqrt{B_{10}^2 - 4A_{10}(a_{11}b_{11} - z)}}{2A_{10}} & a_{10}b_{10} \le z \le a_{11}b_{11} \\ 0, & \text{otherwise.} \end{cases}$$

where $A_i = 25(a_{i+1}-a_i)(b_{i+1}-b_i)$, $i = 1, 2\cdots, 10$ $B_1 = 5[b_1(a_2-a_1)+a_1(b_2-b_1)]$, $B_j = 5[(a_{j+1}-a_j)(jb_j-b_{j+1})+(b_{j+1}-b_j)(ja_j-(j-1)a_{j+1})]$, $j = 2, \cdots, 5$, $B_6 = -5[(a_7-a_6)(5b_7-4b_6)+(b_7-b_6)(5a_7-4a_6)]$, $B_7 = -5[(a_8-a_7)(4b_8-3b_7)+(b_8-b_7)(4_8-2a_7)]$, $B_8 = -5[(a_9-a_8)(3b_9-b_8)+(b_9-b_8)(3b_9-a_8)]$, $B_9 = -5[(a_{10}-a_9)(2b_{10}-b_9)+(2b_{10}-b_9)(2b_{10}-a_9)]$, $B_{10} = -5[b_{10}(a_{11}-a_{10})+a_{10}(b_{10}-b_{11})]$,

Hence, multiplication rule is proved. Therefore we have $\widetilde{A} \otimes \widetilde{B}$=$(a_1b_1,\ a_2b_2,\ a_3b_3,\ a_4b_4,\ a_5b_5,\ a_6b_6,\ a_7b_7,\ a_8b_8,\ a_9b_9,\ a_{10}b_{10}, a_{11}b_{11})$ is a Hendecagonal Fuzzy Number. □

Example 4.7. *Let us consider two H_DFN $\tilde{A} = (1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5, 5.5, 6, 6.5)$ and $\tilde{B} = (2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)$. Then, the multiplication of these two H_DFN is defined as $\tilde{A} \otimes \tilde{B} = (3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78)$ with membership function as follows*

$$\mu_{\widetilde{P}}(x) = \begin{cases} \frac{1}{5}\frac{(x-3)}{3}, & 3 \le x \le 6 \\ \frac{1}{5}\frac{(x-2)}{4}, & 6 \le x \le 10 \\ \frac{1}{5}\frac{(x)}{5}, & 10 \le x \le 15 \\ \frac{1}{5}\frac{(x+3)}{6}, & 15 \le x \le 21 \\ \frac{1}{5}\frac{(x+7)}{7}, & 21 \le x \le 28 \\ \frac{-1}{5}\frac{(x-68)}{8}, & 28 \le x \le 36 \\ \frac{-1}{5}\frac{(x-72)}{9}, & 36 \le x \le 45 \\ \frac{-1}{5}\frac{(x-75)}{10}, & 45 \le x \le 55 \\ \frac{-1}{5}\frac{(x-77)}{11}, & 55 \le x \le 66 \\ \frac{1}{5}\frac{(78-x)}{12}, & 66 \le x \le 78 \\ 0, & \textit{otherwise.} \end{cases}$$

Theorem 4.8. *If $\widetilde{A}$=$(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11})$ and $\widetilde{B}$=$(b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11})$ are the Hendecagonal Fuzzy Numbers, then $\widetilde{D}$=$\widetilde{A}\oslash\widetilde{B}$ is also Hendecagonal fuzzy number*

$\widetilde{A}\oslash\widetilde{B}$=$\left(\frac{a_1}{b_{11}}, \frac{a_2}{b_{10}}, \frac{a_3}{b_9}, \frac{a_4}{b_8}, \frac{a_5}{b_7}, \frac{a_6}{b_6}, \frac{a_7}{b_5}, \frac{a_8}{b_4}, \frac{a_9}{b_3}, \frac{a_{10}}{b_2}, \frac{a_{11}}{b_1}\right)$

Proof. The membership function of H_DFN $\widetilde{D}$=$\widetilde{A}\oslash\widetilde{B}$ can be found by $\alpha-$cut method with the transform $z = x/y$,

$$z = x \div y \in \begin{cases} [5\alpha(a_2-a_1)+a_1, -5\alpha(a_{11}-a_{10})+a_{11}]\div \\ [5\alpha(b_2-b_1)+b_1, -5\alpha(b_{11}-b_{10})+b_{11}], & \text{for } \alpha\in[0,0.2) \\ [5\alpha(a_3-a_2)+2a_2-a_3, -5\alpha(a_{10}-a_9)+2a_{10}-a_9]\div \\ [5\alpha(b_3-b_2)+2b_2-b_3, -5\alpha(b_{10}-b_9)+2b_{10}-b_9], & \text{for } \alpha\in[0.2,0.4) \\ [5\alpha(a_4-a_3)+3a_3-2a_4, -5\alpha(a_9-a_8)+3a_9-2a_8]\div \\ [5\alpha(b_4-b_3)+3b_3-2b_4, -5\alpha(b_9-b_8)+3b_9-2b_8], & \text{for } \alpha\in[0.4,0.6) \\ [5\alpha(a_5-a_4)+4a_4-3a_5, -5\alpha(a_8-a_7)+4a_8-3a_7]\div \\ [5\alpha(b_5-b_4)+4b_4-3b_5, -5\alpha(b_8-b_7)+4b_8-3b_7], & \text{for } \alpha\in[0.6,0.8) \\ [5\alpha(a_6-a_5)+5a_5-3a_6, -5\alpha(a_7-a_6)+5a_7-3a_6]\div \\ [5\alpha(b_6-b_5)+5b_5-3b_6, -5\alpha(b_7-b_6)+5b_7-3b_6], & \text{for } \alpha\in[0.8,1] \end{cases}$$

So, the membership function of $\widetilde{P}$=$\widetilde{A}\oslash\widetilde{B}$ is,

$$\mu_{\widetilde{D}}(x) = \begin{cases} \frac{zb_{11}-a_1}{5[(a_2-a_1)+z(b_{11}-b_{10})]} & \frac{a_1}{b_{11}} \le z \le \frac{a_2}{b_{10}} \\ \frac{z(2b_{10}-b_9)-(a_2-a_3)}{5[(a_3-a_2)+z(b_{10}-b_9)]} & \frac{a_2}{b_{10}} \le z \le \frac{a_3}{b_9} \\ \frac{z(3b_9-2b_8)-(3a_3-2a_4)}{5[(a_4-a_3)+z(b_9-b_8)]} & \frac{a_3}{b_9} \le z \le \frac{a_4}{b_8} \\ \frac{z(4b_8-3b_7)-(4a_4-3a_5)}{5[(a_5-a_4)+z(b_8-b_7)]} & \frac{a_4}{b_8} \le z \le \frac{a_5}{b_7} \\ \frac{z(5b_7-4a_6)-(5b_5-4b_6)}{5[(a_6-a_5)+z(b_7-b_6)]} & \frac{a_5}{b_7} \le z \le \frac{a_6}{b_6} \\ \frac{(5a_7-4a_6)-z(5b_5-4b_6)}{5[(a_7-a_6)+z(b_6-b_5)]} & \frac{a_6}{b_6} \le z \le \frac{a_7}{b_5} \\ \frac{(4a_8-3a_7)-z(4b_4-4b_5)}{5[(a_8-a_7)+z(b_5-b_4)]} & \frac{a_7}{b_5} \le z \le \frac{a_8}{b_4} \\ \frac{(3a_9-2a_6)-z(3b_3-2b_4)}{5[(a_9-a_8)+z(b_4-b_3)]} & \frac{a_8}{b_4} \le z \le \frac{a_9}{b_3} \\ \frac{(2a_{10}-a_9)-z(2b_2-b_3)}{5[(a_{10}-a_9)+z(b_3-b_2)]} & \frac{a_9}{b_3} \le z \le \frac{a_{10}}{b_2} \\ \frac{a_{11}-zb_1}{5[(a_{11}-a_{10})+z(b_2-b_1)]} & \frac{a_{10}}{b_2} \le z \le \frac{a_{11}}{b_1} \\ 0, & \text{otherwise.} \end{cases}$$

Hence, division rule is proved. Therefore we have $\widetilde{A}\oslash\widetilde{B}$=$\left(\frac{a_1}{b_{11}}, \frac{a_2}{b_{10}}, \frac{a_3}{b_9}, \frac{a_4}{b_8}, \frac{a_5}{b_7}, \frac{a_6}{b_6}, \frac{a_7}{b_5}, \frac{a_8}{b_4}, \frac{a_9}{b_3}, \frac{a_{10}}{b_2}, \frac{a_{11}}{b_1}\right)$ is a Hendecagonal Fuzzy Number. □

Example 4.9. *Let us consider two H_DFN $\tilde{A} = (1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5, 5.5, 6, 6.5)$ and $\tilde{B} = (2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)$. Then, the multiplication of these two H_DFN is defined as $\tilde{A}\oslash\tilde{B} = (0.13, 0.18, 0.25, 0.33, 0.44, 0.57, 0.75, 1.00, 1.38, 2.00, 3.25)$*

with membership function as follows

$$\mu_{\widetilde{C}}(x) = \begin{cases} \frac{1}{5}\frac{(x-0.13)}{0.05}, & 0.13 \le x \le 0.18 \\ \frac{1}{5}\frac{(x-0.11)}{0.07}, & 0.18 \le x \le 0.25 \\ \frac{1}{5}\frac{(x-0.09)}{0.08}, & 0.25 \le x \le 0.33 \\ \frac{1}{5}\frac{(x)}{0.11}, & 0.33 \le x \le 0.44 \\ \frac{1}{5}\frac{(x+0.08)}{0.13}, & 0.44 \le x \le 0.57 \\ \frac{-1}{5}\frac{(x-1.47)}{0.18}, & 0.57 \le x \le 0.75 \\ \frac{-1}{5}\frac{(x-1.75)}{0.25}, & 0.75 \le x \le 1 \\ \frac{-1}{5}\frac{(x-2.14)}{0.38}, & 1 \le x \le 1.38 \\ \frac{-1}{5}\frac{(x-2.62)}{0.62}, & 1.38 \le x \le 2 \\ \frac{-1}{5}\frac{(x-3.25)}{1.25}, & 2 \le x \le 3.25 \\ 0, & \textit{otherwise.} \end{cases}$$

5. Distance Measurement Functions for Hendecagonal Fuzzy Number

There are many distance measurement functions were proposed in the literature [1, 22], but the vertex method is an effective and simple method to calculate the distance between any two fuzzy numbers. Therefore, few important properties of the vertex method for hendecagonal fuzzy number are described as follows:

Definition 5.1. *Let* $\widetilde{A}$=(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}) *and* $\widetilde{B}$=(b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}) *be the Hendecagonal Fuzzy Numbers. Then, the vertex method can be defined to calculate the distance between then as*

$$d(\widetilde{A},\widetilde{B}) = \sqrt{\frac{1}{11}\left[(a_1-b_1)^2+(a_2-b_2)^2+(a_3-b_3)^2+\ldots+(a_{11}-b_{11})^2\right]}$$

Definition 5.2. *Let* $\widetilde{A}$=(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}) *and* $\widetilde{B}$=(b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}) *be the Hendecagonal Fuzzy Numbers. Then, the fuzzy number* $\widetilde{A}$ *is closer to* $\widetilde{B}$ *when* $d(\widetilde{A},\widetilde{B})$ *tends to* 0.

Some important properties of the vertex method are described as below,

Theorem 5.3. *If* $\widetilde{A}$ *and* $\widetilde{B}$ *are the Hendecagonal Fuzzy Numbers, then the distance measurement* $d(\widetilde{A},\widetilde{B})$ *is identical to the Euclidean distance.*

Proof. Suppose that $\widetilde{A}$=(a_1, a_2, a_3, …, a_{11}) and $\widetilde{B}$=(b_1, b_2, b_3, …, b_{11}) are two real numbers, then $a_1 = a_2 = a_3 = \ldots = a_{11} = a$; $b_1 = b_2 = b_3 = \ldots = b_{11} = b$.

The distance measurement $d(\widetilde{A},\widetilde{B})$ can be calculated as

$$\begin{aligned} d(\widetilde{A},\widetilde{B}) &= \sqrt{\frac{1}{11}\left[(a_1-b_1)^2+(a_2-b_2)^2+(a_3-b_3)^2+\ldots+(a_{11}-b_{11})^2\right]} \\ &= \sqrt{\frac{1}{11}\left[(a-b)^2+(a-b)^2+(a-b)^2+\ldots+(a-b)^2\right]} \\ &= \sqrt{(a-b)^2} \\ &= |a-b| \end{aligned}$$

□

Theorem 5.4. *Two Hendecagonal Fuzzy Numbers $\widetilde{A}$ and $\widetilde{B}$ are identical if and only if $d(\widetilde{A},\widetilde{B}) = 0$.*

Proof. Let $\widetilde{A}$ and $\widetilde{B}$ be Hendecagonal Fuzzy Numbers. Since it is identical, $a_1 = b_1, a_2 = b_2, a_3 = b_3, \ldots, a_{11} = b_{11}$. The distance between $\widetilde{A}$ and $\widetilde{B}$ is

$$\begin{aligned} d(\widetilde{A},\widetilde{B}) &= \sqrt{\frac{1}{11}\left[(a_1-b_1)^2+(a_2-b_2)^2+(a_3-b_3)^2+\ldots+(a_{11}-b_{11})^2\right]} \\ &= \sqrt{\frac{1}{11}\left[0^2+0^2+0^2+\ldots+0^2\right]} \\ &= 0 \end{aligned}$$

Converse, If $d(\widetilde{A},\widetilde{B}) = 0$, then

$$\begin{aligned} d(\widetilde{A},\widetilde{B}) &= \sqrt{\frac{1}{11}\left[(a_1-b_1)^2+(a_2-b_2)^2+(a_3-b_3)^2+\ldots+(a_{11}-b_{11})^2\right]} = 0 \\ &\Rightarrow (a-b)^2+(a-b)^2+(a-b)^2+\ldots+(a-b)^2 = 0 \\ &\Rightarrow a_1 = b_1, a_2 = b_2, a_3 = b_3, \ldots, a_{11} = b_{11} \end{aligned}$$

Therefore, $\widetilde{A}$ and $\widetilde{B}$ are identical. □

6. Conclusion

In this paper, Hendecagonal fuzzy number and its arithmetic operations have been proposed. Also, linguistic value of (H_DFN) and vertex method to find the distance between two (H_DFN) are introduced. Therefore, decision making techniques such as DEMATEL, TOPSIS and VIKOR techniques can be extended by representing linguistic variable into hendecagonal fuzzy number under uncertain linguistic environment will be the further research.

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