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Analysis of Results that the Class of Metric Spaces is a Proper Subclass of the Class of Fuzzy Metric Spaces

Research Article

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Abstract: In this paper we will establish that every metric space induces a fuzzy metric space using some results and examples of fuzzy metric space from the reference of George and Veeramani [4] and Kramosil and Michalek [9].

Keywords: Metric space, Fuzzy metric space, Stationary fuzzy metric.

1. Introduction

The concept of fuzzy sets and fuzzy logic was introduced by Professor Lofti A Zadeh in 1965. The success of research in fuzzy sets and fuzzy logic has been demonstrated in a variety of fields, such as artificial intelligence, computer science, control engineering, computer applications, robotics and many more. Here, we adopt the notion of fuzzy metric space due to George and Veeramani [4] which is a modification of the notion of fuzzy metric space as studied by Kramosil and Michalek. The notion of fuzzy metric space by George and Veeramani has many advantages in analysis as many notions and results from classical metric spaces can be extended and generalized to the setting of fuzzy metric spaces. In present paper we analyse some results which will prove that every metric space induces a fuzzy metric space.

2. Fundamental Properties of Fuzzy Metric Spaces

Definition 2.1. *A binary operation* $* : [0,1]\times[0,1] \to [0,1]$ *is a continuous triangular norm(t-norm) if for all* $a, b, c, d \in [0,1]$

(1). $a * 1 = a$

(2). $a * b = b * a$ *(commutativity)*

(3). $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$

(4). $a * (b * c) = (a * b) * c$ *(associativity)*

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2.1. Some Examples of Continuous t-norms

Example 2.2. *Define $a * b = ab$, for all $a, b \in [0,1]$. Note that ab is the usual multiplication in $[0,1]$ for all $a, b \in [0,1]$. It follows that is a continuous t-norm.*

Example 2.3. *Define $a * b = \min(a, b)$, for all $a, b \in [0,1]$. Then is a continuous t-norm. Examples on continuous t- norms can be found in [8].*

Definition 2.4. *The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space, where X is an arbitrary set, is continuous t-norm and M is a fuzzy set on $X \times X \times [0,\infty)$ satisfying the following conditions:*

(1). $M(x, y, 0) = 0$, $\forall\, x, y \in X$

(2). $M(x, y, t) = 1$ *iff* $x = y$, $\forall\, t > 0$ *and* $\forall\, x, y \in X$.

(3). $(x, y, t) = M(y, x, t)$, $\forall\, t > 0$ *and* $\forall\, x, y \in X$.

(4). $M(x, z, t+s) \geq M(x, y, t)M(y, z, s)$, $\forall\, s, t > 0$ *and* $\forall\, x, y \in X$.

(5). $M(x, y, \cdot) : [0, \infty \to [0,1]$ *is continuous,* $\forall\, x, y \in X$.

Definition 2.5. *The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space where X is an arbitrary set, is continuous t-norm and M is a fuzzy set on $X \times X \times (0,\infty)$ satisfying the following conditions:*

(1). $M(x, y, t) > 0$, $\forall\, x, y \in X$ *and* $\forall\, t > 0$

(2). $M(x, y, t) = 1$ *iff* $x = y$, $\forall\, t > 0$ *and* $\forall\, x, y \in X$.

(3). $(x, y, t) = M(y, x, t)$, $\forall\, t > 0$ *and* $\forall\, x, y \in X$.

(4). $M(x, z, t+s) \geq M(x, y, t) * M(y, z, s)$, $\forall\, s, t > 0$ *and* $\forall\, x, y \in X$.

(5). $M(x, y, \cdot) : (0, \infty) \to [0,1]$ *is continuous ,* $\forall\, x, y \in X$.

Result 2.6. $M(x, y, \cdot)$ *is nondecreasing* $\forall\, x, y \in X$.

Proof. Suppose that $M(x, y, t) > M(x, y, s)$ for some $0 < t < s$. Then $M(x, y, t)M(y, y, s-t) \leq M(x, y, s) < M(x, y, t)$. By (2) we have $M(y, y, s-t) = 1$. Thus $M(x, y, t) < M(x, y, s) < M(x, y, t)$ which is a contradiction. □

Remark 2.7.

(1). Let $(X, M, *)$ be a fuzzy metric space and let $x, y \in X$, $t > 0$, $0 < r < 1$. Then if $M(x, y, t) > 1 - r$ we can find t_0 with $0 < t_0 < t$, such that $M(x, y, t_0) > 1 - r$.

(2). For any $r_1 > r_2$, we can find an r_3 such that $r_1 r_3 \geq r_2$ and for any r_4 we can find an r_5 such that $r_5 * r_5 \geq r_4$, $(r_1, r_2, r_3, r_4, r_5 \in (0,1))$.

Example 2.8. *Let $X = \mathbb{R}$. Define $a * b = a * b$ for all $a, b \in [0,1]$ and $M(x, y, t) = [\exp(\frac{|x-y|}{t})]^{-1}$ for all $x, y \in X$ and $t \in (0,\infty)$. Then $(X, M, *)$ is a fuzzy metric space. To prove that M is a fuzzy metric.*

Proof.

(1). Consider $x = y \Rightarrow |x-y| = 0$. Hence $[\exp(\frac{|x-y|}{t})]^{-1} = 1$. Therefore $M(x,y,t) = 1$.

Conversely, assume that $M(x,y,t) = 1$. Therefore $[\exp(\frac{|x-y|}{t})]^{-1} = 1 \Rightarrow e^{\frac{|x-y|}{t}} = e^0 \Rightarrow \frac{|x-y|}{t} = 0$, $|x-y| = 0$. Thus $x = y$. Therefore $M(x,y,t) = 1$ iff $x = y$.

(2). To prove that $M(x,y,t) = M(y,x,t)$. Since $|x-y| = |y-x|$, $\forall\ x,y \in \mathbb{R}$. It follows that for all $x,y \in X$ and for all $t > 0$, $M(x,y,t) = M(y,x,t)$.

(3). To prove that $M(x,y,t)M(y,z,s) \leq M(x,z,t+s)$. We know that $x,y,z \in X$ and for all $t,s > 0$, $|x-z| \leq (\frac{t+s}{t})|x-y| + (\frac{t+s}{s})|y-z|$. i.e $\frac{|x-y|}{t+s} \leq \frac{|x-y|}{t} + \frac{|y-z|}{s}$. Thus $e^{\frac{|x-z|}{t+s}} \leq e^{\frac{|x-y|}{t}} e^{\frac{|y-z|}{s}}$. Since e^x is an increasing function for $x > 0$. Therefore

$$\left[\exp(\frac{|x-z|}{t+s})\right]^{-1} \geq \left[\exp(\frac{|x-y|}{t})\right]^{-1} * \left[\exp(\frac{|y-z|}{s})\right]^{-1}.$$

Thus $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$.

(4). Take a sequence $\{t_n\} \in (0,\infty)$ such that the sequence $\{t_n\}$ converges to $t \in (0,\infty)$ where $(0,\infty)$ is equipped with the usual metric. i.e $\lim_n |t_n - t| = 0$. Without the loss of generality, $x,y \in X$. Since the function e^x is continuous on $\mathbb{R}$ we have $e^{\frac{|x-y|}{t_n}}$ converges to $e^{\frac{|x-y|}{t}}$ as t_n converges to t to the usual metric. Therefore $M(x,y,\cdot) : (0,\infty) \to [0,1]$ is continuous.

Hence $(X, M,)$ is a fuzzy metric space. □

Remark 2.9. *In Example 2.3 we can replace $\mathbb{R}$ by any non empty set X and the usual metric on $\mathbb{R}$ by any metric d. Also, Example 2.3 is a fuzzy metric space with the t-norm defined by $a * b = \min(a,b)$ for all $a,b \in [0,1]$.*

Example 2.10. *Let (X,d) be a metric space. Define $a*b = a*b$ for all $a,b \in [0,1]$ and $M(x,y,t) = \frac{kt^n}{kt^n + md(x,y)}$, $k,m,n \in N$. Then $(X,M,*)$ is a fuzzy metric space.*

Remark 2.11. *Note that Example 2.8 holds even with the continuous t-norm $a * b = \min(a,b)$.*

In particular, taking $k = m = n = 1$, we get $M(x,y,t) = \frac{t}{t + d(x,y)}$. We shall call this fuzzy metric induced by a metric d the standard fuzzy metric. It follows M_d denotes a standard fuzzy metric induced by the metric d.

Example 2.12. *Let $X = \mathbb{N}$. Define $a * b = a * b$ and for all $t > 0$, let*

$$M(x,y,t) = \begin{cases} \frac{x}{y} & \text{if } x \leq y \\ \frac{y}{x} & \text{if } y \leq x \end{cases}$$

To show that M is a fuzzy metric.

Proof.

(1). Assume that $x = y$, $\forall\ t > 0$. Then $\frac{x}{y} = \frac{y}{x} = 1$. Hence $M(x,y,t) = 1$. Conversely, assume that $M(x,y,t) = 1$. Then $\frac{x}{y} = 1$, and therefore $x = y$. Similarly if $\frac{y}{x} = 1$ then it follows that $y = x$. Thus $M(x,y,t) = 1$ iff $x = y$.

(2). $\forall\ x,y \in X$ and for all $t > 0$, clearly $M(x,y,t) = M(y,x,t)$.

(3). To prove that $M(x,z,t+s) \geq M(x,y,t)M(y,z,s)$. We consider the following cases:

(i). If $x = y = z$. Then $M(x,y,t) = 1$, $M(y,z,t) = 1$, $M(x,z,t+s) = 1$. Now $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s) = 1$. Which implies that $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$.

(ii). If $x \neq y = z$. Without loss of generality, we may assume that $x < y$ and $y = z$. Then $M(x, y, t) = \frac{x}{y}$. Also, we have $M(y, z, t) = 1$ and $M(x, z, t+s) = \frac{x}{z}$. Now $\frac{x}{y}1 = \frac{x}{y}$ and $\frac{x}{y} = \frac{x}{z}$. Therefore $M(x, z, t+s) \geq M(x, y, t) * M(y, z, s)$.

(iii). If $x = y \neq z$. Without loss of generality, we may assume that $x = y$ and $y < z$. Then $M(x, y, t) = 1$. Also we have $M(y, z, t) = \frac{y}{z}$ and $M(x, z, t+s) = \frac{x}{z}$. Now $1 * \frac{y}{z} = \frac{y}{z}$ and $\frac{y}{z} = \frac{x}{z}$. Therefore $M(x, z, t+s) \geq M(x, y, t) * M(y, z, s)$.

(iv). If $x \neq y \neq z$. Without loss of generality, we may assume that $x < y < z$. Then $M(x, y, t) = \frac{x}{y}$, $M(y, z, s) = \frac{y}{z}$, $M(x, z, t+s) = \frac{x}{z}$. Now $z > y$ implies that $z^2 > y^2$. So $\frac{1}{z^2} < \frac{1}{y^2}$ Thus $\frac{xy}{z^2} < \frac{xy}{y^2}$. Therefore, $\frac{x}{z} * \frac{y}{z} < \frac{x}{y}$. Hence, $M(x, z, t) * M(z, y, t) < M(x, y, t+s)$.

(4). Note that $M(x, y, t)$ is independent of t (that is $M(x, y, t)$ is a constant in terms of t). For any $s, t > 0$, we have $M(x, y, t) = M(x, y, s)$. Thus $M(x, y, \cdot)$ is continuous. Therefore $(X, M, *)$ is a fuzzy metric space.

□

Remark 2.13. *The fuzzy metric in the above Example is independent of t. Such a fuzzy metric is referred to as a stationary fuzzy metric.*

Remark 2.14. *It is interesting to note that there exists no metric d on X satisfying $M(x, y, t) = \frac{t}{t+d(x,y)}$, where $M(x, y, t)$ is as defined in Example 2.1.5. We show that M is not a fuzzy metric on X with the t-norm defined by $a * b = \min(a, b)$. We start by showing that there is no metric d on X satisfying $M(x, y, t) = \frac{t}{t+d(x,y)}$, where M is defined by*

$$M(x, y, t) = \begin{cases} \frac{x}{y} & \text{if } x \leq y \\ \frac{y}{x} & \text{if } y \leq x. \end{cases}$$

Proof. Suppose that there is a metric d on X that induces $M(x, y, t)$. Then $\forall\, t > 0$

$$\begin{aligned} M(x, y, t) &= \frac{t}{t + d(x, y)} \\ (t + d(x, y))M(x, y, t) &= t \\ d(x, y) &= \frac{t(1 - M(x, y, t))}{M(x, y, t)}. \end{aligned}$$

(1). $t > 0$. Assume that $x = y$. Then this implies that $M(x, y, t) = 1$. Therefore, $d(x, y) = \frac{t(1-1)}{1} = 0$. Conversely, assume that $d(x, y) = 0$, then

$$\begin{aligned} \frac{t(1 - M(x, y, t))}{M(x, y, t)} &= 0 \\ t - tM(x, y, t) &= 0 \\ -tM(x, y, t) &= -t \\ M(x, y, t) &= 1. \end{aligned}$$

This implies that $x = y$.

(2). To prove that $d(x, y) = d(y, x)$. We note that $M(x, y, t) = M(y, x, t)$ since M is a fuzzy metric on X. Then

$$d(x, y) = \frac{t(1 - M(x, y, t))}{M(x, y, t)} = \frac{t(1 - M(x, y, t))}{M(x, y, t)} = d(y, x).$$

(3). We now show that the triangle inequality does not hold: Let $t = 2$, $x = 1$, $y = 2$ and $z = 3$. Then $d(x, y) = \frac{t(1-M(x,y,t))}{M(x,y,t)} = \frac{2(1-\frac{1}{2})}{\frac{1}{2}} = 2$, $d(y, z) = \frac{2(1-\frac{2}{3})}{\frac{2}{3}} = 1$, $d(x, z) = \frac{2(1-\frac{1}{3})}{\frac{1}{3}} = 4$. Therefore, $d(x, z) > d(x, y) + d(y, z)$. Thus, (X, d) is not a metric space.

□

We now prove that **M is not a fuzzy metric with the continuous t-norm defined by** $a * b = \min(a, b)$

Proof.

(1). $\forall\, t > 0$. Assume that $x = y$. Then $d(x, y) = 0$. Hence $\frac{t}{t+d(x,y)} = 1$. Therefore, $M(x, y, t) = 1$. Conversely, assume that $M(x, y, t) = 1$. Therefore $\frac{t}{t+d(x,y)} = 1 \Rightarrow t + d(x, y) = t \Rightarrow d(x, y) = 0$. Thus, $x = y$.

(2). To show $M(x, y, t) = M(y, x, t)$ we know that $d(x, y) = d(y, x)$ that is, $M(x, y, t) = \frac{t}{t+d(x,y)} = \frac{t}{t+d(y,x)} = M(y, x, t)$. Thus, $M(x, y, t) = M(y, x, t)$.

(3). To show that the inequality $M(x, y, t)M(y, z, s) \leq M(x, z, t + s)$ does not hold, choose $t = 2$, $x = 1$, $y = 2$ and $z = 3$ Then we obtain; $M(x, y, t) * M(y, z, s) = \frac{2}{3} * \frac{2}{3} = \min(\frac{2}{3} * \frac{2}{3}) = \frac{2}{3}$. Therefore $M(x, z, t + s) < M(x, y, t) * M(y, z, s)$. Thus M is not a fuzzy metric.

□

3. Conclusion:

From the above discussion, we conclude that the class of metric spaces is a proper subclass of the class of fuzzy metric spaces.

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