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Intuitionistic Fuzzy Sequences in Metric Spaces

Research Article

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Abstract: In this paper, the notion of intuitionistic fuzzy sequence in a metric space is introduced and the convergence property of an intuitionistic fuzzy sequence is defined. Also discussed the comparison between crisp sequence and intuitionistic fuzzy sequence in metric space. Theorems related to the convergence of crisp sequence and intuitionistic fuzzy sequence are derived.

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1. Introduction

In every moment of human life, the problems in many fields encounter with uncertain data and are not successfully furnished in classical mathematics. Ever since the inception of fuzzy set theory in 1965 by Lofti A. Zadeh [9], it has been found suitable in mathematical modelling. The intrinsic vagueness and uncertainty involved in the parameters and variables in the models are effectively taken care of by fuzzy mathematical tools. The notion of defining intuitionistic fuzzy sets (IFSs) for fuzzy set generalizations, introduced by Atanassov [1], has proven interesting and useful in various application areas. Since this fuzzy set generalization can present the degrees of membership and non-membership with a degree of hesitancy, the knowledge and semantic representation becomes more meaningful and applicable. Chang (1968) [3] was the first to introduce the concept of a fuzzy topology. In his definition of fuzzy topology, fuzziness in the concept of openness of a fuzzy subset was absent. Hohle(1995)[5] first considered fuzzy topology with fuzziness in the concept of openness. The concept of metric space was introduced in 1906 by Frechet [4] which furnished the common idealization of a large number of mathematical, physical and other scientific constructions in-which the notion of distance appears. In a metric space sequence is a good tool to study the important properties. The closure of a set A can be characterised using convergent sequences in A. The continuity of a function from one metric space to another can be characterised using convergent sequences. In this paper, the notion of an intuitionistic fuzzy sequence in a metric space is introduced and also initiate the new convergent property of an intuitionistic fuzzy sequence which is an generalization of already existing crisp concept.

2. Preliminaries

Definition 2.1 ([4]). *Let X be a non-empty set(crisp) and d a function from $X \times X \to R^+$(non-negative reals) such that for all $x, y, z \in X$ we have*

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M_1 : $d(x, y) = 0$ *if and only if* $x = y$,

M_2 : $d(x, y) = d(y, x)$, *and*

M_1 : $d(x, y) \leq d(x, z) + d(z, y)$.

A function d satisfying the above conditions is said to be a distance function or metric and the pair (X, d) a metric space. We write X for a metric space (X, d).

Definition 2.2 ([9]). *Let $X = \{x_1, x_2, ..., x_n\}$ be a non-empty finite set. A fuzzy set F of X can be defined as $F = \{(x, \mu_F(x))/x \in X\}$, where $\mu_F(x) : X \to [0, 1]$ is the degree of membership of x in X.*

Definition 2.3 ([1]). *An intuitionistic fuzzy set F in X can be formulated as $F = \{(x, \mu_F(x), \nu_F(x))/x \in X\}$, where $\mu_F(x), \nu_F(x) : X \to [0, 1]$ represent the degree of membership and non-membership of x in X, respectively, with the essential condition $0 \leq \mu_F(x) + \nu_F(x) \leq 1$.*

Definition 2.4 ([7]). *Let X be a non-empty set. A fuzzy set A on $N \times X$ is called a fuzzy sequence in X. That is $A : N \times X \to [0, 1]$ is called a fuzzy sequence in X.*

Definition 2.5. *Let X be a non-empty set. An intuitionistic fuzzy set $A = (\mu, \nu)$ on $N \times X$ is called an intuitionistic fuzzy sequence in X. That is (μ, ν) is called an intuitionistic fuzzy sequence in X, where $\mu : N \times X \to [0, 1], \nu : N \times X \to [0, 1]$ with $\mu + \nu \leq 1$*

Example 2.6. *Let $X = N$. Define $\mu, \nu : N \times N \to [0, 1]$ as $\mu(n, x) = \frac{1}{(n+x)}$ and $\nu(n, x) = \frac{n+x-1}{(n+x)}$ for all $n \in N$, for all $x \in N$. Clearly A is an intuitionistic fuzzy sequence in N.*

Theorem 2.7. *Every crisp sequence in X is an intuitionistic fuzzy sequence in X.*

Proof. Let X be a non-empty set. Let S be the set of all crisp sequences on X and let F be the set of all intuitionistic fuzzy sequences on X. Consider a function $T : S \to F$ as follows. Take any $g \in S$. Then $g : N \to X$. Now define $A_g(\mu_g, \nu_g) : N \times X \to [0, 1]$ as

$$\begin{cases} \mu_g(n, x) = 1 \text{ and } \nu_g(n, x) = 0 \text{ if } g(n) = x \\ \mu_g(n, x) = 0 \text{ and } \nu_g(n, x) = 1 \text{ if otherwise} \end{cases} \tag{1}$$

Hence A_g is an intuitionistic fuzzy sequence in X. Let $T(g) = A_g$. We claim that this function T is one one. Let g and h belong to S and let $g \neq h$. Then there exists $n_0 \in N$, such that $g(n_0) \neq h(n_0)$. Let $g(n_0) = x$ and $h(n_0) = y$ then $x \neq y$. Now $A_g(n_0, x) = 1$ (that is $\mu_g(n_0, x) = 1$ and $\nu_g(n_0, x) = 0$) and $A_h(n_0, x) = 0$(that is $\mu_h(n_0, x) = 0$ and $\nu_h(n_0, x) = 1$). Hence $T(g) \neq T(h)$. Therefore T is one one. Hence every crisp sequence in X is uniquely associated to an intuitionistic fuzzy sequence in X. Therefore every crisp sequence is an intuitionistic fuzzy sequence. □

Note 2.8. *The converse of the above theorem is need not be true, which is illustrated with the following example.*

Example 2.9. *Let $X = Z$. Define $\mu, \nu : N \times X \to [0, 1]$ as*

$$A = \begin{cases} \mu(n, x) = \frac{1}{(n+|x|)} \\ \nu(n, x) = \frac{n+|x|-1}{(n+|x|)} \end{cases} \tag{2}$$

Hence $A = (\mu, \nu)$ is an intuitionistic fuzzy sequence but A cannot be considered as a crisp sequence.

Theorem 2.10. *Let X be a non-empty set. An intuitionistic fuzzy sequence $A = (\mu, \nu)$ on X is a crisp sequence if A satisfies the following conditions.*

(1).

$$\begin{cases} \mu(n,x) = 1 \quad and \quad \nu(n,x) = 0 \;\; if \;\; x \in X \\ \mu(n,x) = 0 \quad and \quad \nu(n,x) = 1 \;\; if \;\; x \notin X \end{cases} \tag{3}$$

for all $n \in N$, *for all* $x \in X$.

(2). For each $n \in N$, *there exists unique* x *in* X *such that* $\mu(n,x) = 1$ *and* $\nu(n,x) = 0$

Proof. Let A be an intuitionistic fuzzy sequence on X satisfying the given conditions. It is enough to prove that, there exists a crisp sequence f on X such that $A_f = A$.

Consider a crisp sequence f on X where $f(n) = x$ if $\mu(n,x) = 1, \nu(n,x) = 0$. Since for each $n \in N$, there exists unique x in X such that $\mu(n,x) = 1, \nu(n,x) = 0$, $f(n)$ is defined for all $n \in N$. Hence $f : N \to X$ is a crisp sequence. Now we see that for this sequence f, $A_f = A$. If $\mu_f(n,x) = 1, \nu_f(n,x) = 0$ then $f(n) = x$. Hence $\mu(n,x) = 1, \nu(n,x) = 0$. Therefore $\mu_f(n,x) = 1, \nu_f(n,x) = 0$ implies $\mu_f(n,x) = \mu(n,x)$ and $\nu_f(n,x) = \nu(n,x)$. If $\mu_f(n,x) = 0, \nu_f(n,x) = 1$ then $f(n) \neq x$. Hence $\mu_f(n,x) \neq 1, \nu_f(n,x) \neq 0$. Therefore $\mu(n,x) = 0, \nu(n,x) = 1$. Hence $\mu_f(n,x) = 0, \nu_f(n,x) = 1$ implies $\mu_f(n,x) = \mu(n,x), \nu_f(n,x) = \nu(n,x)$. Therefore there exists $f : N \to X$ such that $A_f = A$. Hence A can be considered as a crisp sequence. □

3. Convergence of an Intuitionistic Fuzzy Sequence

Definition 3.1. *Let* (X,d) *be a metric space and let* A *be an intuitionistic fuzzy sequence on* X. *Let* $\alpha, \beta \in (0,1]$. *Let* $l \in X$ *and* A *is said to be converges to* l *at a level* (α, β) *if*

(1). For each $n \in N$, *there exists atleast one* x *in* X *where* $\mu(n,x) \geq \alpha$ *and* $\nu(n,x) \leq \beta$.

(2). Given $\epsilon > 0$, *there exists* $n_0 \in N$ *such that* $d(x,a) < \epsilon$ *for all* $n \geq n_0$ *and for all* x *with* $\mu(n,x) \geq \alpha$ *and* $\nu(n,x) \leq \beta$.

Therefore given $\epsilon > 0$ *there exists* $n_0 \in N$ *such that* $n \geq n_0$ *and* $\mu(n,x) \geq \alpha$, $\nu(n,x) \leq \beta$ *with* $\alpha + \beta \leq 1 \implies d(x,a) < \epsilon$.

Example 3.2. *Consider* R *with usual metric. Define* $\mu, \nu : N \times X \to [0,1]$ *as*

$$\begin{cases} \mu(n,x) = 1 \quad and \quad \nu(n,x) = 0 \;\; if \;\; x = \frac{1}{n} \\ \mu(n,x) = 0 \quad and \quad \nu(n,x) = 1 \;\; if \;\; otherwise \end{cases} \tag{4}$$

We claim that $A \to 0$. *Take any* $\alpha, \beta > 0$. $\alpha, \beta \in (0,1]$ *with* $\alpha + \beta \leq 1$.

(1). For each $n \in N$, $\frac{1}{n} \in X$ *such that* $\mu(n, 1/n) \geq \alpha$ *and* $\nu(n, 1/n) \leq \beta$

(2). Let $\epsilon > 0$ *be given. Take* $n_0 \in N$ *such that* $n_0 > \frac{1}{\epsilon}$

Let $n \geq n_0$ *and* $\mu(n,x) \geq \alpha, \nu(n,x) \leq \beta$. *Now* $d(x,a) = |x-a| = |x-0| = |x|$. $n \geq n_0$ *and* $\mu(n,x) \geq \alpha, \nu(n,x) \leq \beta \Rightarrow n > \frac{1}{\epsilon}$ *and* $\mu(n,x) = 1, \nu(n,x) = 0 \Rightarrow n > \frac{1}{\epsilon}$ *and* $x = \frac{1}{n}$. *Now* $d(x,a) = |x| = |\frac{1}{n}| = \frac{1}{n} < \epsilon$. *Given* $\epsilon > 0$, *there exists* $n_0 \in N$, *such that* $n \geq n_0$ *and* $\mu(n,x) \geq \alpha, \nu(n,x) \leq \beta \Rightarrow d(x,0) < \epsilon$. *Hence* $A \to 0$.

Theorem 3.3. *The concept of convergence of an intuitionistic fuzzy sequence is an extension of the concept of convergence of crisp sequences.*

Proof. Let f be a crisp sequence in a metric space X. Then this can be considered as an intuitionistic fuzzy sequence $A_f = (\mu_f, \nu_f)$. Now we have to prove that if the crisp sequence f converges then the intuitionistic fuzzy sequence $A_f = (\mu_f, \nu_f)$ converges at some level (α, β). Let f converge to l. Consider $A_f = (\mu_f, \nu_f)$. Take $\alpha, \beta > 0$. Let $\epsilon > 0$ be given. Since the crisp sequence f converges to l, there exists $n_0 \in N$ such that $d(x_n, l) < \epsilon$ for all $n \geq n_0$. Now it is clear that

$$\begin{cases} \mu_f(n,x) = 1 \quad and \quad \nu_f(n,x) = 0 \ \ if \ \ x \in X \\ \mu_f(n,x) = 0 \quad and \quad \nu_f(n,x) = 1 \ \ if \ \ x \notin X \end{cases} \tag{5}$$

for all $n \in N$, for all $x \in X$.

(1). For each $n \in N$, we have $x = x_n$ where $\mu_f(n,x) = 1 \geq \alpha$ and $\nu_f(n,x) = 0 \leq \beta$

(2). Now $n \geq n_0$ and $\mu_f(n,x) \geq \alpha$ and $\nu_f(n,x) \leq \beta$ implies $n \geq n_0$ and $\mu_f(n,x) = 1$,$\nu_f(n,x) = 0$ which implies $x = x_n$. Now $d(x,l) = d(x_n,l) \leq \epsilon$. Given $\epsilon > 0$, there exists $n_0 \in N$ such that $n \geq n_0$ and $\mu_f(n,x) \geq \alpha$ and $\nu_f(n,x) \leq \beta$ implies $d(x,l) < \epsilon$. Hence A_f converges to l. Hence the theorem.

□

Theorem 3.4. *Let f be a crisp sequence in a metric space X. If the corresponding intuitionistic fuzzy sequence A_f converges to l at some level (α, β) then the crisp sequence f converges to l.*

Proof. Let f be a crisp sequence in metric space X. Let A_f be the corresponding intuitionistic fuzzy sequence. Let A_f converge to l at level (α, β). Let $\epsilon > 0$ be given. Since A_f converges to l there exists $n_0 \in N$ such that $n \geq n_0$ and $\mu_f(n,x) \geq \alpha$ and $\nu_f(n,x) \leq \beta$ implies $d(x,l) < \epsilon$. Now $\mu_f(n,x) \geq \alpha, \nu_f(n,x) \leq \beta$ and $\alpha, \beta > 0$ implies that $\mu_f(n,x) = 1$ and $\nu_f(n,x) = 0$ which implies $x = x_n$. Therefore we get $d(x_n,l) < \epsilon$. Hence given $\epsilon > 0$ there exists $n_0 \in N$ such that $d(x_n,l) < \epsilon$ for all $n \geq n_0$. Therefore the crisp sequence f converges to l. □

Theorem 3.5. *Let (a_n) and (b_n) be two crisp sequences in a metric space X converge to same limit l. Let A be the intuitionistic fuzzy sequence defined as*

$$\begin{cases} \mu_f(n,x) = 1, \ \nu_f(n,x) = 0 \ \ if \ x = a_n \ \ or \ \ x = b_n \\ \mu_f(n,x) = 0, \ \nu_f(n,x) = 1 \ \ if \ otherwise \end{cases} \tag{6}$$

Then A converges to l at any level (α, β).

Proof. Let (a_n) converges to l and (b_n) converges to l. By definition of A, for each $n \in N$,there exists $a_n \in X$ such that $\mu_f(n,a_n) = 1$ and $\nu_f(n,a_n) = 0$. Hence $\mu(n,a_n) \geq \alpha$ and $\nu(n,a_n) \leq \beta$. Let $\epsilon > 0$ be given. Since (a_n) converges to l, there exists $n_1 \in N$ such that $d(a_n,l) < \epsilon$ for all $n \geq n_1$. Since (b_n) converges to l, there exists $n_2 \in N$ such that $d(b_n,l) < \epsilon$ for all $n \geq n_2$. Let $n_0 = \max\{n_1, n_2\}$. Now let $n \geq n_0$ and $\mu(n,x) \geq \alpha$ and $\nu(n,x) \leq \beta$. Since $\alpha, \beta > 0, \mu(n,a_n) \geq \alpha$ and $\nu(n,a_n) \leq \beta$ implies that

$$\begin{cases} \mu_f(n,x) = 1 \\ \nu_f(n,x) = 0 \ \text{ if } \ x = a_n \ \text{ or } \ x = b_n \end{cases} \qquad \begin{cases} \mu_f(n,x) = 0 \\ \nu_f(n,x) = 1 \ \text{ if otherwise} \end{cases} \tag{7}$$

Since $n \geq n_1$, $d(a_n,l) < \epsilon$. Since $n \geq n_2, d(b_n,l) < \epsilon$. Hence $d(x,l) < \epsilon$. Therefore given $\epsilon > 0$, there exists $n_0 \in N$ such that $n \geq n_0$ and $\mu(n,a_n) \geq \alpha$, $\nu(n,a_n) \leq \beta$ implies $d(x,l) < \epsilon$. Hence the intuitionistic fuzzy sequence A converges to l at any level (α, β) □

Theorem 3.6. *Let (a_n) and (b_n) be two crisp sequences in a metric space X. Let A be an intuitionistic fuzzy sequence in X defined as*

$$\begin{cases} \mu(n,x)=1 \;\; and \;\; \nu(n,x)=0 \;\; if \;\; x=a_n \;\; or \;\; b_n \\ \mu(n,x)=0 \;\; and \;\; \nu(n,x)=1 \;\; otherwise. \end{cases} \tag{8}$$

Then the intuitionistic fuzzy sequence A converges at some level (α,β) if and only if both (a_n) and (b_n) converge and they converge to the same limit.

Proof. The proof is similar to Theorem 3.5. □

Theorem 3.7. *Let $\{(a_n^k)/k \in K\}$ be a collection of crisp sequences in a metric space X and K be any finite index set. Let A be an intuitionistic fuzzy sequence in M defined as*

$$\begin{cases} \mu_f(n,x)=1 \;\; and \;\; \nu_f(n,x)=0 \;\; if \;\; x=a_n^k \\ \mu_f(n,x)=0 \;\; and \;\; \nu_f(n,x)=1 \;\; otherwise \end{cases} \tag{9}$$

Then the intuitionistic fuzzy sequence A converges if and only if for each k, (a_n^k) converges and all the sequences converge to the same limit in X.

Proof. The proof is similar to the previous theorem. □

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