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Graphs with Equal Total Domination and Inverse Total Domination Numbers

Research Article

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Abstract: Let D be a minimum total dominating set of $G = (V, E)$. If $V - D$ contains a total dominating set D' of G, then D' is called an inverse total dominating set with respect to D. The inverse total domination number $\gamma_t(G)$ of G is the minimum cardinality of an inverse total domination set of G. In this paper, we obtain some graphs for which $\gamma_t(G) = \gamma_t^{-1}(G)$. Also we find some graphs for which $\gamma_t(G) = \gamma_t^{-1}(G) = \frac{p}{2}$.

MSC: 05C.

Keywords: Total dominating set, inverse total dominating set, inverse total dominating number.

1. Introduction

By a graph, we mean a finite, undirected without loops, multiple edges or isolated vertices. Any undefined term in this paper may be found in Kulli [1]. A set D of vertices in a graph G = (V, E) is called a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. Recently several dominating parameters are given in the books by Kulli in [2–4]. Let D be a minimum dominating set of G. If $V - D$ contains a dominating set D' of G, then D' is called an inverse dominating set of G with respect to D. The inverse domination number $\gamma^{-1}(G)$ of G is the minimum cardinality of an inverse dominating set of G. This concept was introduced by Kulli and Sigarkanti in [5]. Many other inverse domination parameters in domination theory were studied, for example, in [6–16]. A set $D \subseteq V$ is a total dominating set of G if every vertex in V is adjacent to some vertex in D. The total domination number $\gamma_t(G)$ of G is the minimum cardinality of a total dominating set of G.

Let $D \subseteq V$ be a minimum total dominating set of G. If $V - D$ contains a total dominating set D' of G, then D' is called an inverse total dominating set with respect to D. The inverse total dominating number $\gamma_t^{-1}(G)$ of G is the minimum cardinality of an inverse total dominating set of G. This concept was introduced by Kulli and Iyer in [17] and was studied in [18]. A γ_t^{-1}-set is a minimum inverse total dominating set. Similarly other sets can be expected. Note that every graph without isolated vertices has a total dominating set. Hence we consider only graphs without isolated vertices. A vertex that is adjacent to a pendant vertex u is called a support of u. If $D = \{u, v\}$ is a total dominating set of G, then u, v are called total dominating vertices of G. A vertex u of G is said to be a γ_t-required vertex of G if u lies in every γ_t-set of G.

An application of inverse total domination is found in a computer network. We consider a computer network in which a core group of file servers has the ability to communicate directly with every computer outside the core group. In addition, each

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file server is directly linked with at least one other backup file server where duplicate information is stored. A minimum core group with this property is a smallest total dominating set for the graph representing the network. If a second important core group is needed then a separate disjoint total dominating set provides duplication in case the first is corrupted in some way. We have $\gamma_t(G) \leq \gamma_t^{-1}(G)$, (See [17]). From the point of networks, one may demand $\gamma_t^{-1}(G) = \gamma_t(G)$, where as many graphs do not enjoy such a property. For Example, we consider the graph G in Figure 1. Then $\gamma_t(G) = 2$ and $\gamma_t^{-1}(G) = p - 2$. In this case, if p is large, then $\gamma_t^{-1}(G)$ is sufficiently large compare to $\gamma_t(G)$.

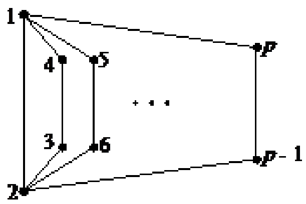

Figure 1.

2. Graphs with $\gamma_t(G) = \gamma_t^{-1}(G)$

Proposition 2.1. *If K_p is a complete graph with $p \geq 2$ vertices, then $\gamma_t(K_p) = \gamma_t^{-1}(K_p) = 2$.*

Proposition 2.2. *If $K_{m,n}$ is a complete bipartite graph with $2 \leq m \leq n$, then $\gamma_t(K_{m,n}) = \gamma_t^{-1}(K_{m,n}) = 2$.*

Proposition 2.3. *If $K_{m,n}$ is a complete bipartite graph with $2 \leq m \leq n$, then $\gamma_t\left(\overline{K_{m,n}}\right) = \gamma_t^{-1}\left(\overline{K_{m,n}}\right) = 4$.*

Proof. Clearly $\overline{K_{m,n}} = K_m \cup K_n$. Therefore $\gamma_t\left(\overline{K_{m,n}}\right) = \gamma_t\left(K_m\right) + \gamma_t\left(K_n\right) = 2 + 2 = 4$.

$$\gamma_t^{-1}\left(\overline{K_{m,n}}\right) = \gamma_t^{-1}\left(K_m\right) + \gamma_t^{-1}\left(K_n\right) = 2 + 2 = 4.$$

□

Thus the result follows.

Theorem 2.4. *Let G be a graph with $\gamma_t(G) = \gamma_t^{-1}(G)$. Then G has no γ_t-required vertex.*

Proof. Let G be a graph with $\gamma_t(G) = \gamma_t^{-1}(G)$. Let D be a γ_t-set and D_1 be a γ_t^{-1}-set of G. Suppose G contains a γ_t-required vertex u. Then u lies in every γ_t-set of G. Thus $u \in D$ and $u \in D_1$, which is a contradiction to $D_1 \subseteq V - D$. □

Theorem 2.5. *If u, v are total dominating vertices of a graph G, then $\gamma_t^{-1}(G) = \gamma_t(G - u - v)$.*

Proof. Since u, v are total dominating vertices of G, $\{u, v\}$ is a γ_t-set of G. Thus any γ_t^{-1}-set of G lies in $G - \{u, v\}$ and is a minimum total dominating set of $G - \{u, v\}$. Hence $\gamma_t^{-1}(G) = \gamma_t(G - u - v)$. □

Theorem 2.6. *Let G be a graph such that G and $\overline{G}$ are connected with at least two pendant vertices a, b in G. Let a', b' be the supports of a and b respectively.*

(1) If $a' \neq b'$, then $\gamma_t(\overline{G} + aa' + bb') = \gamma_t^{-1}(\overline{G} + aa' + bb') = 2$.

(2) If $a' = b'$, then $\gamma_t(\overline{G} + aa') = \gamma_t^{-1}(\overline{G} + aa') = 2$.

Proof. Suppose G and $\overline{G}$ are connected. Then $\Delta(G) \leq p - 2$ and $\Delta(\overline{G}) \leq p - 2$. Thus $\gamma_t(\overline{G}) \geq 2$ and $\gamma_t^{-1}(\overline{G}) \geq 2$. Let a, b be two pendant vertices in G. Let a', b' be the supports of a and b respectively.

(1) Suppose $a' \neq b'$. Let $G_1 = \overline{G} + aa' + bb'$. In G_1, a, a' are adjacent and b, b' are adjacent. Clearly $D = \{a, a'\}$ is a γ_t-set of G_1 and $D_1 = \{b, b'\}$ is a γ_t^{-1}-set of G_1. Hence $\gamma_t(G_1) = \gamma_t^{-1}(G_1) = 2$.

(2) Suppose $a' = b'$. Let $G_2 = \overline{G} + aa'$. In G_2, a, a' are adjacent. Then $D = \{a, a'\}$ is a γ_t-set of G_2. Since G_2 is connected, a' is adjacent to some vertex c in G_2. Thus $D_1 = \{b, c\}$ is a γ_t^{-1}-set of G_2. Thus $\gamma_t(G_2) = \gamma_t^{-1}(G_2) = 2$.

We characterize cycles C_p for which $\gamma_t(C_p) = \gamma_t^{-1}(C_p)$. □

Theorem 2.7. *For any integer $p \geq 4$, $\gamma_t(C_p) = \gamma_t^{-1}(C_p) = \frac{p}{2}$ if and only if $p = 0(mod\ 4)$.*

Proof. Let $V(C_p) = \{1, 2, \ldots, p\}$. Assume $p = 0(mod\ 4)$ and $p \geq 4$. Then $p = 4k$ for some integer $k \geq 1$. When $p = 4k$, the set $D = \{3, 4, 7, 8, \ldots, 4k-1, 4k\}$ is a γ_t-set with $2k = \frac{p}{2}$ vertices and $D' = \{1, 2, 5, 6, \ldots, 4k-3, 4k-2\}$ is a γ_t^{-1}-set with $2k = \frac{p}{2}$ vertices. Hence $\gamma_t(C_p) = \gamma_t^{-1}(C_p) = \frac{p}{2}$.
Conversely suppose $\gamma_t(C_p) = \gamma_t^{-1}(C_p) = \frac{p}{2}$. We now prove that $p = 0(mod\ 4)$. On the contrary, assume $p \neq 0(mod\ 4)$. Then $p = 4k+1$ or $4k+2$ or $4k+3$ for some integer $k \geq 1$. If $p = 4k+1$, then the set $D = \{1, 2, 5, 6, \ldots, 4k+1\}$ is a γ_t-set with $2k+1$ vertices and $D_1 = \{3, 4, 7, 8, \ldots, 4k-1, 4k\}$ is not a γ_t^{-1}-set in C_p. If $p = 4k+2$, then $D = \{1, 2, 5, 6, \ldots, 4k+1, 4k+2\}$ is a γ_t-set with $2k+2$ vertices and $D_1 = \{3, 4, 7, 8, \ldots, 4k-1, 4k\}$ is not a γ_t^{-1}-set in C_p. If $p = 4k+3$, then $D = \{1, 2, 5, 6, \ldots, 4k+1, 4k+2\}$ is a γ_t-set with $2k+2$ vertices and $D_1 = \{3, 4, 7, 8, \ldots, 4k-1, 4k, 4k+3\}$ is not a γ_t^{-1}-set in C_p. Thus $p = 0(mod\ 4)$. □

Theorem 2.8. *For any integer $p \geq 4$, $\gamma_t\left(\overline{C_p} + v_iv_{i+1} + v_jv_{j+1}\right) = \gamma_t^{-1}\left(\overline{C_p} + v_iv_{i+1} + v_jv_{j+1}\right) = 2$.*

Proof. Let $V(C_p) = \{v_1, v_2, \ldots, v_p\}$. Then each vertex v_i in C_p is adjacent to v_{i-1} and v_{i+1} modulo p. Hence each vertex v_i in C_p is adjacent to the remaining $p-3$ vertices. Also v_{i-1} and $v_i + 1$ are adjacent in $\overline{C_p}$. Let $G = \overline{C_p} + v_iv_{i+1} + v_jv_{j+1}$. In G, v_i, v_{i+1} are adjacent and v_j, v_{j+1} are adjacent. Hence $D = \{v_i, v_{i+1}\}$ is a γ_t-set of G and $D_1 = \{v_j, v_{j+1}\}$ is a γ_t^{-1}-set of G. Thus $\gamma_t(G) = \gamma_t^{-1}(G) = 2$. □

Theorem 2.9. *If P_p is a path with $p \geq 4$ vertices, v_1, v_p are end vertices and v_i, v_{i+1} are adjacent non-end vertices, then $\gamma_t(\overline{P_p} + v_iv_{i+1}) = \gamma_t^{-1}(\overline{P_p} + v_iv_{i+1}) = 2$.*

Proof. Let $V(P_p) = \{v_1, v_2, \ldots, v_p\}$. Join v_1 and v_p in P_p. Then $P_p + v_1v_p = C_p$. By Theorem 2.8, $\gamma_t\left\{\left(\overline{P_p + v_1v_p}\right) + v_1v_p + v_iv_{i+1}\right\} = \gamma_t^{-1}\left\{\left(\overline{P_p + v_1v_p}\right) + v_1v_p + v_iv_{i+1}\right\} = 2$. Thus $\gamma_t\left(\overline{P_p} + v_iv_{i+1}\right) = \gamma_t^{-1}\left(\overline{P_p} + v_iv_{i+1}\right) = 2$. □

Theorem 2.10. *For any integers $m, n \geq 2$, $\gamma_t\left(\overline{P_m \cup P_n}\right) = \gamma_t^{-1}\left(\overline{P_m \cup P_n}\right) = 2$.*

Proof. Let $V(P_m) = \{v_1, v_2, \ldots, v_m\}$ and $V(P_n) = \{u_1, u_2, \ldots, u_n\}$. Then each vertex v_i in $\overline{P_m \cup P_n}$ is adjacent to each vertex u_j. Also each vertex u_j in $\overline{P_m \cup P_n}$ is adjacent to each vertex v_i. Then $D = \{v_1, u_1\}$ is a γ_t-set of $\overline{P_m \cup P_n}$ and $D_1 = \{v_2, u_2\}$ is a γ_t^{-1}-set of $\overline{P_m \cup P_n}$. Thus $\gamma_t\left(\overline{P_m \cup P_n}\right) = \gamma_t^{-1}\left(\overline{P_m \cup P_n}\right) = 2$. □

3. Graphs with $\gamma_t(G) = \gamma_t^{-1}(G) = \frac{p}{2}$

In this section, we obtain some results for which $\gamma_t(G) = \gamma_t^{-1}(G) = \frac{p}{2}$.

Theorem 3.1. *If $G = C_{4n}$ or K_4 or $K_4 - e$, then $\gamma_t(G) = \gamma_t^{-1}(G) = \frac{p}{2}$, where p is the number of vertices of G.*

Proof. If $G = C_{4n}$, then by Theorem 2.7, $\gamma_t(G) = \gamma_t^{-1}(G) = \frac{p}{2}$. If $G = K_4$ or $K_4 - e$, then we have $\gamma_t(G) = \gamma_t^{-1}(G) = \frac{p}{2}$, where p is the number of vertices of G. □

Remark 3.2. *Let $G_1, G_2, \ldots, G_m$ be the m connected components of a graph G. Let D_i be a γ_t-set of G_i and D_i' be a γ_t^{-1}-set of G_i for $i = 1, 2, \ldots, m$. Then $\sum_{i=1}^{m} D_i$ is a γ_t-set of G and $\sum_{i=1}^{m} D_i'$ is a γ_t^{-1}-set of G. Therefore $\gamma_t(G) = \sum_{i=1}^{m} \gamma_t(G_i)$ and $\gamma_t^{-1}(G) = \sum_{i=1}^{m} \gamma_t^{-1}(G_i)$.*

Theorem 3.3. *Let $G_1, G_2, \ldots, G_m$ be the m connected components of a graph G. Then $\gamma_t(G) = \gamma_t^{-1}(G)$ if and only if $\gamma_t(G_i) = \gamma_t^{-1}(G_i)$, for $i = 1, 2, \ldots, m$.*

Proof. Let $G_1, G_2, \ldots, G_m$ be the m connected components of G. By Remark 3.2, $\gamma_t(G) = \sum \gamma_t(G_i)$ and $\gamma_t^{-1}(G) = \sum \gamma_t^{-1}(G_i)$. Clearly, $\gamma_t(G) = \gamma_t^{-1}(G)$ if $\gamma_t(G_i) = \gamma_t^{-1}(G_i)$ for $1, 2, \ldots, m$.

Conversely suppose $\gamma_t(G) = \gamma_t^{-1}(G)$. We have $\gamma_t(G_i) \leq \gamma_t^{-1}(G_i)$ for $i = 1, 2, \ldots, m$. We now prove that $\gamma_t(G_i) = \gamma_t^{-1}(G_i)$, for $i = 1, 2, \ldots, m$. On the contrary, assume $\gamma_t(G_i) < \gamma_t^{-1}(G_i)$ for some i . Then $\gamma_t(G_j) > \gamma_t^{-1}(G_j)$, for some j, $j \neq i$, which is a contradiction. Thus $\gamma_t(G_i) = \gamma_t^{-1}(G_i)$ for $i = 1, 2, \ldots, m$. □

Corollary 3.4. *If the connected components G_i of G are either C_{4n} or $K_4 z$ or $K_4 - e$, then $\gamma_t(G) = \gamma_t^{-1}(G) = \frac{p}{2}$.*

Proof. The result follows from Theorem 3.1 and Theorem 3.3. □

Problem 3.5. *Characterize graphs G for which $\gamma_t(G) = \gamma_t^{-1}(G)$.*

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