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Some Types of Generalized H^h-Recurrent in Finsler Spaces

Research Article

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Abstract: The purpose of this paper is to develop some properties of generalized H^h-recurrent affinely connected space, P2-like generalized H^h-recurrent space and P^*-generalized H^h-recurrent space for Berwald curvature tensor H^i_{jkh} which satisfies the condition $H^i_{jkh|l} = \lambda_l H^i_{jkh} + \mu_l(\delta^i_h g_{jk} - \delta^i_k g_{jh})$, where $|l$ is h-covariant differentiation, λ_l and μ_l are non-null covariant vectors field is introduced and such space is called as a generalized H^h-recurrent space and denote it briefly by GH^h-RF_n. Some theorems and conditions have been pointed out which reduce a generalized H^h-recurrent space $F_n(n > 2)$ into a Finsler space of curvature scalar.

Keywords: Finsler space, Generalized H-recurrent space, affinely connected space, P2-like space and P^*-space.

1. Introduction

A Finsler space of recurrent curvature was introduced and studied by R.S.D.Dubey and A.K.Srivastava [3], P.N.Pandy [9], P.N.Pandy and R.B.Misra [10], P.N.Pandy and V.J.Dwivedi [11], P.N.Pandy and S.Pal [12], R.Verma [16], S.Dikshit [2], F.Y.A. Qasem [14], and many others. The concept of C^h-recurrent space have been studied by M.Matsumoto [7], C.K.Mishra and G.Lodhi [8]. U.C.De and M.Guha [1], introduced a generalized recurrent Riemannian manifold. Y.B.Maralabhavi and M.Rathnamma [6], also contributed towards a generalized recurrent and generalized concircular recurrent Riemannian manifolds. P.N.Pandy, S.Saxena and A.Goswami [13] introduced a generalized H-recurrent Finsler space.

Let us consider an n-dimensional Finsler space F_n equipped with the metric function F satisfying the requisite conditions [15]. Let the components of the corresponding metric tensor g_{ij}, Cartan's connection parameters Γ^i_{jk} and Berwald's connection parameters G^{i*}_{jk}. They are symmetric in their lower indices and positively homogeneous of degree zero in the directional arguments. The vectors y_i and y^i satisfies the following relations

$$\begin{aligned} &a)\ \ y_i = g_{ij} y^j, \\ &b)\ \ y_i y^i = F^2, \\ &c)\ \ g_{ij} = \dot{\partial}_i y_j = \dot{\partial}_j y_i, \\ &d)\ \ g_{ij} y^j = \frac{1}{2}\dot{\partial}_i F^2 = F\dot{\partial}_i F \ \text{ and} \\ &e)\ \ \dot{\partial}_j y^i = \delta^i_j. \end{aligned} \tag{1}$$

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The tensor C^*_{ijk} defined by

$$C_{ijk} = \frac{1}{2}\dot{\partial}_i g_{jk} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k F^2 \tag{2}$$

is known as (h)hv-torsion tensor [7]. It is positively homogeneous of degree -1 in the directional arguments and symmetric in all its indices. The (v)hv-torsion tensor C^h_{ik} and its associate (h)hv-torsion tensor C_{ijk} are related by

$$a)\ \ C^h_{ik} := g^{hj}C_{ijk} \ \text{ and } \ b)\ \ C_{ijk} := g_{hj}C^h_{ik}. \tag{3}$$

The (v)hv-torsion tensor C^h_{ik} is also positively homogeneous of degree -1 in the directional arguments and symmetric in its lower indices. E.Cartan deduced the h-covariant derivative for an arbitrary vector filed X^i with respect to x^k given by [15]

$$X^i_{|k} := \partial_k X^i - (\dot{\partial}_r X^i)G^r_k + X^r\Gamma^{*i}_{rk}. \tag{4}$$

The metric tensor g_{ij} and the vector y^i are covariant constant with respect to above process, i.e.

$$a)\ \ g_{ij|k} = 0 \ \text{ and } \ b)\ \ y^i_{|k} = 0. \tag{5}$$

The process of h-covariant differentiation defined above commute with partial differentiation with respect to y^j for arbitrary vector filed X^i, according to

$$\dot{\partial}_j(X^i_{|k}) - (\dot{\partial}_j X^i)_{|k} = X^r(\dot{\partial}_j\Gamma^{*i}_{rk}) - (\dot{\partial}_r X^i)P^r_{jk}, \tag{6}$$

where

$$a)\ \ \dot{\partial}_j\Gamma^{*r}_{hk} = \Gamma^{*r}_{jhk} \ \text{ and } \ b)\ \ P^i_{kh}y^k = 0 = P^i_{kh}y^h. \tag{7}$$

The tensor P^i_{kh} is called v(hv)-torsion tensor and its associate tensor P_{jkh} is given by

$$g_{rj}P^r_{kh} = P_{kjh}. \tag{8}$$

The quantities H^i_{jkh} and H^i_{kh} form the components of tensors and they called h-curvature tensor of Berwald (Berwald curvature tensor) and h(v)-torsion tensor, respectively and defined as follow:

$$a)\ \ H^i_{jkh} := \partial_j G^i_{kh} + G^r_{kh}G^i_{rj} + G^i_{rhj}G^r_k - h/k \ \text{ and } \ b)\ \ H^i_{kh} := \partial_h G^i_k + G^r_k C^i_{rh} - \frac{h}{k}. \tag{9}$$

They are skew-symmetric in their lower indices, i.e. k and h. Also they are positively homogeneous of degree zero and one, respectively in their directional arguments. They are also related by

$$a)\ \ H^i_{jkh}y^j = H^i_{kh}, \ \ b)\ \ H^i_{jkh} = \partial_j H^i_{kh}; \text{ and } \ c)\ \ H^i_{jk} = \partial_j H^i_k. \tag{10}$$

These tensors were constructed initially by mean of the tensor H^i_h, called the deviation tensor, given by

$$H^i_h := 2\partial_h G^i - \partial_r G^i_h y^r + 2G^i_{hs}G^s - G^i_s G^s_h. \tag{11}$$

The deviation tensor H_h^i is positively homogeneous of degree two in the directional arguments. In view of Euler's theorem on homogeneous functions and by contracting the indices i and h in (10) and (11), we have the following:

$$\begin{aligned}
&a)\ \ H^i_{jk}y^j = -H^i_{kj}y^j = H^i_k,\\
&b)\ \ H_{jk} := H^r_{jkr},\\
&c)\ \ H_j := H^r_{jr},\\
&d)\ \ H^r_{rkh} = H_{hk} - H_{kh},\\
&e)\ \ H := \frac{1}{n-1}H^r_r \ \text{ and}\\
&f)\ \ y_i H^i_j = 0.
\end{aligned} \tag{12}$$

The contracted tensor H_{kh} (Ricci tensor), H_k (Curvature vector) and H (curvature scalar) are also connected by

$$a)\ \ H_{kh} = \dot{\partial}_k H_h,\ \ b)\ \ H_{kh}y^k = H_h \ \text{ and }\ c)\ \ H_k y^k = (n-1)H. \tag{13}$$

The quantities H^i_{jkh} and H^i_{kh} are satisfies the following [15]:

$$a)\ \ H_{ijkh} := g_{jr}H^r_{ihk},\ \ b)\ \ H_{jk.h} := g_{jr}H^r_{hk} \ \text{ and }\ c)\ \ H^i_{jkh} + H^i_{hjk} + H^i_{khj} = 0. \tag{14}$$

P.N. Pandey proved [9]

$$y_i H^i_{hk} = 0. \tag{15}$$

Cartan's third curvature tensor R^i_{jkh} satisfies the identity known as Bianchi identity [15]

$$\begin{aligned}
&a)\ \ R^i_{jkh|s} + R^i_{jsk|h} + R^i_{jhs|k} + (R^r_{mhs}P^i_{jkr} + R^r_{mkh}P^i_{jsr} + R^r_{msk}P^i_{jhr})y^m = 0,\\
&b)\ \ R^i_{jkh}y^j = H^i_{kh} = K^i_{jkh}y^j,\ \ c)\ \ R_{ijhk} = g_{rj}R^r_{ihk} \ \text{ and}\\
&d)\ \ R_{jkhm}y^j = H_{kh.m}.
\end{aligned} \tag{16}$$

Also this tensor satisfies the following relation too

$$a)\ \ R^i_{jkh} = K^i_{jkh} + C^i_{js}K^s_{rhk}y^r \ \text{ and }\ b)\ \ R_{ijkh} = K_{ijkh} + C_{ijs}H^s_{kh}, \tag{17}$$

where R_{ijkh} is the associate curvature tensor of R^i_{jkh}. Cartan's fourth curvature tensor K^i_{jkh} and its associate curvature tensor K_{ijkh} satisfy the following identities known as Bianchi identities

$$a)\ \ K^i_{jkh} + K^i_{hjk} + K^i_{khj} = 0 \ \text{ and }\ b)\ \ K_{jrkh} + K_{hrjk} + K_{krhj} = 0. \tag{18}$$

2. An Generalized H^h-Recurrent Space

Let us consider a Finsler space F_n whose Berwald curvature tensor H^i_{jkh}satisfies the condition

$$H^i_{jkh?l} = \lambda_l H^i_{jkh} + \mu_l(\delta^i_h g_{jk} - \delta^i_k g_{jh}),\ \ H^i_{jkh} \neq 0, \tag{19}$$

where λ_l and μ_l are non-null covariant vectors field. We shall call such space as a generalized H^h- recurrent space. We shall denote it briefly by GH^h-RF_n. Now, let us consider a generalized H^h- recurrent space characterized by the condition (19). Transvecting the condition (19) by y^j, using (5b), (10a) and (1a), we get

$$H^i_{kh|l} = \lambda_l H^i_{kh} + \mu_l(\delta^i_h y_k - \delta^i_k y_h). \tag{20}$$

Further, transvecting the condition (20) by y^k, using (5b), (12a) and (1b), we get

$$H^i_{h|l} = \lambda_l H^i_h + \mu_l(\delta^i_h F^2 - y_h y^i). \tag{21}$$

Trausvecting the condition (20) by g_{ip}, using (5a) and (14b), we get

$$H_{kp.h|l} = \lambda_l H_{kp.h} + \mu_l(g_{hp} y_k - g_{kp} y_h). \tag{22}$$

Therefore, we have

Theorem 2.1. *In GH^h-RF_n, the h-covariant derivative of the h(v)-torsion tensor H^i_{kh}, the deviation tensor H^i_h and the tensor $H_{kp.h}$ is given by the conditions (20),(21) and (22), respectively.*

Differentiating (20) partially with respect to y^j, using (10b), (7a), (1c) and using the commutation formula exhibited by (6) for the h(v)-torsion tensor H^i_{jk}, we get

$$H^i_{jkh|l} + H^r_{kh}\Gamma^i_{jrl} - H^i_{rh}\Gamma^r_{jkl} - H^i_{kr}\Gamma^r_{jhl} - H^i_{rkh}P^r_{jl} = (\dot\partial_j\lambda_l)H^i_{kh} + \lambda_l H^i_{jkh} + (\dot\partial_j\mu_l)(\delta^i_h y_k - \delta^i_k y_h) + \mu_l(\delta^i_h g_{jk} - \delta^i_k g_{jh}). \tag{23}$$

This shows that $H^i_{jkh|l} = \lambda_l H^i_{jkh} + \mu_l(\delta^i_h g_{jk} - \delta^i_k g_{jh})$ if and only if

$$H^r_{kh}\Gamma^i_{jrl} - H^i_{rh}\Gamma^r_{jkl} - H^i_{kr}\Gamma^r_{jhl} - H^i_{rkh}P^r_{jl} = (\dot\partial_j\lambda_l)H^i_{kh} + (\dot\partial_j\mu_l)(\delta^i_h y_k - \delta^i_k y_h). \tag{24}$$

Thus, we conclude

Theorem 2.2. *In GH^h-RF_n, Berwald curvature tensor H^i_{jkh} is generalized recurrent if and only if (24) holds good.*

Transvecting (23) by g_{ip}, using (7a) and (14a), we get

$$\begin{aligned} H_{jpkh|l} + g_{ip}(H^r_{kh}\Gamma^i_{jrl} - H^i_{rh}\Gamma^r_{jkl} - H^i_{kr}\Gamma^r_{jhl} - H^i_{rkh}P^r_{jl}) = \lambda_l H_{jpkh} + g_{ip}[(\dot\partial_j\lambda_l)H^i_{kh} + (\dot\partial_j\mu_l)(\delta^i_h y_k - \delta^i_k y_h)] \\ + \mu_l(g_{hp}g_{jk} - g_{kp}g_{jh}). \end{aligned} \tag{25}$$

This shows that $H_{jpkh?l} = \lambda_l H_{jpkh} + \mu_l(g_{jk}g_{hp} - g_{jh}g_{kp})$ if and only if

$$g_{ip}(H^r_{kh}\Gamma^i_{jrl} - H^i_{rh}\Gamma^r_{jkl} - H^i_{kr}\Gamma^r_{jhl} - H^i_{rkh}P^r_{jl}) = g_{ip}[(\dot\partial_j\lambda_l)H^i_{kh} + (\dot\partial_j\mu_l)(\delta^i_h y_k - \delta^i_k y_h)]. \tag{26}$$

Thus, we conclude

Theorem 2.3. *In GH^h-RF_n, the associate tensor H_{jpkh} of Berwald curvature tensor H^i_{jkh} is generalized recurrent if and only if (26) holds good.*

Contracting the indices i and h in the condition (23), using (12b) and (12c), we get

$$H_{jk|l} + H^r_{kp}\Gamma^p_{jrl} - H_r\Gamma^r_{jkl} - H^p_{kr}\Gamma^r_{jpl} - H_{rk}P^r_{jl} = \lambda_l H_{jk} + (\dot{\partial}_j\lambda_l)H_k + (n-1)y_k(\dot{\partial}_j\mu_l) + (n-1)\mu_l g_{jk}. \tag{27}$$

This shows that $H_{jk|l} = \lambda_l H_{jk} + (n-1)\mu_l g_{jk}$ if and only if

$$H^r_{kp}\Gamma^p_{jrl} - H_r\Gamma^r_{jkl} - H^p_{kr}\Gamma^r_{jpl} - H_{rk}P^r_{jl} = (\dot{\partial}_j\lambda_l)H_k + (n-1)y_k(\dot{\partial}_j\mu_l). \tag{28}$$

Thus, we conclude

Theorem 2.4. *In GH^h-RF_n, the H-Ricci tensor H_{jk} is non-vanishing if and only if (28) holds good. Contracting the indices i and j in (23) and using (12d), we get*

$$(H_{hk} - H_{kh})_{|l} + H^r_{kh}\Gamma^p_{prl} - H^p_{rh}\Gamma^r_{pkl} - H^p_{kr}\Gamma^r_{phl} - H^p_{rkh}P^r_{pl} = (\dot{\partial}_p\lambda_l)H^p_{kh} + \lambda_l(H_{hk} - H_{kh}) + (\dot{\partial}_p\mu_l)(\delta^p_h y_k - \delta^p_k y_h). \tag{29}$$

This shows that $(H_{hk} - H_{kh})_{|l} = \lambda_l(H_{hk} - H_{kh})$ if and only if

$$H^r_{kh}\Gamma^p_{prl} - H^p_{rh}\Gamma^r_{pkl} - H^p_{kr}\Gamma^r_{phl} - H^p_{rkh}P^r_{pl} = (\dot{\partial}_p\lambda_l)H^p_{kh} + (\dot{\partial}_p\mu_l)(\delta^p_h y_k - \delta^p_k y_h). \tag{30}$$

Thus, we conclude

Theorem 2.5. *In GH^h-RF_n, the tensor $(H_{hk} - H_{kh})$ behaves as recurrent if and only if (30) holds good .*

Differentiating the condition (21) partially with respect to y^k , using (11c), (1d), (1a), (1c), (1e) and using the commutation formula exhibited by (6) for the h(v) deviation tensor H^i_h, we get

$$H^i_{kh|l} + H^r_h\Gamma^i_{krl} - H^i_r\Gamma^r_{khl} - H^i_{rh}P^r_{kl} = (\dot{\partial}_k\lambda_l)H^i_h + \lambda_l H^i_{kh} + (\dot{\partial}_k\mu_l)(\delta^i_h F^2 - y_h y^i) + \mu_l(2\delta^i_h y_k - g_{kh}y^i - \delta^i_k y_h). \tag{31}$$

The interchange of the indices k and h in (31), the subtraction of the equation thus obtained from (31) and by using (12c), we get

$$\begin{aligned}(\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k)_{|l} + (H^r_h\Gamma^i_{krl} - H^i_r\Gamma^r_{khl} - H^i_{rh}P^r_{kl}\frac{-k}{h}) &= \lambda_l(\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k) + 3\mu_l(\delta^i_h y_k - \delta^i_k y_h)\\ &+ [(\dot{\partial}_k\lambda_l)H^i_h + (\dot{\partial}_k\mu_l)(\delta^i_h F^2 - y_h y^i)\frac{-k}{h}].\end{aligned} \tag{32}$$

This shows that $(\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k)_{|l} = \lambda_l(\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k)$ if and only if

$$\begin{aligned}H^r_h\Gamma^i_{krl} - H^i_r\Gamma^r_{khl} - H^i_{rh}P^r_{kl}\frac{-k}{h} &= 3\mu_l(\delta^i_h y_k - \delta^i_k y_h)\\ &+ [(\dot{\partial}_k\lambda_l)H^i_h(+(\dot{\partial}_k\mu_l)(\delta^i_h F^2 - y_h y^i)\frac{-k}{h}].\end{aligned} \tag{33}$$

Thus, we conclude

Theorem 2.6. *In GH^h-RF_n, the tensor $(\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k)$ behaves as recurrent if and only if (33) holds good.*

3. An Generalized H^h-Recurrent Affinely Connected Space

A Finsler space F_n whose connection parameter G^i_{jk} is independent of y^i is called an affinely connected space (Berwald space). Thus, an affinely connected space is characterized by any one of the following equivalent conditions

$$a)\ \ G^i_{jkh} = 0 \ \text{ and } \ b)\ \ C_{ijk|h} = 0, \tag{34}$$

the connection parameters Γ^i_{kh} of Cartan and G^i_{kh} of Berwald coincides in affinely connected space and they are independent of directional arguments [**?**], i.e.

$$a)\ \ \dot{\partial}_j G^i_{kh} = 0 \ \text{ and } \ b)\ \ \dot{\partial}_j \Gamma^{*i}_{kh} = 0. \tag{35}$$

Definition 3.1. *If the generalized H^h-recurrent space is affinely connected space [satisfies any one of the conditions (34a), (34b), (35a) and (35b)] we called it a generalized H^h-recurrent affinely connected space and denoted briefly by GH^h-RF_n affinely connected space.*

Let us consider a $GH^h - R$ affinely connected space, if $\dot{\partial}_j \lambda_l = 0$ and $\dot{\partial}_j \mu_l = 0$, so in view of the condition (35b) and (7a), the condition (23) reduces to

$$H^i_{jkh?l} = \lambda_l H^i_{jkh} + H^i_{rkh} P^r_{jl} + \mu_l(\delta^i_h g_{jk} - \delta^i_k g_{jh}). \tag{36}$$

This shows that $H^i_{jkh|l} = \lambda_l H^i_{jkh} + \mu_l(\delta^i_h g_{jk} - \delta^i_k g_{jh})$ if and only if $H^i_{rkh} P^r_{jl} = 0$. Thus, we conclude

Theorem 3.2. *In $GH^h - R$ affinely connected space, if the directional derivative of covariant vectors field of one order vanish, then Berwald curvature tensor H^i_{jkh} is generalized recurrent if and only if $H^i_{rkh} P^r_{jl} = 0$.*

Consider a GH^h-R affinely connected space, if $\dot{\partial}_j \lambda_l = 0$ and $\dot{\partial}_j \mu_l = 0$, so in view of the condition (35b), (7a) and (14a), equation (25) reduces to $H_{jpkh|l} = \lambda_l H_{jpkh} + H_{rpkh} P^r_{jl} + \mu_l(g_{jk} g_{hp} - g_{jh} g_{kp})$. This shows that $H_{jpkh|l} = \lambda_l H_{jpkh} + \mu_l(g_{jk} g_{hp} - g_{jh} g_{kp})$ if and only if $H_{rpkh} P^r_{jl} = 0$. Thus, we conclude

Theorem 3.3. *In $GH^h - R$ affinely connected space, if the directional derivative of covariant vectors field of one order vanish, then Berwald associate curvature tensor H_{jpkh} is generalized recurrent if and only if $H_{rpkh} P^r_{jl} = 0$.*

Consider a GH^h-R affinely connected space, if $\dot{\partial}_j \lambda_l = 0$ and $\dot{\partial}_j \mu_l = 0$, so in view of the condition (35b) and (7a), the equation (27) reduces to $H_{jk|l} = \lambda_l H_{jk} + H_{rk} P^r_{jl} + (n-1)\mu_l g_{jk}$. This shows that $H_{jk|l} = \lambda_l H_{jk} + (n-1)\mu_l g_{jk}$ if and only if $H_{rk} P^r_{jl} = 0$. Thus, we conclude

Theorem 3.4. *In GH^h-R affinely connected space, if the directional derivative of covariant vectors field of one order vanish, then H-Ricci tensor H_{jk} is non-vanishing if and only if $H_{rk} P^r_{jl} = 0$.*

Consider a GH^h-R affinely connected space, if $\dot{\partial}_j \lambda_l = 0$ and $\dot{\partial}_j \mu_l = 0$, so in view of the condition (35b) and (7a), the equation (29) reduces to $(H_{hk} - H_{kh})_{?l} = \lambda_l(H_{hk} - H_{kh}) + H^p_{rkh} P^r_{pl}$. This shows that $(H_{hk} - H_{kh})_{|l} = \lambda_l(H_{hk} - H_{kh})$ if and only if $H^p_{rkh} P^r_{pl} = 0$.
Thus, we conclude

Theorem 3.5. *In GH^h-R affinely connected space, if the directional derivative of covariant vectors field of one order vanish, then the tensor $(H_{hk} - H_{kh})$ behaves as recurrent if and only if $H^p_{rkh} P^r_{pl} = 0$.*

Consider a GH^h-R affinely connected space, if $\dot{\partial}_j \lambda_l = 0$ and $\dot{\partial}_j \mu_l = 0$, so in view of the condition (35b) and (7a), the equation (31) reduces to

$$H^i_{kh|l} = \lambda_l H^i_{kh} + H^i_{rh} P^r_{kl} + \mu_l (2\delta^i_h y_k - g_{kh} y^i - \delta^i_k y_h). \tag{37}$$

This shows that

$$H^i_{kh|l} = \lambda_l H^i_{kh} + \mu_l (2\delta^i_h y_k - g_{kh} y^i - \delta^i_k y_h) \tag{38}$$

if and only if $H^i_{rh} P^r_{kl} = 0$.

Thus, we conclude

Theorem 3.6. *In GH^h-R affinely connected space, if the directional derivative of covariant vectors field of one order vanish, then the h-covariant derivative of the h(v)-torsion tensor H^i_{kh} is given by the condition (37) if and only if $H^i_{rh} P^r_{kl} = 0$.*

Transvecting (14) by y^k, using (5b), (12a), (7b), (1b) and (1a), we get

$$H^i_{h|l} = \lambda_l H^i_h + 2\mu_l (\delta^i_h F^2 - y_h y^i). \tag{39}$$

Thus, we conclude

Theorem 3.7. *In GH^h-R affinely connected space, if the directional derivative of covariant vectors field of one order vanish, then the h-covariant derivative of the deviation H^i_h is given by the condition (3.5). Contracting the indices i and h in equation (3.5), using (12e) and (1b), we get*

$$H_{|l} = \lambda_l H + 2\mu_l F^2. \tag{40}$$

Thus, we conclude

Theorem 3.8. *In GH^h-R affinely connected space, if the directional derivative of covariant vectors field of one order vanish, then the covariant scalar H is non-vanishing.*

Trausvecting (3.3) by g_{ip}, using(5a), (14b) and (1a), we get

$$H_{kp.h|l} = \lambda_l H_{kp.h} + H_{rp.h} P^r_{kl} + \mu_l (2g_{hp} y_k - g_{kh} y_p - g_{kp} y_h). \tag{41}$$

This shows that

$$H_{kp.h|l} = \lambda_l H_{kp.h} + \mu_l (2g_{hp} y_k - g_{kh} y_p - g_{kp} y_h) \tag{42}$$

if and only if $H_{rp.h} P^r_{kl} = 0$.

Thus, we conclude

Theorem 3.9. *In GH^h-R affinely connected space, if the directional derivative of covariant vectors field of one order vanish, then the h-covariant derivative of the associate tensor $H_{kp.h}$ of the h(v)-torsion tensor H^i_{kh} is given by the condition (3.8) if and only if $H_{rp.h} P^r_{kl} = 0$.*

Consider a GH^h-R affinely connected space, if $\dot{\partial}_j \lambda_l = 0$ and $\dot{\partial}_j \mu_l = 0$, so in view of the condition (3.2b) and (7a), the equation (2.14) reduces to

$$(\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k)_{|l} = \lambda_l (\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k) + 3\mu_l (\delta^i_h y_k - \delta^i_k y_h) + H^i_{rh} P^r_{kl} - H^i_{rk} P^r_{hl}.$$

This shows that

$$(\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k)_{|l} = \lambda_l (\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k)$$

if and only if

$$3\mu_l(\delta^i_h y_k - \delta^i_k y_h) + H^i_{rh} P^r_{kl} - H^i_{rk} P^r_{hl} = 0. \tag{43}$$

Thus, we conclude

Theorem 3.10. *In GH^h-R affinely connected space, if the directional derivative of covariant vectors field of one order vanish, then the tensor $(\dot{\partial}_k H^i_h - \dot{\partial}_h H^i_k)$ behaves as recurrent if and only if (3.9) holds.*

4. P2-like Generalized H^h-Recurrent Space

A P2-like space [4] is characterized by

$$P^i_{jkh} = \emptyset_j C^i_{kh} - \emptyset^i C_{jkh}, \tag{44}$$

where $\emptyset_j$ and $\emptyset^i$ are non-zero covariant and contravariant vectors field, respectively.

Definition 4.1. *If the generalized H^h-recurrent space is a P2-like space satisfies the condition (4.1) we called it a P2-like generalized H^h-recurrent space and denoted briefly by P2-like $GH^h - RF_n$.*

Let us consider a P2-like $GH^h - RF_n$, then necessarily we have the condition (4.1). Putting the condition (4.1) in the identity (16a) and using (16b), we get

$$R^i_{jkh|s} + R^i_{jsk?h} + R^i_{jhs|k} + \emptyset_j(H^r_{hs} C^i_{kr} + H^r_{kh} C^i_{sr} + H^r_{sk} C^i_{hr}) - \emptyset^i(H^r_{hs} C_{jkr} + H^r_{kh} C_{jsr} + H^r_{sk} C_{jhr}) = 0. \tag{45}$$

Using (17a), (18a), (17b) and (18b) in (4.2), we get

$$R^i_{jkh|s} + R^i_{jsk?h} + R^i_{jhs|k} + \emptyset_j(R^i_{hsk} + R^i_{khs} + R^i_{skh}) - \emptyset^i(R_{jskh} + R_{jhsk} + R_{jkhs}) = 0.$$

Transvecting the above equation by g_{ip}, using (5a) and (16c), we get

$$R_{jpkh|s} + R_{jpsk?h} + R_{jphs|k} + \emptyset_j(R_{hpsk} + R_{kphs} + R_{spkh}) - \emptyset_p(R_{jskh} + R_{jhsk} + R_{jkhs}) = 0, \tag{46}$$

where $g_{ip}\emptyset^i = \emptyset_p$. Transvecting (4.3) by y^j , using (5b) and (16d), we get

$$H_{pk.h|s} + H_{ps.k|h} + H_{ph.s?k} + \emptyset(R_{hpsk} + R_{kphs} + R_{spkh}) - \emptyset_p(H_{sk.h} + H_{hs.k} + H_{kh.s}) = 0, \tag{47}$$

where $\emptyset_j y^j = \emptyset$. Now, differentiating (15) partially with respect to y^j, using (1c) and (10b), we get

$$g_{ij} H^i_{hk} + y_i H^i_{jhk} = 0. \tag{48}$$

Taking skew-symmetric part of (4.5) with respect to the indices j, k and h, using (14b) and (14c), we get

$$H_{kh.j} + H_{jk.h} + H_{hj.k} = 0. \tag{49}$$

putting equation (4.6) in (4.4), we get

$$H_{pk.h|s} + H_{ps.k|h} + H_{ph.s?k} + \emptyset(R_{hpsk} + R_{kphs} + R_{spkh}) = 0. \tag{50}$$

Using the condition (22) in (4.7), we get

$$\lambda_s H_{pk.h} + \lambda_h H_{ps.k} + \lambda_k H_{ph.s} + \mu_s(g_{hk}y_p - g_{pk}y_h) + \mu_h(g_{ks}y_p - g_{ps}y_k) + \mu_k(g_{sh}y_p - g_{ph}y_s) + \emptyset(R_{hpsk} + R_{kphs} + R_{spkh}) = 0. \tag{51}$$

Thus, we conclude

Theorem 4.2. *In P2-like $GH^h - RF_n$, we have the identities (4.7) and (4.8) holds good.*

5. P^*-Generalized H^h-Recurrent Space

A P^*-Finsler space is characterized by the condition ([4, 5])

$$P^i_{kh} = \emptyset C^i_{kh}, \;\; \emptyset \neq 0, \;\; \text{where} \;\; P^i_{jkh}y^j = P^i_{kh} = C^i_{kh|j}y^j. \tag{52}$$

H. Izumi [14] denoted $\emptyset$ by λ.

Definition 5.1. *If the generalized H^h-recurrent space is P^*- space [satisfies the condition (5.1)] we called it P^*-generalized recurrent space and denoted briefly by $P^* - GH^h - RF_n$.*

Now, taking h-covariant derivative of (5.1) covariantly with respect to x^l, we get

$$P^i_{kh|l} = \emptyset_{|l} C^i_{kh} + \emptyset C^i_{kh|l}. \tag{53}$$

If the (v) hv-torsion tensor C^i_{kh} is recurrent, i.e. $C^i_{kh|l} = b_l C^i_{kh}$, then (5.2) can be written as

$$P^i_{kh|l} = \emptyset_{|l} C^i_{kh} + b_l \emptyset C^i_{kh}. \tag{54}$$

Putting equation (5.1) in (5.3), we get $P^i_{kh|l} = \emptyset_{|l} C^i_{kh} + b_l P^i_{kh}$ which implies $P^i_{kh|l} = b_l P^i_{kh}$ if and only if $\emptyset_{|l} C^i_{kh} = 0$. Thus, we conclude

Theorem 5.2. *In $P^* - GH^h - RF_n$, the v(hv)-torsion tensor P^i_{kh} behaves as recurrent if and only if $\emptyset_{|l} C^i_{kh} = 0$ [provided the (v)hv-torsion tensor C^i_{kh} behaves as recurrent].*

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