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Some Unified and Generalized Kummer's Second Summation Theorems with Applications in Laplace Transform Technique

Research Article

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Abstract: Some significant hypergeometric summation theorems with suitable convergence conditions, are obtained in the present study; analogous to summation theorems for Gauss function ${}_2F_1(\frac{1}{2})$ presented by Brychkov, Prudnikov et al. and derived by Fox, Rakha-Rathie. By means of these summation theorems we also find the Laplace transforms of Kummer's confluent hypergeometric function ${}_1F_1$ in closed form.

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1. Introduction, Definitions and Preliminaries

In the usual notation, let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. Also let

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\} \quad , \qquad \mathbb{N} = \{1, 2, 3, \dots\} = \mathbb{N}_0 \backslash \{0\} \quad ,$$

$$\mathbb{Z}_0^- = \{0, -1, -2, \dots\} = \mathbb{Z}^- \cup \{0\} \quad , \qquad \mathbb{Z}^- = \{-1, -2, -3, \dots\}$$

and $\mathbb{Z} = \mathbb{Z}_0^- \cup \mathbb{N}$ being the sets of integers. Here, in our present investigation, we propose to explore several summation and other related formulas for the Gauss and Kummer hypergeometric functions which are, respectively, in the cases

$$p - 1 = q = 1 \quad \text{and} \quad p = q = 1.$$

Here generalized hypergeometric function ${}_pF_q$ with p numerator parameters $\alpha_1, \alpha_2, \dots, \alpha_p$ and q denominator parameters $\beta_1, \beta_2, \dots, \beta_q$ is defined by (see, for example, [14, 15]):

$${}_pF_q\left[\begin{array}{c}\alpha_1, \alpha_2, \dots, \alpha_p; \\ \\ \beta_1, \beta_2, \dots, \beta_q;\end{array} z\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\alpha_j)_n}{\prod_{j=1}^{q}(\beta_j)_n} \frac{z^n}{n!}$$

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$\Big(p, q \in \mathbb{N}_0;\ p \leqq q+1\ ;\ p \leqq q \text{ and } |z| < \infty\ ;\ p = q+1 \text{ and } |z| < 1; p = q+1, |z| = 1 \text{ and } \Re(\omega) > 0\ ;\ p = q+1, |z| = 1, z \neq 1 \text{ and } 0 \geq \Re(\omega) > -1\Big)$, where

$$\omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j \Big(\alpha_j \in \mathbb{C}\ (j = 1, 2, \ldots, p);\ \beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-\,(j = 1, 2, \ldots, q)\Big).$$

In terms of Gamma function $\Gamma(z)$, the widely-used Pochhammer symbol $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ is defined, in general, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}), \end{cases} \tag{1}$$

it is being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ quotient exists (see, for details, [15]).

$$\int_0^\infty e^{-st} t^{\alpha-1} \mathrm{d}t = \frac{\Gamma(\alpha)}{s^\alpha} \tag{2}$$

$$\Big(\Re(s) > 0,\ 0 < \Re(\alpha) < \infty \qquad \text{or} \qquad \Re(s) = 0,\ 0 < \Re(\alpha) < 1\Big).$$

Euler's Beta-type integral representation for the Gauss hypergeometric function ${}_2F_1$ is given by [14]

$${}_2F_1\left[\begin{matrix}\alpha, \beta; \\ \gamma;\end{matrix}\ z\right] = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha} \mathrm{d}t \tag{3}$$

$$\Big(|\arg(1-z)| \leqq \pi - \epsilon\ (0 < \epsilon < \pi); \Re(\gamma) > \Re(\beta) > 0; \alpha \in \mathbb{C}\Big),$$

or, equivalently [14],

$${}_2F_1\left[\begin{matrix}\alpha, \beta; \\ \gamma;\end{matrix}\ z\right] = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-zt)^{-\beta} \mathrm{d}t \tag{4}$$

$$\Big(|\arg(1-z)| \leqq \pi - \epsilon\ (0 < \epsilon < \pi); \Re(\gamma) > \Re(\alpha) > 0; \beta \in \mathbb{C}\Big),$$

the Pfaff-Kummer hypergeometric transformation [14]

$${}_2F_1\left[\begin{matrix}\alpha, \beta; \\ \gamma;\end{matrix}\ z\right] = (1-z)^{-\alpha}{}_2F_1\left[\begin{matrix}\alpha, \gamma-\beta; \\ \gamma;\end{matrix}\ \frac{-z}{1-z}\right] \Big(|\arg(1-z)| \leqq \pi - \epsilon\ (0 < \epsilon < \pi);\ \gamma \in \mathbb{C}\backslash\mathbb{Z}_0^-\Big) \tag{5}$$

The familiar Beta function $B(\alpha, \beta)$ [14] is defined by

$$B(\alpha, \beta) = \begin{cases} \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \mathrm{d}t & (\min\{\Re(\alpha), \Re(\beta)\} > 0) \\ \displaystyle\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha, \beta \in \mathbb{C}\backslash\mathbb{Z}_0^-), \end{cases} \tag{6}$$

and Legendre's duplication formula [15] is given by

$$\sqrt{(\pi)}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\Big(z + \frac{1}{2}\Big) \qquad (2z \in \mathbb{C}\backslash\mathbb{Z}_0^-). \tag{7}$$

In addition to the classical Gauss summation theorem [15] for ${}_2F_1(1)$, there are numerous closed-form summation theorems for ${}_2F_1(z)$ when the argument z takes on some special numerical values [1–3, 11, 13]. Here, for the purpose of our present investigation, we choose to recall the following summation theorem, which is due to Kummer [9].

Kummer's second summation theorem [6]

$$ {}_2F_1\left[\begin{matrix} a, \quad b; \\ \frac{1+a+b}{2}; \end{matrix} \ \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1+a+b}{2})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})} \quad ; \qquad \left(\frac{1+a+b}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right) \tag{8} $$

By citing Gauss's publication dated 1866 [8] (and, at the same time, not at all including a reference to the obviously more relevant second part of Kummer's publication dated 1836 [9]), Kummer's summation Theorem (1.8) was incorrectly attributed to Gauss by Bailey [3], subsequently, by naturally following her teacher Bailey's footsteps, Kummer's summation Theorems (1.8) was (obviously incorrectly) called Gauss's second theorem, by Slater [13] (see also [5] for some of these historical corrections).

2. Known Results Analogous to Kummer's Second Summation Theorem (1.8)

(a): In the 1927, a summation theorem was given by Fox [7] with the help of Mellin-Barnes type contour integral, in the following form

$$ {}_2F_1\left[\begin{matrix} a+m, b; \\ \frac{a+b+1}{2}; \end{matrix} \ \frac{1}{2}\right] = \frac{2^{b-1}\Gamma(\frac{a+b+1}{2})}{\Gamma(b)} \sum_{r=0}^{m} \binom{m}{r} \frac{\Gamma(\frac{b+r}{2})}{\Gamma(\frac{a+r+1}{2})} \left(b, \frac{1+a+b}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-\right). \tag{9} $$

Deductions of summation theorem (2.1): If we replace a by $a-m$ in Fox summation Theorem (2.1), we get the following summation Theorem (2.2) recorded by Prudnikov et al. [10] in the year 1990.

$$ {}_2F_1\left[\begin{matrix} a, \quad b; \\ \frac{1+a+b-m}{2}; \end{matrix} \ \frac{1}{2}\right] = \frac{2^{b-1}\Gamma(\frac{1+a+b-m}{2})}{\Gamma(b)} \sum_{r=0}^{m} \binom{m}{r} \frac{\Gamma(\frac{r+b}{2})}{\Gamma(\frac{1+a+r-m}{2})} \left(b, \frac{1+a+b-m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-; \quad m \in \mathbb{N}_0\right). \tag{10} $$

Further more if we interchange the parameters a and b in summation Theorem (2.2), then replacing a by $a+m$ in the resulting equation, we get the following summation theorem which is equivalent to Fox summation Theorem (2.1).

$$ {}_2F_1\left[\begin{matrix} a+m, \quad b; \\ \frac{1+a+b}{2}; \end{matrix} \ \frac{1}{2}\right] = \frac{2^{a+m-1}\Gamma(\frac{1+a+b}{2})}{\Gamma(a+m)} \sum_{r=0}^{m} \binom{m}{r} \frac{\Gamma(\frac{a+r+m}{2})}{\Gamma(\frac{1+b+r-m}{2})} \left(a+m, \frac{1+a+b}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-; \quad m \in \mathbb{N}_0\right). \tag{11} $$

(b): In the year 2008, a summation theorem was presented by Brychkov [4] in the following form

$$ {}_2F_1\left[\begin{matrix} a, \quad b; \\ \frac{a+b+m}{2}; \end{matrix} \ \frac{1}{2}\right] \stackrel{\circ}{=} \frac{\sqrt{(\pi)}\Gamma(\frac{a+b+m}{2})}{(\frac{-a+b-m}{2})_m} \times \sum_{r=0}^{m} \left\{ \binom{m}{r} (-2)^{-r} (a)_r \left[\frac{1}{\Gamma(\frac{a+r+1}{2})\Gamma(\frac{b+r-m}{2})} + \frac{1}{\Gamma(\frac{a+r}{2})\Gamma(\frac{b+r-m}{2}+1)} \right] \right\}. \tag{12} $$

The symbol $\stackrel{\circ}{=}$ exhibits the fact that the Equation (2.4) does not hold true as stated. In this connection see our summation Theorem (3.4).

(c): In the year 2011, a summation theorem was given by Rakha-Rathie [12] in the following form

$$ {}_2F_1\left[\begin{matrix} a, \quad b; \\ \frac{1+a+b+m}{2}; \end{matrix} \ \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1+a+b+m}{2})\Gamma(\frac{b-a-m+1}{2})}{\Gamma(\frac{a}{2})\Gamma(\frac{a+1}{2})\Gamma(\frac{1-a+b+m}{2})} \sum_{r=0}^{m} \binom{m}{r} \frac{(-1)^r\Gamma(\frac{r+a}{2})}{\Gamma(\frac{1+b+r-m}{2})} \tag{13} $$

$$ \left(a, \frac{1+a+b+m}{2}, \frac{b-a-m+1}{2}, \frac{1-a+b+m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-; \quad m \in \mathbb{N}_0\right). $$

Deductions of summation Theorem (2.5): If we replaces a by $a-m$ in summation Theorem (2.5), and apply Legendre's duplication formula, we get the following summation theorem which is a companion of Fox summation Theorem (2.1).

$$ {}_2F_1\left[\begin{matrix} a-m, & b; & \frac{1}{2} \\ \frac{1+a+b}{2}; & & \end{matrix}\right] = \frac{2^{a-m-1}\Gamma(\frac{1+a+b}{2})\Gamma(\frac{1-a+b}{2})}{\Gamma(a-m)\Gamma(\frac{1-a+b+2m}{2})}\sum_{r=0}^{m}\binom{m}{r}\frac{(-1)^r\Gamma(\frac{a-m+r}{2})}{\Gamma(\frac{1+b-m+r}{2})} \tag{14}$$

$$\left(a-m, \tfrac{1+a+b}{2}, \tfrac{1-a+b+2m}{2}, \tfrac{1-a+b}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-; \quad m \in \mathbb{N}_0\right).$$

Further if we interchange the parameters a and b in summation Theorem (2.6); replace b by $b+m$ and a by $a-m$ in resulting equation, we get the following summation theorem which is another companion of Fox summation Theorem (2.1).

$$ {}_2F_1\left[\begin{matrix} a-m, & b; & \frac{1}{2} \\ \frac{1+a+b}{2}; & & \end{matrix}\right] = \frac{2^{b-1}\Gamma(\frac{1+a+b}{2})\Gamma(\frac{1+a-b-2m}{2})}{\Gamma(b)\Gamma(\frac{1+a-b}{2})}\sum_{r=0}^{m}\binom{m}{r}\frac{(-1)^r\Gamma(\frac{b+r}{2})}{\Gamma(\frac{1+a-2m+r}{2})} \tag{15}$$

$$\left(b, \tfrac{1+a+b}{2}, \tfrac{1+a-b-2m}{2}, \tfrac{1+a-b}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-; \quad m \in \mathbb{N}_0\right).$$

3. New Results Analogous to Kummer's Second Summation Theorem (1.8)

Any values of parameters and variables leading to the result which do not make sense, are tacitly excluded, then

$$ {}_2F_1\left[\begin{matrix} a, & b; & \frac{1}{2} \\ \frac{a+b-m}{2}; & & \end{matrix}\right] = \frac{2^{a-1}\Gamma(\frac{a+b-m}{2})}{\Gamma(a)}\sum_{r=0}^{m}\left\{\binom{m}{r}\left[\frac{\Gamma(\frac{r+a}{2})}{\Gamma(\frac{b+r-m}{2})}+\frac{\Gamma(\frac{r+a+1}{2})}{\Gamma(\frac{b+r-m+1}{2})}\right]\right\}\left(a, \frac{a+b-m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-; \quad m \in \mathbb{N}_0\right) \tag{16}$$

or equivalently,

$$ {}_2F_1\left[\begin{matrix} a, & b; & \frac{1}{2} \\ \frac{a+b-m}{2}; & & \end{matrix}\right] = \sqrt{\pi}\Gamma(\frac{a+b-m}{2})\times$$

$$\times\sum_{r=0}^{m}\left\{\binom{m}{r}(a)_r 2^{-r}\left[\frac{1}{\Gamma(\frac{r+a+1}{2})\Gamma(\frac{b+r-m}{2})}+\frac{1}{\Gamma(\frac{r+a}{2})\Gamma(\frac{b+r-m+1}{2})}\right]\right\}\left(a, \frac{a+b-m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-; \quad m \in \mathbb{N}_0\right), \tag{17}$$

$$ {}_2F_1\left[\begin{matrix} a, & b; & \frac{1}{2} \\ \frac{a+b+m}{2}; & & \end{matrix}\right] = \frac{2^{a-1}\Gamma(\frac{a+b+m}{2})\Gamma(\frac{b-a-m}{2})}{\Gamma(a)\Gamma(\frac{b-a+m}{2})}\sum_{r=0}^{m}\left\{\binom{m}{r}\left[\frac{(-1)^r\Gamma(\frac{r+a}{2})}{\Gamma(\frac{b+r-m}{2})}+\frac{(-1)^r\Gamma(\frac{r+a+1}{2})}{\Gamma(\frac{b+r-m+1}{2})}\right]\right\} \tag{18}$$

$$\left(a, \tfrac{a+b+m}{2}, \tfrac{-a+b-m}{2}, \tfrac{-a+b+m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-; \quad m \in \mathbb{N}_0\right)$$

or equivalently,

$$ {}_2F_1\left[\begin{matrix} a, & b; & \frac{1}{2} \\ \frac{a+b+m}{2}; & & \end{matrix}\right] = \frac{\sqrt{(\pi)}\Gamma(\frac{a+b+m}{2})}{(\frac{-a+b-m}{2})_m}\times\sum_{r=0}^{m}\left\{\binom{m}{r}(-2)^{-r}(a)_r\left[\frac{1}{\Gamma(\frac{a+r+1}{2})\Gamma(\frac{b+r-m}{2})}+\frac{1}{\Gamma(\frac{a+r}{2})\Gamma(\frac{b+r-m+1}{2})}\right]\right\}$$

$$\left(a, \frac{a+b+m}{2}, \frac{-a+b-m}{2}, \frac{-a+b+m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-; \quad m \in \mathbb{N}_0\right). \tag{19}$$

4. Proof of Summation Theorems

In order to evaluate ${}_2F_1\left[\begin{array}{c}a,\ b;\\ \frac{a+b-m}{2};\end{array}\ \frac{1}{2}\right]$ applying Pfaff-Kummer linear hypergeometric transformation (1.5), we get

$$ {}_2F_1\left[\begin{array}{c}a,\ b;\\ \frac{a+b-m}{2};\end{array}\ \frac{1}{2}\right]=2^a\,{}_2F_1\left[\begin{array}{cc}a, & \frac{a-b-m}{2};\\ & \frac{a+b-m}{2};\end{array}\ -1\right] $$

now applying Euler-Beta type integral representation (4) for ${}_2F_1$ in right hand side, we get

$$ \begin{aligned} {}_2F_1\left[\begin{array}{c}a,\quad b;\\ \frac{a+b-m}{2};\end{array}\ \frac{1}{2}\right] &= \frac{2^a\Gamma(\frac{a+b-m}{2})}{\Gamma(a)\Gamma(\frac{b-a-m}{2})}\int_0^1 t^{a-1}(1-t)^{\frac{b-a-m}{2}-1}(1+t)^{\frac{b-a+m}{2}}\mathrm{d}t\\ &= \frac{2^a\Gamma(\frac{a+b-m}{2})}{\Gamma(a)\Gamma(\frac{b-a-m}{2})}\int_0^1 t^{a-1}(1-t^2)^{\frac{b-a-m-2}{2}}(1+t)^m(1+t)\mathrm{d}t\\ &= \frac{2^a\Gamma(\frac{a+b-m}{2})}{\Gamma(a)\Gamma(\frac{b-a-m}{2})}\sum_{r=0}^{m}\binom{m}{r}\left\{\int_0^1 t^{a+r-1}(1-t^2)^{\frac{b-a-m-2}{2}}\mathrm{d}t+\int_0^1 t^{a+r}(1-t^2)^{\frac{b-a-m-2}{2}}\mathrm{d}t\right\}\\ &= \frac{2^{a-1}\Gamma(\frac{a+b-m}{2})}{\Gamma(a)\Gamma(\frac{b-a-m}{2})}\sum_{r=0}^{m}\binom{m}{r}\left\{\int_0^1 y^{\frac{a+r}{2}-1}(1-y)^{\frac{b-a-m}{2}-1}\mathrm{d}y+\int_0^1 y^{\frac{a+r+1}{2}-1}(1-y)^{\frac{b-a-m}{2}-1}\mathrm{d}y\right\} \end{aligned} $$

which, in view of the Definition (1.6), yields the desired Formula (16) by appealing also to the principle of analytic continuation. Similarly we can derive other results of section 3.

5. Applications of Summation Theorems

For the classical Laplace transform defined by

$$ \mathfrak{L}\{f(t):s\}=\int_0^\infty e^{-st}f(t)dt=\mathcal{F}(s) $$

whenever the integral exists in the Lebesgue sense, it is easily seen for Kummer's confluent hypergeometric function ${}_1F_1$ that (see, for example, [15])

$$ \begin{aligned} \mathfrak{L}\left\{t^{\lambda-1}\,{}_1F_1\left[\begin{array}{c}\mu;\\ \nu;\end{array}\ zt\right]:s\right\} &= \int_0^\infty e^{-st}t^{\lambda-1}\,{}_1F_1\left[\begin{array}{c}\mu;\\ \nu;\end{array}\ zt\right]\mathrm{d}t\\ &= \frac{\Gamma(\lambda)}{s^\lambda}\,{}_2F_1\left[\begin{array}{c}\lambda,\mu;\\ \nu;\end{array}\ \frac{z}{s}\right] \end{aligned} \tag{20} $$

$\Big(|s|>|z|\ ;\ |s|=|z|,\Re(\nu-\mu-\lambda)>0\ ;\ |s|=|z|\ ,\ s\neq z\ ,0\geq\Re(\nu-\mu-\lambda)>-1\ ;\Re(\lambda)>0\ ;\quad \nu\in\mathbb{C}\backslash\mathbb{Z}_0^-$ and $\Re(s)>\max\{\Re(z),0\}\Big)$.

In this section, we apply the summation formulas (9), (13), (16) and (18) in order to derive several closed-form expressions for the Laplace transforms of Kummer's confluent hypergeometric function ${}_1F_1$.

In each of the following results, any exceptional values of the parameters and variables, which would make the results invalid, are tacitly excluded.

If we set $\lambda=b,\ \mu=a+m,\ \nu=\frac{1+a+b}{2}$ and $z=\frac{s}{2}$ in Equation (20) and apply Equation (9) we get

$$ \begin{aligned} \mathfrak{L}\left\{t^{b-1}\,{}_1F_1\left[\begin{array}{c}a+m;\\ \frac{1+a+b}{2};\end{array}\ \frac{st}{2}\right]:s\right\} &= \int_0^\infty e^{-st}t^{b-1}\,{}_1F_1\left[\begin{array}{c}a+m;\\ \frac{1+a+b}{2};\end{array}\ \frac{st}{2}\right]\mathrm{d}t\\ &= \frac{2^{b-1}\Gamma(\frac{a+b+1}{2})}{s^b}\sum_{r=0}^{m}\binom{m}{r}\frac{\Gamma(\frac{b+r}{2})}{\Gamma(\frac{a+r+1}{2})} \end{aligned} \tag{21} $$

$$\left(b,\ \tfrac{1+a+b}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-\quad;\Re(b)>0,\Re(s)>0\right).$$

If we assume $\lambda=a+m,\ \mu=b,\ \nu=\frac{1+a+b}{2}$ and $z=\frac{s}{2}$ in Equation (20) and use Equation (9) we obtain

$$\begin{aligned}\mathfrak{L}\left\{t^{a+m-1}{}_1F_1\left[\begin{array}{r}b;\\ \frac{1+a+b}{2};\end{array}\ \frac{st}{2}\right]:s\right\}&=\int_0^\infty e^{-st}t^{a+m-1}{}_1F_1\left[\begin{array}{r}b;\\ \frac{1+a+b}{2};\end{array}\ \frac{st}{2}\right]\mathrm{d}t\\&=\frac{\Gamma(a+m)}{s^{a+m}}\frac{2^{b-1}\Gamma(\frac{a+b+1}{2})}{\Gamma(b)}\sum_{r=0}^{m}\binom{m}{r}\frac{\Gamma(\frac{b+r}{2})}{\Gamma(\frac{a+r+1}{2})}\end{aligned}\tag{22}$$

$$\left(b,\ \tfrac{1+a+b}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-\quad;\Re(a)>-m,\Re(s)>0\right).$$

If we let $\lambda=b,\ \mu=a,\ \nu=\frac{1+a+b+m}{2}$ and $z=\frac{s}{2}$ in Equation (20) and use Equation (13) we find

$$\begin{aligned}\mathfrak{L}\left\{t^{b-1}{}_1F_1\left[\begin{array}{r}a;\\ \frac{1+a+b+m}{2};\end{array}\ \frac{st}{2}\right]:s\right\}&=\int_0^\infty e^{-st}t^{b-1}{}_1F_1\left[\begin{array}{r}a;\\ \frac{1+a+b+m}{2};\end{array}\ \frac{st}{2}\right]\mathrm{d}t\\&=\frac{\Gamma(b)}{s^{b}}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1+a+b+m}{2})\Gamma(\frac{b-a-m+1}{2})}{\Gamma(\frac{a}{2})\Gamma(\frac{a+1}{2})\Gamma(\frac{1-a+b+m}{2})}\sum_{r=0}^{m}\binom{m}{r}\frac{(-1)^r\Gamma(\frac{r+a}{2})}{\Gamma(\frac{1+b+r-m}{2})}\end{aligned}\tag{23}$$

$$\left(a,\ \tfrac{1+a+b+m}{2},\ \tfrac{b-a-m+1}{2},\ \tfrac{1-a+b+m}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-;\quad m\in\mathbb{N}_0\quad;\Re(b)>0,\Re(s)>0\right).$$

If we select $\lambda=a,\ \mu=b,\ \nu=\frac{1+a+b+m}{2}$ and $z=\frac{s}{2}$ in Equation (20) and use Equation (13) we get

$$\begin{aligned}\mathfrak{L}\left\{t^{a-1}{}_1F_1\left[\begin{array}{r}b;\\ \frac{1+a+b+m}{2};\end{array}\ \frac{st}{2}\right]:s\right\}&=\int_0^\infty e^{-st}t^{a-1}{}_1F_1\left[\begin{array}{r}b;\\ \frac{1+a+b+m}{2};\end{array}\ \frac{st}{2}\right]\mathrm{d}t\\&=\frac{2^{a-1}}{s^{a}}\frac{\Gamma(\frac{1+a+b+m}{2})\Gamma(\frac{b-a-m+1}{2})}{\Gamma(\frac{1-a+b+m}{2})}\sum_{r=0}^{m}\binom{m}{r}\frac{(-1)^r\Gamma(\frac{r+a}{2})}{\Gamma(\frac{1+b+r-m}{2})}\end{aligned}\tag{24}$$

$$\left(a,\ \tfrac{1+a+b+m}{2},\ \tfrac{b-a-m+1}{2},\ \tfrac{1-a+b+m}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-;\quad m\in\mathbb{N}_0\quad;\Re(a)>0,\Re(s)>0\right).$$

If we assume $\lambda=b,\ \mu=a,\ \nu=\frac{a+b-m}{2}$ and $z=\frac{s}{2}$ in Equation (20) and use Equation (16) we obtain

$$\begin{aligned}\mathfrak{L}\left\{t^{b-1}{}_1F_1\left[\begin{array}{r}a;\\ \frac{a+b-m}{2};\end{array}\ \frac{st}{2}\right]:s\right\}&=\int_0^\infty e^{-st}t^{b-1}{}_1F_1\left[\begin{array}{r}a;\\ \frac{a+b-m}{2};\end{array}\ \frac{st}{2}\right]\mathrm{d}t\\&=\frac{\Gamma(b)}{s^{b}}\frac{2^{a-1}\Gamma(\frac{a+b-m}{2})}{\Gamma(a)}\sum_{r=0}^{m}\left\{\binom{m}{r}\left[\frac{\Gamma(\frac{r+a}{2})}{\Gamma(\frac{b+r-m}{2})}+\frac{\Gamma(\frac{r+a+1}{2})}{\Gamma(\frac{b+r-m+1}{2})}\right]\right\}\end{aligned}\tag{25}$$

$$\left(a,\ \tfrac{a+b-m}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-;\quad m\in\mathbb{N}_0\quad;\Re(b)>0,\Re(s)>0\right).$$

If we choose $\lambda=a,\ \mu=b,\ \nu=\frac{a+b-m}{2}$ and $z=\frac{s}{2}$ in Equation (20) and use Equation (16) we deduce

$$\begin{aligned}\mathfrak{L}\left\{t^{a-1}{}_1F_1\left[\begin{array}{r}b;\\ \frac{a+b-m}{2};\end{array}\ \frac{st}{2}\right]:s\right\}&=\int_0^\infty e^{-st}t^{a-1}{}_1F_1\left[\begin{array}{r}b;\\ \frac{a+b-m}{2};\end{array}\ \frac{st}{2}\right]\mathrm{d}t\\&=\frac{2^{a-1}\Gamma(\frac{a+b-m}{2})}{s^{a}}\sum_{r=0}^{m}\left\{\binom{m}{r}\left[\frac{\Gamma(\frac{r+a}{2})}{\Gamma(\frac{b+r-m}{2})}+\frac{\Gamma(\frac{r+a+1}{2})}{\Gamma(\frac{b+r-m+1}{2})}\right]\right\}\end{aligned}\tag{26}$$

$$\left(a,\ \tfrac{a+b-m}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-;\quad m\in\mathbb{N}_0\quad;\Re(a)>0,\Re(s)>0\right).$$

If we let $\lambda = b,\ \mu = a,\ \nu = \frac{a+b+m}{2}$ and $z = \frac{s}{2}$ in Equation (20) and use Equation (18) we obtain

$$\mathfrak{L}\left\{t^{b-1}{}_1F_1\left[\begin{array}{c} a;\ \frac{st}{2} \\ \frac{a+b+m}{2};\end{array}\right] : s\right\} = \int_0^\infty e^{-st}t^{b-1}{}_1F_1\left[\begin{array}{c} a;\ \frac{st}{2} \\ \frac{a+b+m}{2};\end{array}\right]\mathrm{d}t$$
$$= \frac{2^{a-1}\Gamma(b)\Gamma(\frac{a+b+m}{2})\Gamma(\frac{b-a-m}{2})}{s^b\ \Gamma(a)\Gamma(\frac{b-a+m}{2})}\sum_{r=0}^{m}\left\{\binom{m}{r}\left[\frac{(-1)^r\Gamma(\frac{r+a}{2})}{\Gamma(\frac{b+r-m}{2})} + \frac{(-1)^r\Gamma(\frac{r+a+1}{2})}{\Gamma(\frac{b+r-m+1}{2})}\right]\right\} \tag{27}$$

$$\left(a, \tfrac{a+b+m}{2}, \tfrac{-a+b-m}{2}, \tfrac{-a+b+m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-;\quad m \in \mathbb{N}_0\ \ ; \Re(b) > 0, \Re(s) > 0\right).$$

If we choose $\lambda = a,\ \mu = b,\ \nu = \frac{a+b+m}{2}$ and $z = \frac{s}{2}$ in Equation (20) and apply Equation (18) we get

$$\mathfrak{L}\left\{t^{a-1}{}_1F_1\left[\begin{array}{c} b;\ \frac{st}{2} \\ \frac{a+b+m}{2};\end{array}\right] : s\right\} = \int_0^\infty e^{-st}t^{a-1}{}_1F_1\left[\begin{array}{c} b;\ \frac{st}{2} \\ \frac{a+b+m}{2};\end{array}\right]\mathrm{d}t$$
$$= \frac{2^{a-1}\Gamma(\frac{a+b+m}{2})\Gamma(\frac{b-a-m}{2})}{s^a\ \Gamma(\frac{b-a+m}{2})}\sum_{r=0}^{m}\left\{\binom{m}{r}\left[\frac{(-1)^r\Gamma(\frac{r+a}{2})}{\Gamma(\frac{b+r-m}{2})} + \frac{(-1)^r\Gamma(\frac{r+a+1}{2})}{\Gamma(\frac{b+r-m+1}{2})}\right]\right\} \tag{28}$$

$$\left(a, \tfrac{a+b+m}{2}, \tfrac{-a+b-m}{2}, \tfrac{-a+b+m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-;\quad m \in \mathbb{N}_0\ \ ; \Re(a) > 0, \Re(s) > 0\right).$$

We conclude our present investigation by observing that several other corollaries and consequences of the remaining summation formulas (10), (11), (14), (15), (17), (19) of Sections 2, 3 and its applications in Laplace transforms of Kummer's confluent hypergeometric functions, can also be deduced in an analogous manner.

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