Ciric Type Coincidence and Fixed Point Results For Nonexpansive Multi-Valued and Single Valued Maps

Research Article

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Abstract: In this paper we consider the existence of coincidences and fixed points of nonexpansive type conditions satisfied by multivalued and single valued maps and prove some fixed point theorems for nonexpansive type single and multivalued mappings.

MSC: 47H10, 54H25.

1. Introduction

Fixed point theory in the framework of metric spaces is one of the most powerful and useful tools in nonlinear functional analysis. The intrinsic subject of this theory is concerned with the conditions for the existence, uniqueness and exact methods of evaluation of fixed point of a mapping. The application of fixed point theorems is remarkable in a wide scale of mathematical, engineering, economic, physical, computer science and other fields of science. The Banach contraction principle [1] is a simplest and limelight result in this direction. Fixed point theorems for contractive, non-expansive, contractive type and non-expansive type mappings provide techniques for solving a variety of applied problems in mathematical and engineering sciences. It is one of the reason that many authors have studied various classes of contractive type or non-expansive type mappings.

In this paper, X always denotes a metric space. H denotes the Housdorff (resp. generalized Housdorff) metric on $CB(X)$(resp. $CL(X)$) induced by the metric d, where $CB(X)$(resp. $CL(X)$) is the collection of all nonempty closed and bounded (resp. closed), subsets of X. For these definitions one may refer [7, 15, 18]. For $y \in X$ and $A \subset X, d(y, A)$ will denote the ordinary distance between y and A. If T is such that for all x, y in X

$$d(Tx, Ty) \leq \lambda d(x, y) \tag{1}$$

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where $0 < \lambda < 1$, then T is said to be a contraction mapping. If T satisfies (1) with $\lambda = 1$, then T is called a non-expansive mapping. If T satisfies any conditions of type

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx) \tag{2}$$

where a_i $(i = 1, 2, 3, 4, 5)$ are nonnegative real numbers such that $a_1 + a_2 + a_3 + a_4 + a_5 < 1$, then T is said to be a contractive type mapping. If T satisfies (2) with $a_1 + a_2 + a_3 + a_4 + a_5 = 1$, then T is said to be a non-expansive type mapping. Similar terminology is used for multi-valued mappings.

Bogin [3] proved the following result:

Theorem 1.1. *Let X be a nonempty complete metric space and $T : X \to X$ a mapping satisfying*

$$d(Tx, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)] + c[d(x, Ty) + d(y, Tx)] \tag{3}$$

where $a \geq 0$, $b > 0$, $c > 0$ and

$$a + 2b + 2c = 1 \tag{4}$$

Then T has a unique fixed point.

This result was generalized by Rhoades [17] and Ciric [6, 8]. Iseki [10] studied a family of commuting mappings $T_1, T_2, \ldots T_n$ which satisfy (3) with $a \geq 0$, $b \geq 0$, $c \geq 0$ and $a + 2b + 2c = 1$. For Banach spaces the famous is Gregus's Fixed Point Theorem [9] for non-expansive type single-valued mappings, which satisfy (3) with $c = 0, a < 1$. Ciric [8] introduced and investigated a new class of self-mappings T on X which satisfy an inequality of type (3) with $b \geq 0$ and still have a fixed point. Also proved that by an example if the mapping T satisfies (3) with $b = 0$ and if a and c are such that (4) holds, then T need not have a fixed point. Therefore, a contractive condition for T, which shall guarantee a fixed point of T in the case $b = 0$ and $a + 2c = 1$, must be stricter then (3).

In [12] Jungck introduced the concept of commuting maps. In [13] Jungck introduced the concept of compatible mappings which generalize the concept of commuting maps.

Definition 1.2 ([13]). *Two self maps T and f of a metric space X are said to be compatible if $\lim_{n \to +\infty} d(Tfx_n, fTx_n) = 0$, whenever $\{x_n\}$ is a sequence such that $\lim_{n \to +\infty} Tx_n = \lim_{n \to +\infty} fx_n = t \in X$.*

Definition 1.3 ([7]). *An orbit of the multi-valued map T at a point x_0 in X is a sequence $\{x_n : x_n \in Tx_{n-1}\}$. A space X is $T-$ orbitally complete if every Cauchy sequence of the form $\{x_{n_i} : x_{n_i} \in Tx_{n_i-1}\}$ converges in X.*

Definition 1.4 ([18]). *If for a point x_0 in X, there exists a sequence $x_n \in X$ such that $fx_{n+1} \in Tx_n$, $n = 0, 1, 2, \ldots$ then $O_f(x_0) = \{fx_n : n = 1, 2, \ldots\}$ is an orbit of (T, f) at x_0. A space X is called $(T, f)-$orbitally complete if every Cauchy sequence of the form $\{x_{n_i} : x_{n_i} \in Tx_{n_i-1}\}$ converges in X.*

Let $T, f : X \to X$ be two self mappings on X. For each $x, y \in X$, denote

$$M(x, y) = \max\{d(fx, Ty), d(fy, Tx)\}$$

$$m(x, y) = \min\{d(fx, Ty), d(fy, Tx)\}$$

In this paper, we shall investigate a new class of self-mappings T, f on X which satisfy the following non-expansive type condition:

$$d(Tx, Ty) \leq a(x, y) \max d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{1}{2}[M(x, y) + m(x, y)] + c(x, y)[M(x, y) + hm(x, y)] \qquad (5)$$

for all $x, y \in X$, where $0 < h < 1$, $a(x, y) \geq 0$,

$$\beta = \inf\{c(x, y) : x, y \in X\} > 0 \qquad (6)$$

and

$$\sup_{x,y \in X} (a(x, y) + 2c(x, y)) = 1. \qquad (7)$$

2.   Main Results

Now, we give our main results.

Theorem 2.1. *Let (X, d) be a metric space, T, f are self maps of X satisfying condition (5), where a and c satisfying (6) and (7) with $T(X) \subseteq f(X)$ and either (a) X is complete and f is surjective; or (b) X is complete, f is continuous and T, f are compatible; or (c) $f(X)$ is complete ; or (d) $T(X)$ is complete. Then f and T have a coincidence point in X. Further, the coincidence value is unique, i.e. $fp = fq$ whenever $fp = Tp$ and $fq = Tq$, $(p, q \in X)$.*

Proof.   First, we shall prove that f and T have at most one coincidence point. On the contrary, suppose that f and T have two coincidence points p and q. Then from (5) with a and c evaluated at (p, q), we have

$$d(Tp, Tq) \leq a \max \left\{ d(fp, fq), d(fp, Tp), d(fq, Tq), \frac{1}{2}[M(p, q) + m(p, q)] \right\} + c[M(p, q) + hm(p, q)]$$

$$= [a + c(1 + h)]d(Tp, Tq).$$

Hence by (7), $(Tp, Tq) \leq [1 - c(1 - h)]d(Tp, Tq)$ implying $Tp = Tq$ by (6) and hence $fp = fq$.

Pick $x_0 \in X$. We construct two sequences $\{x_n\}$ and $\{y_n\}$ as follows: Since $T(X) \subseteq f(X)$, choose x_1 so that $y_1 = fx_1 = Tx_0$. In general, choose x_{n+1} so that $y_{n+1} = fx_{n+1} = Tx_n$. Applying (5), we have

$$d(Tx_n, Tx_{n+1}) \leq a \max \left\{ d(fx_n, fx_{n+1}), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1}), \frac{1}{2}[M(x_n, x_{n+1}) + m(x_n, x_{n+1})] \right\}$$

$$+ c[M(x_n, x_{n+1}) + hm(x_n, x_{n+1})]$$

$$= a \max \left\{ d(fx_n, Tx_n), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1}), \frac{1}{2}[M(x_n, x_{n+1}) + m(x_n, x_{n+1})] \right\}$$

$$+ c[M(x_n, x_{n+1}) + hm(x_n, x_{n+1})] \qquad (8)$$

where a and c are evaluated at (x_n, x_{n+1}). Since $m(x_n, x_{n+1}) = 0$ and $M(x_n, x_{n+1}) = d(fx_n, Tx_{n+1})$. If we suppose that for some n, $d(fx_{n+1}, Tx_{n+1}) > d(fx_n, Tx_n)$. Then $M(x_n, x_{n+1}) \leq 2d(fx_{n+1}, Tx_{n+1})$ and the inequality (8) gives $d(fx_{n+1}, Tx_{n+1}) \leq (a + 2c)d(fx_{n+1}, Tx_{n+1})$ a contradiction. Therefore, for all n we have

$$d(fx_{n+1}, Tx_{n+1}) \leq d(fx_n, Tx_n). \qquad (9)$$

Again from (5), we have

$$
\begin{aligned}
d(y_{n-1}, Tx_n) &= d(Tx_{n-2}, Tx_n) \\
&\leq a \max \left\{ d(fx_{n-2}, fx_n), d(fx_{n-2}, Tx_{n-2}), d(fx_n, Tx_n), \frac{1}{2}[M(x_{n-2}, x_n) + m(x_{n-2}, x_n)] \right\} \\
&\quad + c[M(x_{n-2}, x_n) + hm(x_{n-2}, x_n)] \\
&= a \max \left\{ d(fx_{n-2}, fx_n), d(fx_{n-2}, Tx_{n-2}), d(fx_n, Tx_n), \frac{1}{2}[M(x_{n-2}, x_n) + m(x_{n-2}, x_n)] \right\} \\
&\quad + c[M(x_{n-2}, x_n) + hm(x_{n-2}, x_n)]
\end{aligned}
\tag{10}
$$

where a and c are evaluated at (x_{n-2}, x_n). Since

$$
\begin{aligned}
d(fx_{n-2}, fx_n) &\leq d(fx_{n-2}, fx_{n-1}) + d(fx_{n-1}, fx_n) \\
&= d(fx_{n-2}, Tx_{n-2}) + d(fx_{n-1}, Tx_{n-1}) \\
&\leq 2d(fx_{n-2}, Tx_{n-2}) \\
d(fx_{n-2}, Tx_n) &\leq d(fx_{n-2}, fx_{n-1}) + d(fx_{n-1}, Tx_n) \\
&\leq d(fx_{n-2}, fx_{n-1}) + d(fx_{n-1}, fx_n) + d(fx_n, Tx_n) \\
&= d(fx_{n-2}, Tx_{n-2}) + d(fx_{n-1}, Tx_{n-1}) + d(fx_n, Tx_n) \\
&\leq 3d(fx_{n-2}, Tx_{n-2})
\end{aligned}
$$

and

$$
d(fx_n, Tx_{n-2}) = d(Tx_{n-1}, fx_{n-1}) \leq d(fx_{n-2}, Tx_{n-2})
$$

Hence

$$
\begin{aligned}
m(x_{n-2}, x_n) &= \min\{d(fx_{n-2}, Tx_n), d(fx_n, Tx_{n-2})\} \\
&\leq \min\{3d(fx_{n-2}, Tx_{n-2}), d(fx_{n-2}, Tx_{n-2})\} \\
&= d(fx_{n-2}, Tx_{n-2})
\end{aligned}
$$

and

$$
\begin{aligned}
M(x_{n-2}, x_n) &= \max\{d(fx_{n-2}, Tx_n), d(fx_n, Tx_{n-2})\} \\
&\leq \max\{3d(fx_{n-2}, Tx_{n-2}), d(fx_{n-2}, Tx_{n-2})\} \\
&= 3d(fx_{n-2}, Tx_{n-2})
\end{aligned}
$$

Using (9), the inequality (10) gives

$$
\begin{aligned}
d(Tx_{n-2}, Tx_n) &\leq 2ad(fx_{n-2}, Tx_{n-2}) + c[3d(fx_{n-2}, Tx_{n-2}) + hd(fx_{n-2}, Tx_{n-2})] \\
&= [2a + c(3 + h)]d(fx_{n-2}, Tx_{n-2}) \\
&= [2 - c(1 - h)]d(fx_{n-2}, Tx_{n-2})
\end{aligned}
\tag{11}
$$

Again from (5), we have

$$d(y_n, y_{n+1}) = d(Tx_{n-1}, Tx_n)$$
$$\leq a \max\left\{d(fx_{n-1}, fx_n), d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n), \frac{1}{2}[M(x_{n-1}, x_n) + m(x_{n-1}, x_n)]\right\}$$
$$+ c[M(x_{n-1}, x_n) + hm(x_{n-1}, x_n)]$$
$$= a \max\left\{d(fx_{n-1}, Tx_{n-1}), d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n), \frac{1}{2}[M(x_{n-1}, x_n) + m(x_{n-1}, x_n)]\right\}$$
$$+ c[M(x_{n-1}, x_n) + hm(x_{n-1}, x_n)] \tag{12}$$

where a and c are evaluated at (x_{n-1}, x_n). Since $m(x_{n-1}, x_n) = 0$ and $M(x_{n-1}, x_n) = d(Tx_{n-2}, Tx_n)$. Using (9) and (11), the inequality (12) gives

$$d(y_n, y_{n+1}) \leq ad(fx_{n-2}, Tx_{n-2}) + cd(Tx_{n-2}, Tx_n)$$
$$\leq ad(fx_{n-2}, Tx_{n-2}) + c[2 - b(1-h)]d(fx_{n-2}, Tx_{n-2})$$
$$= [1 - c^2(1-h)]d(fx_{n-2}, Tx_{n-2})$$

Hence $d(y_n, y_{n+1}) \leq [1 - \beta^2(1-h)]d(y_{n-2}, y_{n-1})$. Proceeding in this manner we obtain

$$d(Tx_{n-1}, Tx_n) \leq (1 - \beta^2(1-h))^{[\frac{n}{2}]} d(y_0, y_1) \tag{13}$$

where $[\frac{n}{2}]$ stands for the greatest integer not exceeding $\frac{n}{2}$. Since $\beta = \inf\{c(x, y) : x, y \in X\} > 0$ and $h \in (0, 1)$, $\{y_n\}$ is Cauchy, hence converges to a point p in X and then $fx_n \to p$ and $Tx_n \to p$ as $n \to +\infty$.

Case (a): Suppose that f is surjective. Then there exists a point z in X such that $p = fz$. From (5), we have

$$d(fz, Tz) \leq d(fz, y_{n+1}) + d(y_{n+1}, Tz)$$
$$= d(fz, y_{n+1}) + d(Tx_n, Tz)$$
$$\leq d(fz, y_{n+1}) + a \max\left\{d(fx_n, fz), d(fx_n, Tx_n), d(fz, Tz), \frac{1}{2}[M(x_n, z) + m(x_n, z)]\right\}$$
$$+ c[M(x_n, z) + hm(x_n, z)] \tag{14}$$

Since $M(x_n, z) \to d(fz, Tz)$ and $m(x_n, z) \to 0$ as $n \to +\infty$. Taking limit $n \to +\infty$ in the above inequality, we have $d(fz, Tz) \leq \sup_{x,y \in X} (a + c)d(fz, Tz)$ implies that $fz = Tz = p$.

Case (b): Suppose f is continuous and f and T are compatible. Then since $\lim_{n \to +\infty} y_n = p$, we have $\lim_{n \to +\infty} fy_n = fp$. Now using triangle inequality, we have

$$d(fp, Tp) \leq d(fp, fy_{n+1}) + d(fy_{n+1}, Tz)$$
$$\leq d(fp, fy_{n+1}) + d(fTx_n, Tfx_n) + d(Tfx_n, Tp)$$

Note that since $\lim_{n \to +\infty} fx_n = \lim_{n \to +\infty} Tx_n = p$ and f, T are compatible, $\lim_{n \to +\infty} d(Tfx_n, fTx_n) = 0$. On letting $n \to +\infty$, in the above inequality, we have

$$d(fp, Tp) \leq \lim_{n \to +\infty} d(Tfx_n, Tp) \tag{15}$$

From (5), we have

$$d(Tfx_n, Tp) \leq a \max \left\{ d(ffx_n, fp), d(ffx_n, Tfx_n), d(fp, Tp), \frac{1}{2}[M(fx_n, p) + m(fx_n, p)] \right\}$$
$$+ c[M(fx_n, p) + hm(fx_n, p)] \tag{16}$$

Note that

$$d(ffx_n, Tfx_n) \leq d(ffx_n, fTx_n) + d(fTx_n, Tfx_n)$$
$$= d(ffx_n, ffx_{n+1}) + d(fTx_n, Tfx_n).$$

Using the continuity of f and compatibility of f and T, it follows that $\lim\limits_{n \to +\infty} d(ffx_n, Tfx_n) = 0$.
Since $\lim\limits_{n \to +\infty} ffx_n = fp$, it follows that $\lim\limits_{n \to +\infty} Tfx_n = fp$ and

$$\lim_{n \to +\infty} M(fx_n, p) = \lim_{n \to +\infty} \max\{d(ffx_n, Tp), d(fp, Tfx_n)\} = d(fp, Tp)$$

and

$$\lim_{n \to +\infty} m(fx_n, p) = \lim_{n \to +\infty} \min\{d(ffx_n, Tp), d(fp, Tfx_n)\} = 0$$

Taking limit $n \to +\infty$ in the inequality (15), we have $d(fp, Tp) \leq \sup\limits_{x, y \in X} (a + c)d(fp, Tp)$ implies that $fp = Tp$.

Case (c): In this case $p \in f(X)$. Let $z \in f^{-1}(p)$. Then $p = fz$ and the proof is complete by case (a).

Case (d): In this case $p \in T(X) \subseteq f(X)$ and the proof is complete by case (c). $\qquad\square$

Corollary 2.2. *Let (X, d) be a complete metric space and T is self mapping of X satisfying (5) with $f = I$, the identity map on X, where $h = 1$, a and c satisfying (6) and (7). Then T has a unique fixed point and at this fixed point T is continuous.*

Proof.   The existence and uniqueness of the fixed point comes from Theorem 2.1 by setting $f = I$. To prove continuity, let $\{y_n\} \subset X$ with $\lim\limits_{n \to +\infty} y_n = p$, p the unique fixed point of T. Using (5), we have

$$d(Ty_n, Tp) \leq a \max \left\{ d(y_n, p), d(y_n, Ty_n), d(p, Tp), \frac{1}{2}[M(y_n, p) + m(y_n, p)] \right\}$$
$$+ c[M(y_n, p) + m(y_n, p)]$$
$$\leq a(d(y_n, p) + d(p, Ty_n)) + c[d(p, Ty_n) + d(y_n, p)]$$
$$= (a + c)d(y_n, p) + (a + c)d(p, Ty_n)$$
$$= (1 - c)d(y_n, p) + (1 - c)d(p, Ty_n)$$

Hence

$$d(Ty_n, Tp) \leq (1 - \beta)d(y_n, p) + (1 - \beta)d(p, Ty_n)$$

Since $\beta = \inf\{c(x, y) : x, y \in X\} > 0$. Hence we get

$$d(Ty_n, Tp) \leq \left(\frac{1}{\beta} - 1 \right) d(y_n, p)$$

Taking the limit as $n \to +\infty$ yields $\lim\limits_{n \to +\infty} Ty_n = Tp$. Therefore T is continuous at p. $\qquad\square$

Next we establish some results when T is a multi-valued map from a metric space X to the collection of nonempty subset of X, and f is a self map of X.

In this theorem, we use the following non-expansive type condition: let $T : X \to C(X)$ be a multi-valued map and $f : X \to X$ be a single valued map, which satisfy the following condition:

$$H(Tx, Ty) \leq a(x,y) \max\left\{ d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{1}{2}[M(x,y)+m(x,y)] \right\}$$
$$+ c(x,y)[M(x,y)+hm(x,y)] \tag{17}$$

for all $x, y \in X$, where $0 < h < 1$, with a and c satisfy (6) and (7).

Theorem 2.3. *Let (X, d) be a metric space, T a multi-valued map from X to $C(X)$ and f be a self map of X satisfying (17), where a and c satisfying (6) and (7), with $T(X) \subseteq f(X)$. Then f and T have a coincidence point in X either (a) X is $(T, f)-$orbitally complete and f is surjective or (b) $f(X)$ is $(T, f)-$orbitally complete or (c) $T(X)$ is $(T, f)-$orbitally complete.*

Proof.   Pick $x_0 \in X$. We construct two sequences $\{x_n\}$ and $\{y_n\}$ as follows: Since $T(X) \subseteq f(X)$, choose x_1 so that $y_1 = fx_1 \in Tx_0$. If $Tx_0 = Tx_1$, choose $y_2 = fx_2 \neq Tx_1$ such that $y_1 = y_2$. If $Tx_0 \neq Tx_1$, choose $y_2 = fx_2 \in Tx_1$ such that $d(y_1, y_2) \leq H(Tx_0, Tx_1)$. Such a choice is possible since Tx is compact for each x in X. In general, choose $y_{n+2} = fx_{n+2} \in Tx_{n+1}$ such that $y_{n+1} = y_{n+2}$ if $Tx_{n+1} = Tx_{n+2}$ and $d(y_{n+1}, y_{n+2}) \leq H(Tx_n, Tx_{n+1})$ otherwise.
From (17), we have

$$d(y_{n+1}, y_{n+2}) \leq H(Tx_n, Tx_{n+1})$$
$$\leq a \max\left\{ d(fx_n, fx_{n+1}), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1}), \frac{1}{2}[M(x_n, x_{n+1})+m(x_n, x_{n+1})] \right\}$$
$$+ c[M(x_n, x_{n+1})+hm(x_n, x_{n+1})]$$
$$= a \max\left\{ d(fx_n, Tx_n), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1}), \frac{1}{2}[M(x_n, x_{n+1})+m(x_n, x_{n+1})] \right\}$$
$$+ c[M(x_n, x_{n+1})+hm(x_n, x_{n+1})] \tag{18}$$

where a and c are evaluated at (x_n, x_{n+1}). Since $m(x_n, x_{n+1}) = 0$ and $M(x_n, x_{n+1}) = d(fx_n, Tx_{n+1})$. If we suppose that for some n, $d(fx_{n+1}, Tx_{n+1}) > d(fx_n, Tx_n)$. Then $M(x_n, x_{n+1}) \leq 2d(fx_{n+1}, Tx_{n+1})$ and the inequality (18) gives

$$d(fx_{n+1}, Tx_{n+1}) \leq (a+2c)d(fx_{n+1}, Tx_{n+1})$$

a contradiction. Therefore, for all n we have

$$d(fx_{n+1}, Tx_{n+1}) \leq d(fx_n, Tx_n). \tag{19}$$

Again from (17), we have

$$d(y_{n-1}, y_{n+1}) \leq H(Tx_{n-2}, Tx_n)$$
$$\leq a \max\left\{ d(fx_{n-2}, fx_n), d(fx_{n-2}, Tx_{n-2}), d(fx_n, Tx_n), \frac{1}{2}[M(x_{n-2}, x_n)+m(x_{n-2}, x_n)] \right\}$$
$$+ c[M(x_{n-2}, x_n)+hm(x_{n-2}, x_n)]$$
$$= a \max\left\{ d(fx_{n-2}, fx_n), d(fx_{n-2}, Tx_{n-2}), d(fx_n, Tx_n), \frac{1}{2}[M(x_{n-2}, x_n)+m(x_{n-2}, x_n)] \right\}$$
$$+ c[M(x_{n-2}, x_n)+hm(x_{n-2}, x_n)] \tag{20}$$

where a and c are evaluated at (x_{n-2}, x_n). Since

$$d(fx_{n-2}, fx_n) \leq d(fx_{n-2}, fx_{n-1}) + d(fx_{n-1}, fx_n)$$

$$= d(fx_{n-2}, Tx_{n-2}) + d(fx_{n-1}, Tx_{n-1})$$

$$\leq 2d(fx_{n-2}, Tx_{n-2})$$

$$d(fx_{n-2}, Tx_n) \leq d(fx_{n-2}, fx_{n-1}) + d(fx_{n-1}, Tx_n)$$

$$\leq d(fx_{n-2}, fx_{n-1}) + d(fx_{n-1}, fx_n) + d(fx_n, Tx_n)$$

$$= d(fx_{n-2}, Tx_{n-2}) + d(fx_{n-1}, Tx_{n-1}) + d(fx_n, Tx_n)$$

$$\leq 3d(fx_{n-2}, Tx_{n-2})$$

and

$$d(fx_n, Tx_{n-2}) = d(Tx_{n-1}, fx_{n-1}) \leq d(fx_{n-2}, Tx_{n-2}).$$

Hence

$$m(x_{n-2}, x_n) = \min\{d(fx_{n-2}, Tx_n), d(fx_n, Tx_{n-2})\}$$

$$\leq \min\{3d(fx_{n-2}, Tx_{n-2}), d(fx_{n-2}, Tx_{n-2})\}$$

$$= d(fx_{n-2}, Tx_{n-2})$$

and

$$M(x_{n-2}, x_n) = \max\{d(fx_{n-2}, Tx_n), d(fx_n, Tx_{n-2})\}$$

$$\leq \max\{3d(fx_{n-2}, Tx_{n-2}), d(fx_{n-2}, Tx_{n-2})\}$$

$$= 3d(fx_{n-2}, Tx_{n-2})$$

Using (18), the inequality (19) gives

$$d(Tx_{n-2}, Tx_n) \leq 2ad(fx_{n-2}, Tx_{n-2}) + c[3d(fx_{n-2}, Tx_{n-2}) + hd(fx_{n-2}, Tx_{n-2})]$$

$$= [2a + c(3 + h)]d(fx_{n-2}, Tx_{n-2})$$

$$= [2 - c(1 - h)]d(fx_{n-2}, Tx_{n-2}) \tag{21}$$

Again from (17), we have

$$d(y_n, y_{n+1}) \leq Hd(Tx_{n-1}, Tx_n)$$

$$\leq a \max\left\{d(fx_{n-1}, fx_n), d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n), \frac{1}{2}[M(x_{n-1}, x_n) + m(x_{n-1}, x_n)]\right\}$$

$$+ c[M(x_{n-1}, x_n) + hm(x_{n-1}, x_n)]$$

$$= a \max\left\{d(fx_{n-1}, Tx_{n-1}), d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n), \frac{1}{2}[M(x_{n-1}, x_n) + m(x_{n-1}, x_n)]\right\}$$

$$+ c[M(x_{n-1}, x_n) + hm(x_{n-1}, x_n)] \tag{22}$$

where a and c are evaluated at (x_{n-1}, x_n). Since $m(x_{n-1}, x_n) = 0$ and $M(x_{n-2}, x_n) = d(Tx_{n-2}, Tx_n)$. Using (19) and (21), the inequality (22) gives

$$d(y_n, y_{n+1}) \leq ad(fx_{n-2}, Tx_{n-2}) + cd(Tx_{n-2}, Tx_n)$$

$$\leq ad(fx_{n-2}, Tx_{n-2}) + c[2 - c(1-h)]d(fx_{n-2}, Tx_{n-2})$$

$$= [1 - c^2(1-h)]d(fx_{n-2}, Tx_{n-2})$$

Hence

$$d(y_n, y_{n+1}) \leq [1 - \beta^2(1-h)]d(y_{n-2}, y_{n-1})$$

Proceeding in this manner we obtain

$$d(Tx_{n-1}, Tx_n) \leq (1 - \beta^2(1-h))^{[\frac{n}{2}]}d(y_0, y_1) \tag{23}$$

where $[\frac{n}{2}]$ stands for the greatest integer not exceeding $\frac{n}{2}$. Since $\beta = \inf\{c(x,y) : x,y \in X\} > 0$ and $h \in (0,1)$, $\{y_n\}$ is Cauchy, hence converges to a point p in X in cases (a)-(c).

If f is surjective, there exists a point z such that $p = fz$. This is obviously true in cases (b) and (c) as well,

$$d(fz, Tz) \leq d(fz, y_{n+1}) + d(y_{n+1}, Tz)$$

$$\leq d(fz, y_{n+1}) + H(Tx_n, Tz)$$

$$\leq d(fz, y_{n+1}) + a \max\left\{d(fx_n, fz), d(fx_n, Tx_n), d(fz, Tz), \frac{1}{2}[M(x_n, z) + m(x_n, z)]\right\} + c[M(x_n, z) + hm(x_n, z)]$$

Since $M(x_n, z) \to d(fz, Tz)$ and $m(x_n, z) \to 0$ as $n \to +\infty$. On letting $n \to +\infty$ in the above inequality, we obtain

$$d(fz, Tz) \leq \sup_{x,y \in X}(a+c)d(fz, Tz)$$

which implies that $fz \in Tz$. □

Theorem 2.4. *Let X, T and f satisfy the hypotheses of Theorem 2.3 with $C(X)$ replaced by $CL(X)$ and $\delta = \sup_{x,y \in X}(a(x,y) + 2c(x,y)) < 1$. Then T and f have a coincidence point in X.*

Proof. Choose $x_0 \in X$. We construct two sequences $\{x_n\}$ and $\{y_n\}$ as follows: Since $T(X) \subseteq f(X)$, choose x_1 so that $y_1 = fx_1 \in Tx_0$. If $Tx_0 = Tx_1$, choose $y_2 = fx_2 \in Tx_1$ such that $y_1 = y_2$. If $Tx_0 \neq Tx_1$, choose $y_2 = fx_2 \in Tx_1$ such that $d(y_1, y_2) \leq \lambda H(Tx_0, Tx_1)$, where $\lambda > 1$ and $\lambda\delta < 1$. In general, choose $y_{n+2} = fx_{n+2} \in Tx_{n+1}$ such that $d(y_{n+1}, y_{n+2}) \leq \lambda H(Tx_n, Tx_{n+1})$. From (17) we have

$$d(y_{n+1}, y_{n+2}) \leq \lambda H(Tx_n, Tx_{n+1})$$

$$\leq \lambda a \max\left\{d(fx_n, fx_{n+1}), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1}), \frac{1}{2}[M(x_n, x_{n+1}) + m(x_n, x_{n+1})]\right\}$$

$$+ \lambda c[M(x_n, x_{n+1}) + hm(x_n, x_{n+1})]$$

$$= \lambda a \max\left\{d(fx_n, Tx_n), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1}), \frac{1}{2}[M(x_n, x_{n+1}) + m(x_n, x_{n+1})]\right\}$$

$$+ \lambda c[M(x_n, x_{n+1}) + hm(x_n, x_{n+1})] \tag{24}$$

where a and c are evaluated at (x_n, x_{n+1}). Since $m(x_n, x_{n+1}) = 0$ and $M(x_n, x_{n+1}) = d(fx_n, Tx_{n+1})$. If we suppose that for some n, $d(fx_{n+1}, Tx_{n+1}) > d(fx_n, Tx_n)$. Then $M(x_n, x_{n+1}) \leq 2d(fx_{n+1}, Tx_{n+1})$ and the inequality (24) gives

$$d(fx_{n+1}, Tx_{n+1}) \leq \lambda(a + 2c)d(fx_{n+1}, Tx_{n+1})$$

a contradiction. Therefore, for all n we have

$$d(fx_{n+1}, Tx_{n+1}) \leq d(fx_n, Tx_n). \tag{25}$$

and then from (24), we obtain

$$d(y_{n+1}, y_{n+2}) \leq \lambda(a + 2c)d(fx_n, Tx_n)$$
$$= \lambda(a + 2c)d(y_n, y_{n+1})$$
$$\leq kd(y_n, y_{n+1}) \leq k^n d(y_0, y_1)$$

where $k = \sup_{x,y \in X} \lambda(a + 2c)$, Therefore $\{y_n\}$ is Cauchy, hence converges to some point p in X. Since f is surjective, there exists a point z such that $p = fz$. Now

$$d(fz, Tz) \leq d(fz, y_{n+1}) + d(y_{n+1}, Tz)$$
$$\leq d(fz, y_{n+1}) + \lambda H(Tx_n, Tz)$$
$$\leq d(fz, y_{n+1}) + \lambda a \max\left\{ d(fx_n, fz), d(fx_n, Tx_n), d(fz, Tz), \frac{1}{2}[M(x_n, z) + m(x_n, z)] \right\}$$
$$+ \lambda c[M(x_n, z) + hm(x_n, z)].$$

Since $M(x_n, z) \to d(fz, Tz)$ and $m(x_n, z) \to 0$ as $n \to +\infty$. On letting $n \to +\infty$ in the above inequality, we obtain

$$d(fz, Tz) \leq \sup_{x,y \in X} \lambda(a + c)d(fz, Tz)$$

which implies that $fz \in Tz$. $\qquad \square$

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