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Some Common Fixed Point Theorems For Nonexpansive Type Mappings In 2-Metric Spaces

Research Article

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Abstract: The aim of this paper is to prove some common fixed point theorems for weakly compatible mappings under nonexpansive type conditions in the setting of 2-metric spaces. Our result extend and generalizes corresponding results of Singh, Adiga and Giniswami [9] and Liu and Zhang [7].

MSC: 47H10, 54H25.

Keywords: 2-metric spaces, nonexpansive mapping,compatible mappings compatible mapping of type (A).

1. Introduction

The concept of 2-metric space is a natural generalization of the classical one of metric space. It has been investigated, initially, by Gähler and has been developed extensively by Gähler and many other mathematicians [2–4]. The topology induced by 2-metric space is called 2-metric topology, which is generated by the set of all open spheres with two centers. Many authors used the topology in many applications; for example, El Naschie used this sort of the topology in physical applications [1]. Iseki [5] studied the fixed point theorems in 2-metric spaces. A number of fixed point theorems has been proved for 2-metric spaces.

Liu and Zhang [7] proved a few necessary and sufficient conditions for the existence of a common fixed point of a pair of mappings in 2-metric spaces. These results have generalized and improved by a number of mathematicians. Singh, Adiga and Giniswami [9] proved a fixed point theorem in 2-metric spaces for nonexpansive type mappings.

In this paper, we prove some common fixed point theorems for weakly compatible mappings under nonexpansive type conditions in the setting of 2-metric spaces. Our result extend and generalizes corresponding results of Singh, Adiga and Giniswami [9] and Liu and Zhang [7].

2. Preliminaries

Now we shall recall some basic definitions and lemmas which are frequently used to prove our main result.

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Definition 2.1. *A 2-metric space on a set X with at least three points in a non-negative real valued mapping $d : X\times X\times X \to R$ satisfying the following properties:*

(a). To each pair of points a, b with $a \neq b$ in X there is a point $c \in X$ such that $d(a, b, c) \neq 0$.

(b). $d(a, b, c) = 0$ if at least two of the points are equal.

(c). $d(a, b, c) = d(b, c, a) = d(a, c, b,)$

(d). $d(a, b, c) \leq d(a, b, u) + d(a, u, c) + d(u, b, c)$ for all $a, b, c, u \in X$.

The pair (X,d) is called a 2-metric space

Definition 2.2. *The sequence x_n is convergent to $x \in X$ and x is the limit of this sequence of $\lim_{n\to\infty} d(x_n, x, u) = 0$ for each $a \in X$. A sequence x_n is called Cauchy sequence if $\lim_{n,m\to\infty} d(x_n, x_m, u) = 0$ for all $u \in X$*

Definition 2.3. *A 2-metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.*

Note that in a 2-metric space (X, d) is convergent. sequence need not be Cauchy sequence but every convergent sequence is a Cauchy sequence when the 2-metric d is continuous on X.

Definition 2.4. *Let f and g be two self maps of a 2-metric space (X, d).Then f and g are said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n, u) = 0$ for each $x \in X$, whenever x_n is a sequence such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \in X$.*

Definition 2.5. *Let f and g are said to be compatible of type (A) if $\lim_{n\to\infty} d(fgx_n, ggx_n, u) = \lim_{n\to\infty} d(gfx_n, ffx_n, u) = 0$ for all $u \in X$ whenever $x_n \subset X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.*

Definition 2.6. *A mapping f from a 2-metric space (X, d) into itself is said to be continuous at $x \in X$ if for every sequence x_n such that $\lim_{n\to\infty} d(x_n, x, u) = 0$ for all $u \in X$, $\lim_{n\to\infty} d(fx_n, fx, u) = 0$, f is called continuous on X if it is so at all points of X.*

Lemma 2.7. *Let f and g are compatible mappings from a 2-metric space (X, d) into itself, if $ft = gt$ for some $t \in X$, then $fgt = ggt = gft = fft$.*

Lemma 2.8. *Let f and g are compatible mappings. if f is continuous at $t \in X$ and if $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, then $\lim_{n\to\infty} gfx_n = ft$*

Singh, Adiga and Giniswami [9] proved the following theorem in 2-metric spaces for nonexpansive type mappings.

Theorem 2.9. *Let (X, d) be a 2-metric space and $T : X \to X$ be a self mapping satisfying the following nonexpansive type condition:*

$$(Tx, Ty, u) \leq a \max\left\{d(x, y, u), d(x, Tx, u), d(y, Ty, u), \frac{1}{2}[d(x, Ty, u) + d(y, Tx, u)]\right\}$$
$$+ b\max\{d(x, Tx, u), d(y, Ty, u)\} + c[d(x, Ty, u) + d(y, Tx, u)]$$

for all $x, y, u \in X$, where a, b, c are real numbers such that $a + b + 2c = 1$ and $a \geq 0$, $b > 0$, $c > 0$. Then T has a unique fixed point and T is continuous at this point.

Liu and Zhang [7] proved the following theorems:

Theorem 2.10. *Let (X, d) be a complete 2-metric space with d continuous on X and let h and t be two mappings of X into itself. Then the following conditions are equivalent:*

(1). h and t have a common fixed point;

(2). there exists $r \in (0,1)$, $f : X \to t(X)$, $g : X \to h(X)$ such that

(a_1) the pairs f, h and g, t are compatible,

(a_2) one of f, g, h and t is continuous,

(a_3) $d(fx, gy, u) \le r \max\left\{d(hx, ty, u), d(hx, fx, u), d(ty, gy, u), \frac{1}{2}[d(hx, gy, u) + d(ty, fx, u)]\right\}$ for all $x, y, u \in X$,

(3). there exist $w \in W$, $f : X \to t(X)$, $g : X \to h(X)$ satisfying (a_1), (a_2) and

(a_4)

$$
\begin{aligned}
d(fx, gy, u) \le{}& \max\left\{d(hx, ty, u), d(hx, fx, u), d(ty, gy, u), \frac{1}{2}\left[d(hx, gy, u) + d(ty, fx, u)\right]\right\}\\
&- w\left[\max\left\{d(hx, ty, u), d(hx, fx, u), d(ty, gy, u), \frac{1}{2}\left[d(hx, gy, u) + d(ty, fx, u)\right]\right\}\right]
\end{aligned}
$$

for all $x, y, u \in X$, where $W = \{w : w : R^+ \to R^+ \ \text{ is continuous and satisfy } \ 0 < w(r) < r \ \text{ for } \ r > 0\}$

Theorem 2.11. *Let (X, d) be a complete 2-metric space with d continuous on X and let h and t be two mappings of X into itself. Then condition (1) of Theorem 2.9 is equivalent to each of the following condition:*

(4) There exists $r \in (0,1)$, $f : X \to t(X) \cap h(X)$ such that

(a_5) the pairs f, h and f, t are compatible,

(a_6) one of f, h and t is continuous,

(a_7) $d(fx, fy, u) \le r \max\left\{d(hx, ty, u), d(hx, fx, u), d(ty, fy, u), \frac{1}{2}\left[d(hx, fy, u) + d(ty, fx, u)\right]\right\}$ for all $x, y, u \in X$,

(5) there exist $w \in W$, $f : X \to t(X) \cap h(X)$ satisfying (a_5), (a_6) and

(a_8)

$$
\begin{aligned}
d(fx, fy, u) \le{}& \max\left\{d(hx, ty, u), d(hx, fx, u), d(ty, fy, u), \frac{1}{2}[d(hx, fy, u) + d(ty, fx, u)]\right\}\\
&- w[\max\left\{d(hx, ty, u), d(hx, fx, u), d(ty, fy, u), \frac{1}{2}[d(hx, fy, u) + d(ty, fx, u)]\right\}]
\end{aligned}
$$

for all $x, y, u \in X$, where $W = \{w : w : R^+ \to R^+ \ \text{ is continuous and satisfy } \ 0 < w(r) < r \ \text{ for } \ r > 0\}$.

3. Main Result

Throughout this section, N and N_0 denote the set of positive and non-negative integers, respectively. Let $R^+ = [0, \infty)$.

Theorem 3.1. *Let (X, d) be a complete 2-metric space with d continuous on X and let f and g be two mapping of X into itself, there exists $w \in W$, $f : X \to t(X)$ and $g : X \to h(X)$ satisfying:*

(a) The pair (f, h) and (g, t) are compatible.

(b) One of f, g, h and t is continuous

(c)

$$
\begin{aligned}
d(fx,gy,u) \leq\; & a\max\{d(hx,ty,u),d(ty,gy,u)\} + b\max\{d(hx,fx,u),d(ty,gy,u),d(ty,fx,u)\} \\
& + c[d(hx,gy,u)+d(ty,fx,u)] - w[a\max\{d(hx,ty,u),d(ty,gy,u)\} \\
& + b\max\{d(hx,fx,u),d(ty,gy,u),d(ty,fx,u)\} + c[d(hx,gy,u)+d(ty,fx,u)]]
\end{aligned} \tag{1}
$$

where $a \geq 0$, $b > 0$, $c > 0$ such that $a+b+2c = 1$ for all $x,y,u \in X$, then f,g,h and t have a common fixed point.

Proof. Let x_0 be an arbitrary point in X. Since $f(X) \subset t(X)$ and$g(X) \subset h(X)$, then there exist sequence ${x_n}_{n\in N}$ and ${y_n}_{n\in N}$ in X satisfying,

$$
\begin{aligned}
y_{2n} &= tx_{2n+1} = fx_{2n} \\
y_{2n+1} &= hx_{2n+2} = gx_{2n+1} \quad \text{for } \; n \in N_0.
\end{aligned}
$$

Define $d_n(a) = d(y_n, y_{n+1}, a)$ for $a \in X$ and $n \in N_0$. We claim that for any $i,j,k \in N_0$, $d(y_i,y_j,y_k) = 0$. Suppose that $d_{2n}(y_{2n+1}) > 0$ then using (1), we have

$$
\begin{aligned}
d_{2n}(y_{2n+2}) &= d(y_{2n},y_{2n+1},y_{2n+2}) \\
&= d(fx_{2n+2},gx_{2n+1},y_{2n}) \\
&\leq a\max\{d(hx_{2n+2},tx_{2n+1},y_{2n}),d(tx_{2n+1},gx_{2n+1},y_{2n})\} \\
&+ b\max\{d(hx_{2n+2},fx_{2n+2},y_{2n}),d(tx_{2n+1},gx_{2n+2},y_{2n}),d(tx_{2n+1},fx_{2n+2},y_{2n})\} \\
&+ c[d(hx_{2n+2},gx_{2n+1},y_{2n})+d(tx_{2n+1},fx_{2n+2},y_{2n})] - w[a\max\{d(hx_{2n+2},tx_{2n+1},y_{2n}),d(tx_{2n+1},gx_{2n+1},y_{2n})\} \\
&+ b\max\{d(hx_{2n+2},fx_{2n+2},y_{2n}),d(tx_{2n+1},gx_{2n+1},y_{2n}),d(tx_{2n+1},fx_{2n+2},y_{2n})\} \\
&+ c[d(hx_{2n+2},gx_{2n+1},y_{2n})+d(tx_{2n+1},fx_{2n+2},y_{2n})]] \\
&\leq a\max\{d(y_{2n+1},y_{2n},y_{2n}),d(y_{2n},y_{2n+1},y_{2n})\} \\
&+ b\max\{d(y_{2n+1},y_{2n+2},y_{2n}),d(y_{2n},y_{2n+1},y_{2n}),d(y_{2n},y_{2n+2},y_{2n})\} \\
&+ c[d(y_{2n+1},y_{2n+1},y_{2n})+d(y_{2n},y_{2n+2},y_{2n})] - w[a\max\{d(y_{2n+1},y_{2n},y_{2n}),d(y_{2n},y_{2n+1},y_{2n})\} \\
&+ b\max\{d(y_{2n+1},y_{2n+2},y_{2n}),d(y_{2n},y_{2n+1},y_{2n}),d(y_{2n},y_{2n+2},y_{2n})\} + c[d(y_{2n+1},y_{2n+1},y_{2n})+d(y_{2n},y_{2n+2},y_{2n})]] \\
&\leq bd(y_{2n+1},y_{2n+2},y_{2n}) - wbd(y_{2n+1},y_{2n+2},y_{2n}) \\
&\leq bd_{2n}(y_{2n+2}) - wbd_{2n}(y_{2n+2}) \\
&< bd_{2n}(y_{2n+2}) \\
&< d_{2n}(y_{2n+2})
\end{aligned}
$$

a contradiction. Hence $d_{2n}(y_{2n+2}) = 0$. Similarly, we have $d_{2n+1}(y_{2n+3}) = 0$. Consequently, for all $n \in N_0$,

$$
d_n(y_{n+2}) = 0 \tag{2}
$$

Using (2) we have

$$
\begin{aligned}
d(y_n,y_{n+2},u) &\leq d(y_n,y_{n+1},y_{n+2}) + d(y_n,y_{n+1},u) + d(y_{n+1},y_{n+2},u) \\
&\leq d_n(y_{n+2}) + d_n(u) + d_{n+1}(u) \\
&= d_n(u) + d_{n+1}(u)
\end{aligned} \tag{3}
$$

Now applying (1) again and using (3), we have

$$\begin{aligned}
d_{2n+1}(u) &= d(y_{2n+1}, y_{2n+2}, u)\\
&= d(fx_{2n+2}, gx_{2n+1}, u)\\
&\leq a\max\{d(hx_{2n+2}, tx_{2n+1}, u), d(gx_{2n+1}, tx_{2n+1}, u)\}\\
&+ b\max\{d(fx_{2n+2}, hx_{2n+2}, u), d(gx_{2n+1}, tx_{2n+1}, u), d(tx_{2n+1}, fx_{2n+2}, u)\}\\
&+ c[d(hx_{2n+2}, gx_{2n+1}, u) + d(tx_{2n+1}, fx_{2n+2}, u)] - w[a\max\{d(hx_{2n+2}, tx_{2n+1}, u), d(gx_{2n+1}, tx_{2n+1}, u)\}\\
&+ b\max\{d(fx_{2n+2}, hx_{2n+2}, u), d(gx_{2n+1}, tx_{2n+1}, u), d(tx_{2n+1}, fx_{2n+2}, u)\}\\
&+ c[d(hx_{2n+2}, gx_{2n+1}, u) + d(tx_{2n+1}, fx_{2n+2}, u)]]\\
&\leq a\max\{d(y_{2n+1}, y_{2n}, u), d(y_{2n+1}, y_{2n}, u)\} + b\max\{d(y_{2n+2}, y_{2n+1}, u), d(y_{2n+1}, y_{2n}, u), d(y_{2n}, y_{2n}, u)\}\\
&+ c[d(y_{2n+1}, y_{2n+1}, u) + d(y_{2n}, y_{2n+2}, u)] - w[a\max\{d(y_{2n+1}, y_{2n}, u), d(y_{2n+1}, y_{2n}, u)\}\\
&+ b\max\{d(y_{2n+2}, y_{2n+1}, u), d(y_{2n+1}, y_{2n}, u), d(y_{2n}, y_{2n}, u)\} + c[d(y_{2n+1}, y_{2n+1}, u) + d(y_{2n}, y_{2n+2}, u)]\\
&\leq a\max\{d_{2n}(u), d_{2n}(u)\} + b\max d_{2n+1}(u), d_{2n}(u) + cd(y_{2n}, y_{2n+2}, u) - w[a\max\{d_{2n}(u), d_{2n}(u)\}\\
&+ b\max\{d_{2n+1}(u), d_{2n}(u)\} + cd(y_{2n}, y_{2n+1}, u)\\
&\leq a\max\{d_{2n}(u), d_{2n}(u)\} + b\max d_{2n+1}(u), d_{2n}(u) + c[d_{2n}(u) + d_{2n+1}(u)]\\
&- w[a\max\{d_{2n}(u), d_{2n}(u)\} + b\max\{d_{2n+1}(u), d_{2n}(u)\} + c[d_{2n}(u) + d_{2n+1}(u)]]
\end{aligned}$$

Suppose that $d_{2n}(u) < d_{2n+1}(u)$, then

$$\begin{aligned}
d_{2n+1}(u) &< [ad_{2n+1}(u) + bd_{2n+1}(u) + 2cd_{2n+1}(u)] - w[ad_{2n+1}(u) + bd_{2n+1}(u) + 2cd_{2n+1}(u)]\\
&= (a+b+2c)d_{2n+1}(u) - w[(a+b+2c)d_{2n+1}(u)]\\
&= d_{2n+1}(u) - wd_{2n+1}(u)
\end{aligned}$$

a contradiction. Hence

$$\begin{aligned}
d_{2n+1}(u) &\leq d_{2n}(u)\\
d_{2n+1}(u) &\leq d_{2n}(u) - wd_{2n}(u)\\
&\leq d_{2n}(u)
\end{aligned}$$

Similarly, we have $d_{2n}(u) \leq d_{2n-1}(u)$. That is, for all $n \in N$,

$$d_{n+1}(u) \leq d_n(u) \tag{4}$$

Letn, m be in N_0 if $n \geq m$, then

$$d_n(y_m) \leq d_m(y_m) = 0 \tag{5}$$

If $n < m$ then

$$\begin{aligned}
d_n(y_m) &= d_n(y_n, y_{n+1}, y_m)\\
&\leq d(y_n, y_{n+1}, y_{m-1}) + d(y_n, y_{m-1}, y_m) + d(y_{m-1}, y_{n+1}, y_m)\\
&= d_n(y_{m-1}) + d_{m-1}(y_n) + d_{m-1}(y_{n+1})\\
&= d_n(y_{m-1})\\
&\leq d_n(y_{m-2}) \leq d_n(y_{m-3}) \ldots < d_n(y_{n+1}) = 0
\end{aligned}$$

Thus for any $n, m \in N_0$,

$$d_n(y_m) = 0 \tag{6}$$

For all $i, j, k \in N_0$, we may without loss of generality. Assume that $i < j$ it follows from (6)

$$\begin{aligned} d(y_i, y_j, y_k) &\le d_i(y_i) + d_j(y_k) + d(y_{i+1}, y_j, y_k) \\ &= d(y_{i+1}, y_j, y_k) \\ &\le d(y_{i+2}, y_j, y_k) \end{aligned}$$

And inductively, we have

$$d(y_i, y_j, y_k) \le d(y_{j-1}, y_j, y_k) = d_{j-1}(y_k) = 0$$

Therefore

$$d(y_i, y_j, y_k) = 0 \tag{7}$$

Applying (1) Again and using (4), (5), (6), we have

$$\begin{aligned} d_{2n}(u) &= d(y_{2n}, y_{2n+1}, u) \\ &= d(fx_{2n}, gx_{2n+1}, u) \\ &\le a \max\{d(hx_{2n}, tx_{2n+1}, u), d(tx_{2n+1}, gx_{2n+1}, u)\} + b \max\{d(hx_{2n}, fx_{2n}, u), d(tx_{2n+1}, gx_{2n+1}, u), d(tx_{2n+1}, fx_{2n}, u)\} \\ &+ c[d(hx_{2n}, gx_{2n+1}, u) + d(tx_{2n+1}, fx_{2n}, u)] - w[a \max\{d(hx_{2n}, tx_{2n+1}, u), d(tx_{2n+1}, gx_{2n+1}, u)\} \\ &+ b \max\{d(hx_{2n}, fx_{2n}, u), d(tx_{2n+1}, gx_{2n+1}, u), d(tx_{2n+1}, fx_{2n}, u)\} + c[d(hx_{2n}, gx_{2n+1}, u) + d(tx_{2n+1}, fx_{2n}, u)]] \\ &\le a \max\{d(y_{2n-1}, y_{2n}, u), d(y_{2n}, y_{2n+1}, u)\} + b \max\{d(y_{2n-1}, y_{2n}, u), d(y_{2n}, y_{2n+1}, u), d(y_{2n}, y_{2n}, u)\} \\ &+ c[d(y_{2n-1}, y_{2n+1}, u) + d(y_{2n}, y_{2n}, u)] - w[a \max\{d(y_{2n-1}, y_{2n}, u), d(y_{2n}, y_{2n+1}, u)\} \\ &+ b \max\{d(y_{2n-1}, y_{2n}, u), d(y_{2n}, y_{2n+1}, u), d(y_{2n}, y_{2n}, u)\} + c[d(y_{2n-1}, y_{2n+1}, u) + d(y_{2n}, y_{2n}, u)]] \\ &\le a \max\{d_{2n-1}(u), d_{2n}(u)\} + bd_{2n}(u) + cd(y_{2n-1}, y_{2n+1}, u) \\ &- w[a \max\{d_{2n-1}(u), d_{2n}(u)\} + bd_{2n}(u) + cd(y_{2n-1}, y_{2n+1}, u)] \\ &\le a \max\{d_{2n-1}(u), d_{2n}(u)\} + bd_{2n}(u) + c[d_{2n-1}(u) + d_{2n}(u) + d_{2n+1}(y_{2n-1})] \\ &- w[a \max\{d_{2n-1}(u), d_{2n}(u)\} + bd_{2n}(u) + c[d_{2n-1}(u) + d_{2n}(u) + d_{2n+1}(y_{2n-1})] \\ &\le a \max\{d_{2n-1}(u), d_{2n-1}(u)\} + bd_{2n-1}(u) + c[d_{2n-1}(u) + d_{2n-1}(u)] \\ &- w[a \max\{d_{2n-1}(u), d_{2n-1}(u)\} + bd_{2n-1}(u) + c[d_{2n-1}(u) + d_{2n-1}(u)] \\ &\le (a + b + 2c)d_{2n-1}(u) - w(a + b + 2c)d_{2n-1}(u) \\ &= d_{2n-1}(u) - wd_{2n-1}(u) \end{aligned}$$

Similarly we have

$$d_{2n+1}(u) \le d_{2n}(u) - wd_{2n}(u)$$

It follows that

$$\sum_{i=0}^{n} w(d_i(u)) \leq \sum_{i=0}^{n} [d_i(u) - d_{i+j}(u)]$$
$$\sum_{i=0}^{n} w(d_i(u)) \leq d_0(u) - d_{n+1}(u)$$
$$\vdots \qquad \vdots$$
$$\leq d_0(u)$$

So the series of non negative, terms $\sum_{n=0}^{\infty} w(d_n(u))$ is convergent. This means that

$$\lim_{n\to\infty} w(d_n(u)) = 0. \tag{8}$$

Using (4), we have $d_n(u)_{n\in N_0}$ converges to some $r \geq 0$. By continuity of w and (8), we have

$$w(r) = \lim_{n\to\infty} w(d_n(u)) = 0$$

Which implies that $r = 0$. Hence

$$\lim_{n\to\infty} d_n(u) = 0. \tag{9}$$

In order to show that y_n is a Cauchy sequence it is sufficient to show that $y_{2n_{n\in N_0}}$ is a Cauchy sequence. Suppose not; then there exist $\varepsilon > 0$ and $u \in X$ such that for each positive integer k,there are positive integers $2m(k)$ and $2n(k)$ with $2m(k) > 2n(k) > 2k$ and $d(y_{2m(k)}, y_{2n(k)}, u) \geq \varepsilon$. For each positive integer k, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying the above inequality, so that

$$d(y_{2m(k)-2}, y_{2n(k)}, u) \leq \varepsilon \;\; d(y_{2m(k)}, y_{2n(k)}, u) > \varepsilon. \tag{10}$$

For each positive integer k, from (7) and (10), we have

$$\begin{aligned}
\varepsilon &< d(y_{2m(k)}, y_{2n(k)}, u)\\
&\leq d(y_{2m(k)-2}, y_{2n(k)}, u) + d(y_{2m(k)}, y_{2m(k)-2}, u) + d(y_{2m(k)}, y_{2n(k)}, y_{2m(k)-2})\\
&\leq \varepsilon + d(y_{2m(k)-2}, y_{2m(k)}, y_{2m(k)-1}) + d(y_{2m(k)-2}, y_{2m(k)-1}, u) + d(y_{2m(k)-1}, y_{2m(k)}, u)\\
&= \varepsilon + d(y_{2m(k)-2}(u), y_{2m(k)-1}(u))
\end{aligned}$$

Which implies

$$\lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}, u) = \varepsilon \tag{11}$$

It follows from (10)

$$\begin{aligned}
0 &< d(y_{2n(k)}, y_{2m(k)}, u) - d(y_{2n(k)}, y_{2m(k)-2}, u))\\
&\leq d(y_{2m(k)-2}, y_{2m(k)}, u)\\
&\leq d_{2m(k)-2}(u) + d_{2m(k)-1}(u)
\end{aligned}$$

Then by (9) and (11), we have

$$\lim_{n\to\infty} d(y_{2n(k)}, y_{2m(k)-2}, u) = \varepsilon \tag{12}$$

Using triangular inequality, we have

$$|d(y_{2n(k)}, y_{2m(k)-1}, u) - d(y_{2n(k)}, y_{2m(k)}, u)| \le d_{2m(k)-1}(u) + d_{2m(k)-1}(y_{2n(k)})$$
$$|d(y_{2n(k)+1}, y_{2m(k)}, u) - d(y_{2n(k)}, y_{2m(k)}, u)| \le d_{2n(k)}(u) + d_{2n(k)}(y_{2m(k)})$$
$$|d(y_{2n(k)+1}, y_{2m(k)-1}, u) - d(y_{2n(k)}, y_{2m(k)-1}, u)| \le d_{2n(k)}(u) + d_{2n(k)}(y_{2m(k)-1})$$

It is easy to see that

$$\lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)-1}, u) = \lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)}, u)$$
$$\lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)-1}, u) = \varepsilon \tag{13}$$

It follows from (5)

$$\begin{aligned}
d(y_{2m(k)}, y_{2n(k)+1}, u) &= d(fx_{2m(k)}, gx_{2n(k)+1}, u) \\
&\le a\max\left\{d(hx_{2m(k)}, tx_{2n(k)+1}, u), d(tx_{2n(k)+1}, gx_{2n(k)+1}, u)\right\} \\
&+ b\max\left\{d(hx_{2m(k)}, fx_{2m(k)}, u), d(tx_{2n(k)+1}, gx_{2n(k)+1}, u), d(tx_{2n(k)+1}, fx_{2m(k)}, u)\right\} \\
&+ c[d(hx_{2m(k)}, gx_{2n(k)+1}, u) + d(tx_{2n(k)+1}, fx_{2m(k)}, u)] \\
&- w[a\max\left\{d(hx_{2m(k)}, tx_{2n(k)+1}, u), d(tx_{2n(k)+1}, gx_{2n(k)+1}, u)\right\} \\
&+ b\max\left\{d(hx_{2m(k)}, fx_{2m(k)}, u), d(tx_{2n(k)+1}, gx_{2n(k)+1}, u), d(tx_{2n(k)+1}, fx_{2m(k)}, u)\right\} \\
&+ c[d(hx_{2m(k)}, gx_{2n(k)+1}, u) + d(tx_{2n(k)+1}, fx_{2m(k)}, u)]] \\
&\le a\max\left\{d(y_{2m(k)-1}, y_{2n(k)}, u), d(y_{2n(k)}, y_{2n(k)+1}, u)\right\} \\
&+ b\max\left\{d(y_{2m(k)-1}, y_{2m(k)}, u), d(y_{2n(k)}, y_{2n(k)+1}, u), d(y_{2n(k)}, y_{2m(k)}, u)\right\} \\
&+ c[d(y_{2m(k)-1}, y_{2n(k)+1}, u) + d(y_{2n(k)}, y_{2m(k)}, u)] - w[a\max\left\{d(y_{2m(k)-1}, y_{2n(k)}, u), d(y_{2n(k)}, y_{2n(k)+1}, u)\right\} \\
&+ b\max\left\{d(y_{2m(k)-1}, y_{2m(k)}, u), d(y_{2n(k)}, y_{2n(k)+1}, u), d(y_{2n(k)}, y_{2m(k)}, u)\right\} \\
&+ c[d(y_{2m(k)-1}, y_{2n(k)+1}, u) + d(y_{2n(k)}, y_{2m(k)}, u)]] \\
&\le a\max\left\{d(y_{2m(k)-1}, y_{2n(k)}, u), d_{2n(k)}(u)\right\} + b\max\left\{d_{2m(k)-1}(u), d_{2n(k)}(u), d(y_{2n(k)}, y_{2m(k)}, u)\right\} \\
&+ c[d(y_{2m(k)-1}, y_{2n(k)+1}, u) + d(y_{2n(k)}, y_{2m(k)}, u)] - w[a\max\left\{d(y_{2m(k)-1}, y_{2n(k)}, u), d_{2n(k)}(u)\right\} \\
&+ b\max\left\{d_{2m(k)-1}(u), d_{2n(k)}(u), d(y_{2n(k)}, y_{2m(k)}, u)\right\} + c[d(y_{2m(k)-1}, y_{2n(k)+1}, u) + d(y_{2n(k)}, y_{2m(k)}, u)]]
\end{aligned}$$

Letting $k\to\infty$ and using (13), (11), (9), we have

$$\begin{aligned}
\varepsilon &\le [a\varepsilon + b\varepsilon + 2\varepsilon c] - w[a\varepsilon + b\varepsilon + 2\varepsilon c] \\
&\le [a+b+2c]\varepsilon - w[a+b+2c]\varepsilon \\
&= \varepsilon - w\varepsilon
\end{aligned}$$

A contradiction. Therefore ${y_{2n}}_{n\in N_0}$ is a Cauchy sequence in X. It follows from completeness of (X,d) that ${y_{2n}}_{n\in N_0}$ converge to a point $z\in X$. Now suppose that $t$is continuous. Since f and t are compatible and ${gx_{2n+1}}_{n\in N_0}$ and ${tx_{2n+1}}_{n\in N_0}$ converge to the point z, by Lemma 2.8, we get $gtx_{2n+1}, tgx_{2n+1} \to tz$ as $h \to \infty$. Applying in equality (1), we have

$$\begin{aligned}
d(fx_{2n}, gtx_{2n+1}, u) \leq\ & a \max\{d(hx_{2n}, ttx_{2n+1}, u), d(ttx_{2n+1}, gtx_{2n+1}, u)\} \\
& + b \max\{d(hx_{2n}, fx_{2n}, u), d(ttx_{2n+1}, gtx_{2n+1}, u), d(ttx_{2n+1}, fx_{2n}, u)\} \\
& + c[d(hx_{2n}, gtx_{2n+1}, u) + d(ttx_{2n+1}, fx_{2n}, u)] - w[a \max\{d(hx_{2n}, ttx_{2n+1}, u), d(ttx_{2n+1}, gtx_{2n+1}, u)\} \\
& + b \max\{d(hx_{2n}, fx_{2n}, u), d(ttx_{2n+1}, gtx_{2n+1}, u), d(ttx_{2n+1}, fx_{2n}, u)\} \\
& + c[d(hx_{2n}, gtx_{2n+1}, u) + d(ttx_{2n+1}, fx_{2n}, u)]]
\end{aligned}$$

Letting $n \to \infty$ we get

$$\begin{aligned}
d(z, tz, u) \leq\ & a \max\{d(z, tz, u), d(tz, tz, u)\} + b \max\{d(z, z, u), d(tz, tz, u), d(ttz, z, u)\} \\
& + c[d(z, tz, u) + d(ttz, z, u)] - w[a \max\{d(z, tz, u), d(tz, tz, u)\} \\
& + b \max\{d(z, z, u), d(tz, tz, u), d(ttz, z, u)\} + c[d(z, tz, u) + d(ttz, z, u)]] \\
& \leq ad(z, tz, u) + bd(tz, z, u) + c[d(z, tz, u) + d(tz, z, u)] - w[ad(z, tz, u) + bd(tz, z, u) + c[d(z, tz, u) + d(tz, z, u)]] \\
& \leq (a + b + 2c)d(z, tz, u) - w[(a + b + 2c)d(z, tz, u)] \\
& \leq d(z, tz, u)
\end{aligned}$$

Implies $d(z, tz, u) = 0 \Rightarrow z = tz$. Again from (1), we have

$$\begin{aligned}
d(fx_{2n}, gz, u) \leq\ & a \max\{d(hx_{2n}, tz, u), d(tz, gz, u)\} + b \max\{d(hx_{2n}, fx_{2n}, u), d(tz, gz, u), d(tz, fx_{2n}, u)\} \\
& + c[d(hx_{2n}, gz, u) + d(tz, fx_{2n}, u)] - w[a \max\{d(hx_{2n}, tz, u), d(tz, gz, u)\} \\
& + b \max\{d(hx_{2n}, fx_{2n}, u), d(tz, gz, u), d(tz, fx_{2n}, u)\} + c[d(hx_{2n}, gz, u) + d(tz, fx_{2n}, u)]]
\end{aligned}$$

Letting $n \to \infty$ we get

$$\begin{aligned}
d(z, gz, u) \leq\ & a \max\{d(z, z, u), d(z, gz, u)\} + b \max\{d(z, z, u), d(z, gz, u), d(z, z, u)\} \\
& + c[d(z, gz, u) + d(z, z, u)] - w[a \max\{d(z, z, u), d(z, gz, u)\} \\
& + b \max\{d(z, z, u), d(z, gz, u), d(z, z, u)\} + c[d(z, gz, u) + d(z, z, u)]] \\
& \leq (a + b + c)d(z, gz, u) - w[(a + b + c)d(z, gz, u)] \\
& < d(z, gz, u)
\end{aligned}$$

Hence $z = gz$ i.e. z is a fixed point of g. Similarly, we can show that z is a fixed point of f and h i.e. z is a common fixed point of f, g, h and t. Similarly, we can complete the proof when f or g or h is continuous. □

Theorem 3.2. *Let (X, d) be a complete 2-metric space with d continuous on X and let h and t be two mapping of X into itself, there exists $w \in N_0$, $f : X \to t(X) \to h(X)$ satisfying:*

(a) The pair (f, h) and (f, t) are compatible.

(b) One of f, h and t is continuous

(c)

$$d(fx, fy, u) \leq a \max\{d(hx, ty, u), d(ty, fy, u)\} + b \max\{d(hx, fx, u), d(ty, fy, u), d(ty, fx, u)\}$$
$$+ c[d(hx, fy, u) + d(ty, fx, u)] - w[a \max\{d(hx, ty, u), d(ty, fy, u)\}$$
$$+ b \max\{d(hx, fx, u), d(ty, fy, u), d(ty, fx, u)\} + c[d(hx, fy, u) + d(ty, fx, u)]] \quad (14)$$

Where $a \geq 0$, $b > 0$, $c > 0$ such that $a + b + 2c = 1$ for all $x, y, u \in X$ then f, h and t have a common fixed point.

Proof. The proof of this theorem is identical to the proof of Theorem 3.1. □

Remark 3.3. *Theorem 3.1 and 3.2 are still true even though the condition of the compatibility is replaced by the compatibility of type(A).*

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