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Parametric Metric Space, Parametric b-metric Space and Expansive Type Mapping

Research Article

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Abstract: In this article, we present some fixed point theorems under various expansive conditions in parametric metric spaces and parametric b-metric spaces. The presented theorems extend, generalize and improve many existing results in [13]. Also we prove a common fixed point theorem, which is generalization of Theorem 2.2 of Jain et al. [24] in the setting of parametric b-metric space. We introduce an example the support the validity of our result.

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1. Introduction

Fixed point theory has fascinated many researchers since 1922 with the celebrated Banach's fixed point theorem. There exists a vast literature on the topic and is a very active field of research at present. Theorems concerning the existence and properties of fixed points are known as fixed point theorems. Such theorems are very important tool for proving the existence and eventually the uniqueness of the solutions to various mathematical models (integral and partial differential equations, variational inequalities).

The study of expansive mappings is a very interesting research area in fixed point theory. Wang et.al [20] introduced the concept of expanding mappings and proved some fixed point theorems in complete metric spaces. Daffer and Kaneko [19] defined an expanding condition for a pair of mappings and proved some common fixed point theorems for two mappings in complete metric spaces. Aage and Salunke [21] introduced several meaningful fixed point theorems for one expanding mapping. Jain et al. [13] proved some fixed point theorems for continuous and surjective expansive mappings in dislocated metric spaces.

The concept of metric spaces has been generalized in many directions. The notion of a b-metric space was studied by Czerwik in [9, 10] and a lot of fixed point results for single and multi-valued mappings by many authors have been obtained in (ordered) b-metric spaces. Alghamdi, et al. [11] obtained some fixed point nt and coupled fixed point theorems on b-metric-like spaces. In 2004, Park introduced the notion of intuitionistic fuzzy metric space [6]. He showed that, for each

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intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$, the topology generated by the intuitionistic fuzzy metric (M, N) coincides with the topology generated by the fuzzy metric M. The notion of parametric metric spaces being a natural generalization of metric spaces was recently introduced and studied by Hussain et al. [7]. Hussain et al. [8] introduced a new type of generalized metric space, called parametric b-metric space, as a generalization of both metric and b-metric spaces. For more details on parametric metric space, parametric b-metric space and related results we refer the reader to [7, 8, 24].

In this paper, we present some fixed point and common point theorems under various expansive conditions in parametric metric spaces and parametric b-metric spaces. These results improve and generalize some important known results in [13]. Also we prove a common fixed point theorem, which is generalization of Theorem 2.2 of Jain et al. [24] in the setting of parametric b-metric space. We introduce an example the support the validity of our result.

2. Preliminaries

Throughout this paper \mathbb{R} and \mathbb{R}^+ will represents the set of real numbers and nonnegative real numbers, respectively. In 2014, Hussain et al. [7] defined and studied the concept of parametric metric space as follows.

Definition 2.1. Let X be a nonempty set and $\mathcal{P} : X \times X \times (0, +\infty) \rightarrow [0, +\infty)$ be a function. We say \mathcal{P} is a parametric metric on X if,

- (1). $\mathcal{P}(x, y, t) = 0$ for all t > 0 if and only if x = y;
- (2). $\mathcal{P}(x, y, t) = \mathcal{P}(y, x, t)$ for all t > 0;

(3). $\mathcal{P}(x, y, t) \leq \mathcal{P}(x, z, t) + \mathcal{P}(z, y, t)$ for all $x, y, z \in X$ and all t > 0:

and one says the pair (X, \mathcal{P}) is a parametric metric space.

The following definitions and results are required in the sequel which can be found in [7, 24].

Definition 2.2. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a parametric metric space (X, \mathcal{P}) .

- (1). $\{x_n\}_{n=1}^{\infty}$ is said to be convergent to $x \in X$, written as $\lim_{n \to \infty} x_n = x$, for all t > 0, if $\lim_{n \to \infty} \mathcal{P}(x_n, x, t) = 0$.
- (2). $\{x_n\}_{n=1}^{\infty}$ is said to be a Cauchy sequence in X if for all t>0, if $\lim_{n \to \infty} \mathcal{P}(x_n, x_m, t) = 0$.

(3). (X,\mathcal{P}) is said to be complete if every Cauchy sequence is a convergent sequence.

Definition 2.3. Let (X,\mathcal{P}) be a parametric metric space and $T:X \to X$ be a mapping. We say T is a continuous mapping at x in X, if for any sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\lim_{n \to \infty} x_n = x$, then $\lim_{n \to \infty} Tx_n = Tx$.

Example 2.4. Let X denote the set of all functions $f:(0,+\infty) \to \mathbb{R}$. Define $\mathcal{P}: X \times X \times (0,+\infty) \to [0,+\infty)$ by $\mathcal{P}(f,g,t) = |f(t) - g(t)|, \forall f, g \in X$ and all t > 0. Then \mathcal{P} is a parametric metric on X and the pair (X, \mathcal{P}) is a parametric metric space.

Lemma 2.5. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a parametric metric space (X, \mathcal{P}) such that $\mathcal{P}(x_n, x_{n+1}, t) \preceq \lambda \mathcal{P}(x_{n-1}, x_n, t)$ where $\lambda \in [0, 1)$ and $n = 1, 2, \ldots$ Then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, \mathcal{P}) .

Hussain et al. [8] introduced the concept of parametric b-metric space as follows.

Definition 2.6. Let X be a nonempty set, $s \ge 1$ be a real number and $\mathcal{P} : X \times X \times (0, +\infty) \to [0, +\infty)$ be a function. We say \mathcal{P} is a parametric b-metric on X if,

(1). $\mathcal{P}(x, y, t) = 0$ for all t > 0 if and only if x=y;

(2). $\mathcal{P}(x, y, t) = \mathcal{P}(y, x, t)$ for all t > 0;

(3). $\mathcal{P}(x, y, t) \leq s \left[\mathcal{P}(x, z, t) + \mathcal{P}(z, y, t)\right]$ for all $x, y, z \in X$ and all t > 0, where $s \geq 1$.

and one says the pair $(X, \mathcal{P},)$ is a parametric metric space with parameter $s \ge 1$. Obviously, for s = 1, parametric b-metric reduces to parametric metric.

The following definitions and results will be needed in the sequel which can be found in [8, 24].

Definition 2.7. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a parametric b-metric space (X, \mathcal{P}, s) .

(1). $\{x_n\}_{n=1}^{\infty}$ is said to be convergent to $x \in X$, written as $\lim_{n \to \infty} x_n = x$, for all t > 0, if $\lim_{n \to \infty} \mathcal{P}(x_n, x, t) = 0$.

(2). $\{x_n\}_{n=1}^{\infty}$ is said to be a Cauchy sequence in X if for all t>0, if $\lim_{n,m\to\infty} \mathcal{P}(x_n, x_m, t) = 0$.

(3). (X, \mathcal{P}, s) is said to be complete if every Cauchy sequence is a convergent sequence.

Example 2.8. Let $X = [0, +\infty)$ and define $\mathcal{P} : X \times X \times (0, +\infty) \to [0, +\infty)$ by $\mathcal{P}(x, y, t) = t(x-y)^p$. Then \mathcal{P} is a parametric b-metric with constant $s=2^p$.

Definition 2.9. Let (X, \mathcal{P}, s) be a parametric b-metric space and $T: X \to X$ be a mapping. We say T is a continuous mapping at x in X, if for any sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\lim_{n \to \infty} x_n = x$, then $\lim_{n \to \infty} Tx_n = Tx$.

Lemma 2.10. Let (X, \mathcal{P}, s) be a b-metric space with the coefficient s = 1 and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. if $\{x_n\}_{n=1}^{\infty}$ converges to x and also $\{x_n\}_{n=1}^{\infty}$ converges to y, then x=y. That is, the limit of $\{x_n\}_{n=1}^{\infty}$ is unique.

Lemma 2.11. Let (X, \mathcal{P}, s) be a b-metric space with the coefficient s = 1 and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. If $\{x_n\}_{n=1}^{\infty}$ converges to x. Then

$$\frac{1}{s}\mathcal{P}\left(x,y,t\right) \leq \lim_{n \to +\infty} \mathcal{P}\left(x_{n},y,t\right) \leq s \mathcal{P}\left(x,y,t\right)$$

 $\forall y \in X and all t > 0.$

Lemma 2.12. Let (X, \mathcal{P}, s) be a b-metric space with the coefficient s = 1 and let $\{x_k\}_{k=0}^n \subset X$. Then

$$\mathcal{P}(x_n, x_0, t) \le s\mathcal{P}(x_0, x_1, t) + s^2 \mathcal{P}(x_2, x_3, t) + \ldots + s^{n-1} \mathcal{P}(x_{n-2}, x_{n-1}, t) + s^{n-1} \mathcal{P}(x_{n-1}, x_n, t)$$

Lemma 2.13. Let (X, \mathcal{P}, s) be a parametric metric space with the coefficient s = 1. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points of X such that

$$\mathcal{P}(x_n, x_{n+1}, t) \preceq \lambda \mathcal{P}(x_{n-1}, x_n, t)$$

where $\lambda \in [0, \frac{1}{s})$ and $n = 1, 2, \dots$ Then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, \mathcal{P}, s) .

3. Fixed Point Theorems in Parametric Metric Space

In this section, we prove some fixed point theorem for continuous and surjective mapping satisfying expansion condition, which is generalization of results of [13] in the setting of complete parametric metric space. **Theorem 3.1.** Let (X, \mathcal{P}) be a complete parametric metric space and T a continuous mapping satisfying the following condition:

$$\mathcal{P}(Tx, Ty, t) + \alpha \max\{\mathcal{P}(x, Ty, t), \mathcal{P}(y, Tx, t)\} \ge \beta \frac{\mathcal{P}(x, Tx, t)\left[1 + \mathcal{P}(y, Ty, t)\right]}{1 + \mathcal{P}(x, y, t)} + \gamma \mathcal{P}(x, y, t)$$
(1)

for all $x, y \in X$, $x \neq y$, and for all t > 0, where $\alpha, \beta, \gamma \ge 0$ are real constants and $\beta + \gamma > 1 + 2\alpha, \gamma > 1 + \alpha$. Then T has a fixed point in X.

Proof. Choose $x_0 \in X$ be arbitrary, to define the iterative sequence $\{x_n\}_{n \in \mathbb{N}}$ as follows, $Tx_n = x_{n-1}$ for $n = 1, 2, 3, \ldots$. Taking $x = x_n$ and $y = x_{n+1}$ in (1), we obtain

$$\mathcal{P}(Tx_{n+1}, Tx_{n+2}, t) + \alpha \max \{ \mathcal{P}(x_{n+1}, Tx_{n+2}, t), \mathcal{P}(x_{n+2}, Tx_{n+1}, t) \}$$

$$\geq \beta \frac{\mathcal{P}(x_{n+1}, Tx_{n+1}, t) \left[1 + \mathcal{P}(x_{n+2}, Tx_{n+2}, t) \right]}{1 + \mathcal{P}(x_{n+1}, x_{n+2}, t)} + \gamma \mathcal{P}(x_{n+1}, x_{n+2}, t)$$

$$\Rightarrow \mathcal{P}(x_n, x_{n+1}, t) + \alpha \max \{ \mathcal{P}(x_{n+1}, x_{n+1}, t), \mathcal{P}(x_{n+2}, x_n, t) \}$$

$$\geq \beta \frac{\mathcal{P}(x_{n+1}, x_n, t) \left[1 + \mathcal{P}(x_{n+2}, x_{n+1}, t) \right]}{1 + \mathcal{P}(x_{n+1}, x_{n+2}, t)} + \gamma \mathcal{P}(x_{n+1}, x_{n+2}, t)$$

$$\Rightarrow \mathcal{P}(x_n, x_{n+1}, t) + \alpha \max \{ \mathcal{P}(x_{n+1}, x_{n+1}, t), \mathcal{P}(x_{n+2}, x_n, t) \}$$

$$\geq \beta \frac{\mathcal{P}(x_{n+1}, x_n, t) \left[1 + \mathcal{P}(x_{n+1}, x_{n+2}, t) \right]}{1 + \mathcal{P}(x_{n+1}, x_{n+2}, t)} + \gamma \mathcal{P}(x_{n+1}, x_{n+2}, t)$$

$$\Rightarrow \mathcal{P}(x_n, x_{n+1}, t) + \alpha \mathcal{P}(x_n, x_{n+2}, t) \ge \beta \mathcal{P}(x_n, x_{n+1}, t) + \gamma \mathcal{P}(x_{n+1}, x_{n+2}, t)$$

$$\Rightarrow \mathcal{P}(x_n, x_{n+1}, t) + \alpha \mathcal{P}(x_n, x_{n+1}, t) + \alpha \mathcal{P}(x_{n+1}, x_{n+2}, t) \ge \beta \mathcal{P}(x_n, x_{n+1}, t) + \gamma \mathcal{P}(x_{n+1}, x_{n+2}, t)$$

$$\Rightarrow (1+\alpha-\beta) \mathcal{P}(x_n, x_{n+1}, t) \ge (\gamma-\alpha) \mathcal{P}(x_{n+1}, x_{n+2}, t)$$

for all t>0. The last inequality gives

$$\mathcal{P}(x_{n+1}, x_{n+2}, t) \leq \frac{1 + \alpha - \beta}{\gamma - \alpha} \mathcal{P}(x_n, x_{n+1}, t)$$

$$= \lambda \mathcal{P}(x_n, x_{n+1}, t)$$
(2)

for all t>0, where $\lambda = \frac{1+\alpha-\beta}{\gamma-\alpha} < 1$. Hence by induction, we obtain

$$\mathcal{P}\left(x_{n+1}, x_{n+2}, t\right) \leq \lambda^{n+1} \mathcal{P}\left(x_0, x_1, t\right) \tag{3}$$

By Lemma 2.5, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. But X is a complete parametric metric space; hence, $\{x_n\}_{n\in\mathbb{N}}$ is converges. Call the limit $x^*\in X$. Then, $x_n\to x^*$ as $n\to +\infty$. By continuity of T we have,

$$Tx^{*} = T\left(\lim_{n \to \infty} x_{n}\right)$$

$$= \lim_{n \to \infty} Tx_{n}$$

$$= \lim_{n \to \infty} x_{n-1} = x^{*}$$
(4)

That is, $Tx^* = x^*$; thus, T has a fixed point in X.

Uniqueness

Let y^* be another fixed point of T in X, then $Ty^* = y^*$ and $Tx^* = x^*$. Now,

$$\mathcal{P}\left(Tx^{*},Ty^{*},t\right)+\alpha \max\left\{\mathcal{P}\left(x^{*},Ty^{*},t\right),\mathcal{P}\left(y^{*},Tx^{*},t\right)\right\} \geq \beta \frac{\mathcal{P}\left(x^{*},Tx^{*},t\right)\left[1+\mathcal{P}\left(y^{*},Ty^{*},t\right)\right]}{1+\mathcal{P}\left(x^{*},y^{*},t\right)}+\gamma \mathcal{P}\left(x^{*},y^{*},t\right)$$

This implies that

$$\mathcal{P}(x^*, y^*, t) + \alpha \mathcal{P}(x^*, y^*, t) \ge \gamma \mathcal{P}(x^*, y^*, t)$$

That is

$$\mathcal{P}(x^*, y^*, t) \ge (\gamma - \alpha) \mathcal{P}(x^*, y^*, t)$$

$$\Rightarrow \mathcal{P}(x^*, y^*, t) \le \frac{1}{\gamma - \alpha} \mathcal{P}(x^*, y^*, t)$$
(5)

This is true only when $\mathcal{P}(x^*, y^*, t) = 0$. So $x^* = y^*$. Hence T has a unique fixed point in X.

Next we prove Theorem 3.1 for surjective mapping.

Theorem 3.2. Let (X, \mathcal{P}) be a complete parametric metric space and T a surjective mapping satisfying the condition (1) for all $x, y \in X$, $x \neq y$, and all t > 0, where $\alpha, \beta, \gamma \ge 0$ are real constants and $\alpha + \beta > 1 + 2\alpha, \gamma > 1 + \alpha$. Then, T has a fixed point in X.

Proof. Choose $x_0 \in X$ to be arbitrary, and define the iterative sequence $\{x_n\}_{n \in \mathbb{N}}$ as follows: $Tx_n = x_{n-1}$ for $n = 1, 2, 3, \ldots$. Then, using (1), we obtain, sequence $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in X. But X is a complete parametric metric space; hence $\{x_n\}_{n \in \mathbb{N}}$ is converges. Call the limit $x^* \in X$. Then, $x_n \to x^*$ as $n \to +\infty$.

Existence of Fixed Point

Since T is a surjective map, so there exists a point y in X, such that x=Ty. Consider

$$\mathcal{P}(x_{n}, x, t) = \mathcal{P}(Tx_{n+1}, Ty, t)$$

$$\geq -\alpha \max \left\{ \mathcal{P}(x_{n+1}, Ty, t), \mathcal{P}(y, Tx_{n+1}, t) \right\} + \beta \frac{\mathcal{P}(x_{n+1}, Tx_{n+1}, t) \left[1 + \mathcal{P}(y, Ty, t)\right]}{1 + \mathcal{P}(x_{n+1}, y, t)} + \gamma \mathcal{P}(x_{n+1}, y, t) .$$
(6)

Taking $n \to +\infty$, we get

$$\mathcal{P}(x,x,t) \ge -\alpha \max\left\{\mathcal{P}(x,x,t), \mathcal{P}(y,x,t)\right\} + \beta \frac{\mathcal{P}(x,x,t)\left[1 + \mathcal{P}(y,x,t)\right]}{1 + \mathcal{P}(x,y,t)} + \gamma \mathcal{P}(x,y,t) \tag{7}$$

$$\Rightarrow 0 \ge -\alpha \mathcal{P}(x, y, t) + \gamma \mathcal{P}(y, x, t)$$
$$\Rightarrow (\gamma - \alpha) \mathcal{P}(x, y, t) \le 0$$
$$\Rightarrow \mathcal{P}(x, y, t) = 0 \text{ as } \gamma > \alpha.$$

Hence x=y and so Tx=x, that is, x is a fixed point of T.

Uniqueness

Let y^* be another fixed point of T in X, then $Ty^* = y^*$ and $Tx^* = x^*$. Now,

$$\mathcal{P}\left(Tx^{*}, Ty^{*}, t\right) + \alpha \max\left\{\mathcal{P}\left(x^{*}, Ty^{*}, t\right), \mathcal{P}\left(y^{*}, Tx^{*}, t\right)\right\} \ge \beta \frac{\mathcal{P}\left(x^{*}, Tx^{*}, t\right)\left[1 + \mathcal{P}\left(y^{*}, Ty^{*}, t\right)\right]}{1 + \mathcal{P}\left(x^{*}, y^{*}, t\right)} + \gamma \mathcal{P}\left(x^{*}, y^{*}, t\right) \tag{8}$$

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This implies that

$$\mathcal{P}\left(\boldsymbol{x}^{*},\boldsymbol{y}^{*},t\right) + \alpha \mathcal{P}\left(\boldsymbol{x}^{*},\boldsymbol{y}^{*},t\right) \geq \gamma \mathcal{P}\left(\boldsymbol{x}^{*},\boldsymbol{y}^{*},t\right)$$

That is

$$\mathcal{P}(x^*, y^*, t) \ge (\gamma - \alpha) \mathcal{P}(x^*, y^*, t)$$

$$\Rightarrow \mathcal{P}(x^*, y^*, t) \le \frac{1}{\gamma - \alpha} \mathcal{P}(x^*, y^*, t)$$
(9)

This is true only when $\mathcal{P}(x^*, y^*, t) = 0$. Hence $x^* = y^*$. Hence T has a unique fixed point in X. The proof is completed. \Box

4. Fixed Point Theorems in Parametric b-Metric Space

In this section, we prove some fixed point theorem for continuous and surjective mapping satisfying expansion condition, which is generalization of results of [13] in the setting of complete parametric b-metric space.

Theorem 4.1. Let (X, \mathcal{P}, s) be a complete parametric b-metric space and T a continuous mapping satisfying the following condition:

$$\mathcal{P}(Tx, Ty, t) + \alpha \max\{\mathcal{P}(x, Ty, t), \mathcal{P}(y, Tx, t)\} \ge \beta \frac{\mathcal{P}(x, Tx, t)\left[1 + \mathcal{P}(y, Ty, t)\right]}{1 + \mathcal{P}(x, y, t)} + \gamma \mathcal{P}(x, y, t)$$
(10)

for all $x, y \in X$, $x \neq y$, and for all t > 0, where $\alpha, \beta, \gamma \ge 0$ are real constants and $s\beta + \gamma > (1+\alpha) s + s^2 \alpha, \gamma > 1 + \alpha$. Then T has a fixed point in X.

Proof. Choose $x_0 \in X$ be arbitrary, to define the iterative sequence $\{x_n\}_{n \in \mathbb{N}}$ as follows: $Tx_n = x_{n-1}$ for $n = 1, 2, 3, \ldots$. Taking $x = x_{n+1}$ and $y = x_{n+2}$ in (10), we obtain

$$\mathcal{P}(Tx_{n+1}, Tx_{n+2}, t) + \alpha \max \{ \mathcal{P}(x_{n+1}, Tx_{n+2}, t), \mathcal{P}(x_{n+2}, Tx_{n+1}, t) \}$$

$$\geq \beta \frac{\mathcal{P}(x_{n+1}, Tx_{n+1}, t) \left[1 + \mathcal{P}(x_{n+2}, Tx_{n+2}, t) \right]}{1 + \mathcal{P}(x_{n+1}, x_{n+2}, t)} + \gamma \mathcal{P}(x_{n+1}, x_{n+2}, t)$$

$$\Rightarrow \mathcal{P}(x_n, x_{n+1}, t) + \alpha \max \{ \mathcal{P}(x_{n+1}, x_{n+1}, t), \mathcal{P}(x_{n+2}, x_n, t) \}$$

$$\geq \beta \frac{\mathcal{P}(x_{n+1}, x_n, t) \left[1 + \mathcal{P}(x_{n+2}, x_{n+1}, t) \right]}{1 + \mathcal{P}(x_{n+1}, x_{n+2}, t)} + \gamma \mathcal{P}(x_{n+1}, x_{n+2}, t)$$

$$\Rightarrow \mathcal{P}(x_n, x_{n+1}, t) + \alpha \max \{ \mathcal{P}(x_{n+1}, x_{n+1}, t), \mathcal{P}(x_{n+2}, x_n, t) \}$$

$$\geq \beta \frac{\mathcal{P}(x_{n+1}, x_n, t) \left[1 + \mathcal{P}(x_{n+1}, x_{n+2}, t) \right]}{1 + \mathcal{P}(x_{n+1}, x_{n+2}, t)} + \gamma \mathcal{P}(x_{n+1}, x_{n+2}, t)$$

$$\Rightarrow \mathcal{P}(x_{n}, x_{n+1}, t) + \alpha \mathcal{P}(x_{n}, x_{n+2}, t) \ge \beta \mathcal{P}(x_{n}, x_{n+1}, t) + \gamma \mathcal{P}(x_{n+1}, x_{n+2}, t)$$

$$\Rightarrow \mathcal{P}(x_{n}, x_{n+1}, t) + \alpha s \left[\mathcal{P}(x_{n}, x_{n+1}, t) + \mathcal{P}(x_{n+1}, x_{n+2}, t) \right] \ge \beta \mathcal{P}(x_{n}, x_{n+1}, t) + \gamma \mathcal{P}(x_{n+1}, x_{n+2}, t)$$

$$\Rightarrow (1 + s\alpha - \beta) \mathcal{P}(x_{n}, x_{n+1}, t) \ge (\gamma - s\alpha) \mathcal{P}(x_{n+1}, x_{n+2}, t)$$

for all t>0. The last inequality gives

$$\mathcal{P}(x_{n+1}, x_{n+2}, t) \leq \frac{1 + s\alpha - \beta}{\gamma - s\alpha} \mathcal{P}(x_n, x_{n+1}, t)$$

$$= \lambda \mathcal{P}(x_n, x_{n+1}, t)$$
(11)

for all t>0, where $\lambda = \frac{1+s\alpha-\beta}{\gamma-s\alpha} < \frac{1}{s}$. Hence by induction, we obtain

$$\mathcal{P}\left(x_{n+1}, x_{n+2}, t\right) \leq \lambda^{n+1} \mathcal{P}\left(x_0, x_1, t\right) \tag{12}$$

By Lemma 2.13, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. But X is a complete parametric b-metric space; hence, $\{x_n\}_{n\in\mathbb{N}}$ is converges. Call the limit $x^*\in X$. Then, $x_n\to x^*$ as $n\to +\infty$. By continuity of T we have

$$Tx^* = T\left(\lim_{n \to \infty} x_n\right) = \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} x_{n-1} = x^*$$

That is, $Tx^* = x^*$; thus, T has a fixed point in X.

Uniqueness

Let y^* be another fixed point of T in X, then $Ty^* = y^*$ and $Tx^* = x^*$. Now,

$$\mathcal{P}\left(Tx^{*}, Ty^{*}, t\right) + \alpha \max\left\{\mathcal{P}\left(x^{*}, Ty^{*}, t\right), \mathcal{P}\left(y^{*}, Tx^{*}, t\right)\right\} \geq \beta \frac{\mathcal{P}\left(x^{*}, Tx^{*}, t\right)\left[1 + \mathcal{P}\left(y^{*}, Ty^{*}, t\right)\right]}{1 + \mathcal{P}\left(x^{*}, y^{*}, t\right)} + \gamma \mathcal{P}\left(x^{*}, y^{*}, t\right) + \alpha \sum_{i=1}^{n} \left[1 + \mathcal{P}\left(y^{*}, y^{*}, t\right)\right] + \alpha \sum_{i=1}^{$$

This implies that

$$\mathcal{P}\left(x^{*}, y^{*}, t\right) + \alpha \mathcal{P}\left(x^{*}, y^{*}, t\right) \geq \gamma \mathcal{P}\left(x^{*}, y^{*}, t\right)$$

That is

$$\mathcal{P}(x^*, y^*, t) \ge (\gamma - \alpha) \mathcal{P}(x^*, y^*, t)$$

$$\Rightarrow \mathcal{P}(x^*, y^*, t) \le \frac{1}{\gamma - \alpha} \mathcal{P}(x^*, y^*, t)$$
(13)

This is true only when $\mathcal{P}(x^*, y^*, t) = 0$. So $x^* = y^*$. Hence T has a unique fixed point in X.

Next we prove Theorem 4.1 for surjective mapping.

Theorem 4.2. Let (X, \mathcal{P}) be a complete parametric b-metric space and T a surjective mapping satisfying the condition (10) for all $x, y \in X$, $x \neq y$, and all t > 0, where $\alpha, \beta, \gamma \ge 0$ are real constants and $s\beta + \gamma > (1+\alpha) s + s^2 \alpha$, $\gamma > 1 + \alpha$. Then, T has a fixed point in X.

Proof. Choose $x_0 \in X$ to be arbitrary, and define the iterative sequence $\{x_n\}_{n \in \mathbb{N}}$ as follows: $Tx_n = x_{n-1}$ for n = 1, 2, 3, ...Then, using (10), we obtain, sequence $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in X. But X is a complete parametric metric space; hence $\{x_n\}_{n \in \mathbb{N}}$ is converges. Call the limit $x^* \in X$. Then, $x_n \to x^*$ as $n \to +\infty$.

Existence of Fixed Point

Since T is a Surjective map, so there exists a point y in X, such that x=Ty. Consider

$$\mathcal{P}(x_{n}, x, t) = \mathcal{P}(Tx_{n+1}, Ty, t)$$

$$\geq -\alpha \max \left\{ \mathcal{P}(x_{n+1}, Ty, t), \mathcal{P}(y, Tx_{n+1}, t) \right\} + \beta \frac{\mathcal{P}(x_{n+1}, Tx_{n+1}, t) \left[1 + \mathcal{P}(y, Ty, t)\right]}{1 + \mathcal{P}(x_{n+1}, y, t)} + \gamma \mathcal{P}(x_{n+1}, y, t).$$
(14)

Taking $n \rightarrow +\infty$, we get

$$\mathcal{P}(x,x,t) \ge -\alpha \max\left\{\mathcal{P}(x,x,t), \mathcal{P}(y,x,t)\right\} + \beta \frac{\mathcal{P}(x,x,t)\left[1 + \mathcal{P}(y,x,t)\right]}{1 + \mathcal{P}(x,y,t)} + \gamma \mathcal{P}(x,y,t)$$
(15)

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 $\Rightarrow 0 \ge -\alpha \mathcal{P}(x, y, t) + \gamma \mathcal{P}(y, x, t)$ $\Rightarrow (\gamma - \alpha) \mathcal{P}(x, y, t) \le 0$ $\Rightarrow \mathcal{P}(x, y, t) = 0 \text{ as } \gamma > \alpha.$

Hence x=y and so Tx=x, that is, x is a fixed point of T.

Uniqueness Let y^* be another fixed point of T in X, then $Ty^* = y^*$ and $Tx^* = x^*$. Now,

$$\mathcal{P}(Tx^{*}, Ty^{*}, t) + \alpha \max \left\{ \mathcal{P}(x^{*}, Ty^{*}, t), \mathcal{P}(y^{*}, Tx^{*}, t) \right\} \ge \beta \frac{\mathcal{P}(x^{*}, Tx^{*}, t) \left[1 + \mathcal{P}(y^{*}, Ty^{*}, t)\right]}{1 + \mathcal{P}(x^{*}, y^{*}, t)} + \gamma \mathcal{P}(x^{*}, y^{*}, t)$$
(16)

This implies that

$$\mathcal{P}\left(\boldsymbol{x}^{*},\boldsymbol{y}^{*},t\right)\!+\!\alpha\mathcal{P}\left(\boldsymbol{x}^{*},\boldsymbol{y}^{*},t\right)\!\geq\!\!\gamma\mathcal{P}\left(\boldsymbol{x}^{*},\boldsymbol{y}^{*},t\right)$$

That is

$$\mathcal{P}(x^*, y^*, t) \ge (\gamma - \alpha) \mathcal{P}(x^*, y^*, t)$$

$$\Rightarrow \mathcal{P}(x^*, y^*, t) \le \frac{1}{\gamma - \alpha} \mathcal{P}(x^*, y^*, t)$$
(17)

This is true only when $\mathcal{P}(x^*, y^*, t) = 0$. Hence $x^* = y^*$. Hence T has a unique fixed point in X. The proof is completed. Now, we prove the following common fixed point theorem, which is generalization of Theorem 4.4 of Jain et al. [24] in the setting of parametric b-metric space.

Theorem 4.3. Let $T, S: X \to X$ be two surjective mappings of a complete parametric b-metric space (X, \mathcal{P}, s) . Suppose that T and S satisfying the following inequalities

$$\mathcal{P}\left(T\left(Sx\right), Sx, t\right) + k\mathcal{P}\left(T\left(Sx\right), x, t\right) \ge a \ \mathcal{P}\left(Sx, x, t\right)$$
(18)

and

$$\mathcal{P}\left(S\left(Tx\right), Tx, t\right) + k\mathcal{P}\left(S\left(Tx\right), x, t\right) \ge b\mathcal{P}\left(Tx, x, t\right)$$
(19)

for all $x \in X$, all t>0 and some nonnegative real numbers a, b and k with $a>s(1+k)+s^2k$ and $b>s(1+k)+s^2k$. If T or S is continuous. Then T and S have a common fixed point.

Proof. Let x_0 be an arbitrary point in X. since T is surjective, there exists $x_1 \in X$ such that $x_0 = Tx_1$. Also since S is surjective, there exists $x_2 \in X$ such that $x_2 = Sx_1$. Continuing this process, we construct a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that

$$x_{2n} = Tx_{2n+1}$$
 and $x_{2n+1} = Sx_{2n+2}$ (20)

for all $n \in \mathbb{N} \cup \{0\}$. Now for $n \in \mathbb{N} \cup \{0\}$, we have

$$\mathcal{P}\left(T\left(Sx_{2n+2}\right), Sx_{2n+2}, t\right) + k\mathcal{P}\left(T\left(Sx_{2n+2}\right), x_{2n+2}, t\right) \ge a\mathcal{P}\left(Sx_{2n+2}, x_{2n+2}, t\right)$$
(21)

Thus we have

$$\mathcal{P}(x_{2n}, x_{2n+1}, t) + k\mathcal{P}(x_{2n}, x_{2n+2}, t) \ge a\mathcal{P}(x_{2n+1}, x_{2n+2}, t)$$
(22)

Since

$$\mathcal{P}(x_{2n}, x_{2n+2}, t) \leq s \left[\mathcal{P}(x_{2n}, x_{2n+1}, t) + \mathcal{P}(x_{2n+1}, x_{2n+2}, t) \right]$$

Hence from (22),

$$\mathcal{P}(x_{2n+1}, x_{2n+2}, t) \leq \frac{1+sk}{a-sk} \mathcal{P}(x_{2n}, x_{2n+2}, t)$$
(23)

On other hand, we have

$$\mathcal{P}(S(Tx_{2n+1}), Tx_{2n+1}, t) + k \mathcal{P}(S(Tx_{2n+1}), x_{2n+1}, t) \ge b\mathcal{P}(Tx_{2n+1}, x_{2n+1}, t)$$
(24)

Thus, we have

$$\mathcal{P}(x_{2n-1}, x_{2n}, t) + k\mathcal{P}(x_{2n-1}, x_{2n+1}, t) \ge b\mathcal{P}(x_{2n}, x_{2n+1}, t)$$
(25)

Since $\mathcal{P}(x_{2n-1}, x_{2n+1}, t) \leq s \left[\mathcal{P}(x_{2n-1}, x_{2n}, t) + \mathcal{P}(x_{2n}, x_{2n+1}, t) \right]$. Hence from (25), we have

$$\mathcal{P}(x_{2n}, x_{2n+1}, t) \le \frac{1+sk}{b-sk} \mathcal{P}(x_{2n-1}, x_{2n}, t)$$
(26)

Let

$$\lambda = \max\left\{\frac{1+sk}{a-sk}, \frac{1+sk}{b-sk}\right\}$$

Obviously, $\lambda < \frac{1}{s}$. Then by combining (23) and (26), we have

$$\mathcal{P}\left(x_{n}, x_{n+1}, t\right) \leq \lambda \mathcal{P}\left(x_{n-1}, x_{n}, t\right) \tag{27}$$

for all $n \in \mathbb{N} \cup \{0\}$ and for all t > 0. By Lemma 2.13, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the complete parametric b-metric space (X, \mathcal{P}, s) . Then there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to +\infty$. Therefore $x_{2n+1} \to x^*$ and $x_{2n+2} \to x^*$ as $n \to +\infty$. Without loss of generality, we may assume that T is continuous, then $Tx_{2n+1} \to Tx^*$ as $n \to +\infty$. But $Tx_{2n+1} = x_{2n} \to x^*$ as $n \to +\infty$. Thus, we have $Tx^* = x^*$. Since S is surjective, there exists $u \in X$ such that $Su = x^*$. Now

$$\mathcal{P}\left(T\left(Su\right), Su, t\right) + k\mathcal{P}\left(T\left(Su\right), u, t\right) \ge a\mathcal{P}\left(Su, u, t\right)$$
(28)

implies that $k\mathcal{P}(x^{\star}, u, t) \ge a\mathcal{P}(x^{\star}, u, t)$. Thus

$$\mathcal{P}\left(x^{\star}, u, t\right) \leq \frac{k}{a} \mathcal{P}\left(x^{\star}, u, t\right) \tag{29}$$

Since a > k, we conclude that $\mathcal{P}(x^*, u, t) = 0$. So $x^* = u$. Hence $Tx^* = Sx^* = x^*$. Therefore x^* is a common fixed point of T and S.

By taking b=a in above Theorem 4.3, we have the following result.

Corollary 4.4. Let $T, S: X \to X$ be two surjective mappings of a complete parametric b-metric space (X, \mathcal{P}, s) . Suppose that T and S satisfying the following inequalities

$$\mathcal{P}\left(T\left(Sx\right), Sx, t\right) + k\mathcal{P}\left(T\left(Sx\right), x, t\right) \ge a \mathcal{P}\left(Sx, x, t\right)$$
(30)

and

$$\mathcal{P}\left(S\left(Tx\right), Tx, t\right) + k\mathcal{P}\left(S\left(Tx\right), x, t\right) \ge a\mathcal{P}\left(Tx, x, t\right) \tag{31}$$

for all $x \in X$, all t > 0 and some nonnegative real numbers a and k with $a > s(1+k) + s^2k$. If T or S is continuous. Then T and S have a common fixed point.

By taking S=T in above Corollary 4.4, we have the following result.

Corollary 4.5. Let $T:X \to X$ be a surjective mapping of a complete parametric b-metric space (X, \mathcal{P}, s) . Suppose that T satisfying the following inequality

$$\mathcal{P}\left(T\left(Tx\right), Tx, t\right) + k\mathcal{P}\left(T\left(Tx\right), x, t\right) \ge a \mathcal{P}\left(Tx, x, t\right)$$

$$(32)$$

for all $x \in X$, all t > 0 and some nonnegative real numbers a and k with $a > s(1 + k) + s^2k$. If T is continuous. Then T has a fixed point.

Remark 4.6. From Theorem 4.3 and Corollary 4.4, Corollary 4.5, we obtain Theorem 4.4 and Corollary 4.5, Corollary 4.6 of [24] if s = 1, respectively.

Now, we give an example.

Example 4.7. Let $X = [0, +\infty)$ and define $\mathcal{P} : X \times X \times (0, +\infty) \rightarrow [0, +\infty)$ by

$$\mathcal{P}(x, y, t) = \begin{cases} t (x - y)^2, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

 $\forall x, y \in X \text{ and all } t > 0.$ It is obvious that \mathcal{P} is a parametric b-metric on X with s = 2 > 1 and (X, \mathcal{P}) is complete. Also, \mathcal{P} is not a parametric metric on X.

Define a mapping $T: X \to X$ by Tx = 6x, $\forall x$. Clearly T is a surjection on X. Now we consider following

$$\mathcal{P}(T(Tx), Tx, t) + \mathcal{P}(T(Tx), x, t) = t(36x - 6x)^2 + t(36x - x)^2$$
$$\geq 84t(6x - x)^2$$
$$= a \mathcal{P}(Tx, x, t)$$

Hence in all above cases $\mathcal{P}(T(Tx), Tx, t) + k\mathcal{P}(T(Tx), x, t) \ge a \mathcal{P}(Tx, x, t) \quad \forall x, y \in X \text{ and all } t > 0, \text{ where } a = 84 > s(1+k) + s^2k = 2(1+k) + 4k = 6k + 2, \text{ and } 0 \le k \le \frac{27}{2}.$ Then (32) is satisfied. Thus all conditions of Corollary 4.4 are satisfied and $x^* = 0 \in X$ is a fixed point of T.

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