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Complementary Tree Domination in Unicyclic Graphs

Research Article

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Abstract: A set D of a graph $G = (V, E)$ is a dominating set of every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set. A dominating set D is called a complementary tree dominating set if the induced subgraph $< V - D >$ is a tree. The minimum cardinality of a complementary tree dominating set is called the complementary tree domination number of G and is denoted by $\gamma_{ctd}(G)$. In this paper, connected unicyclic graphs for which $\gamma_{ctd}(G) = \gamma(G)$ nad $\gamma_{ctd}(G) = \gamma(G) + 1$ are characterized.

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1. Introduction

Graphs discussed in this paper are undirected and simple. For a graph $G(V, E)$, let V and E denotes its vertex set and edge set repectively. A graph G is unicyclic if it contains exactly one cycle. L. Volkman has studied graphs having equal domination number and edge independence number [5]. He has also investigated graphs with equal domination number and covering number. In this paper, connected unicyclic graphs for which $\gamma_{ctd}(G) = \gamma(G)$ and $\gamma_{ctd}(G) = \gamma(G) + 1$ are established.

2. Prior Results

Definition 2.1. *A dominating set $D \subseteq V$ of a connected graph $G = (V, E)$ is said to be a complementary tree dominating set of a connected graph G, if the induced subgraph $< V - D >$ is a tree. The minimum cardinality of a complementary tree dominating set is called the complementary tree domination number of G and is denoted by $\gamma_{ctd}(G)$. A set corresponding to the complementary tree dominating number is called γ_{ctd}-set of G. A complementary tree dominating set is denoted as a ctd-set in brief.*

Here, it is assumed as K_1, the complete graph on a single vertex is connected. Therefore, a complementary tree dominating set can have atmost $(p - 1)$ vertices and hence, $\gamma_{ctd}(G) \leq p - 1$ and γ_{ctd}-set exists for all connected graphs. Since every ctd-set is a dominating set, $\gamma(G) \leq \gamma_{ctd}(G)$.

A complementary tree dominating set D of G is said to be minimal, if no proper subset of D is a complementary tree dominating set of G.

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Notation 2.2. *Let P_m be a path on m $(m \geq 2)$ vertices and let $P_1 = K_1$ and $P_m^+ = P_m \circ K_1$ $(m \geq 1)$ be the Corona of P_m and K_1.*

(a) By joining P_m^+ $(m \geq 1)$ at a vertex v of C_n, $(n \geq 3)$, it is meant that, joining a vertex of degree 2 of P_m^+ to v with an edge.

(b) By joining $K_{1,n}$ $(n \geq 1)$ at a vertex v of C_n, it is meant that, joining the central vertex of $K_{1,n}$ to v with an edge.

(c) By attaching a pendant edge (or a path P_n, $n \geq 3$) at a vertex v of a graph G, it is meant that, merging a vertex of the pendant edge (or a pendant vertex of P_n, $n \geq 3$) with v.

(d) By attaching a tree to a vertex v of a graph G, it is meant that, merging a pendant vertex of the tree with v.

Notation 2.3. *The following classes of unicyclic graphs can be defined.*
Let $H_1^{(t)}$ be the graph obtained from C_n $(n \geq 5)$ by attaching a pendant edge at each of the t vertices of C_n such that $(n-t)$ consecutive vertices of C_n have degree 2 $(t \leq n)$.

(a) Let $\mathcal{G}_1^{(t)}$ be the class of unicyclic graphs $H_1^{(t)}$.

(b) Let $\mathcal{G}_2^{(t)}$ be the class of unicyclic graphs obtained from $H_1^{(t)}$ by joining atleast one P_m^+ $(m \geq 1)$ at atleast one vertex of t consecutive vertices $(t \leq n)$ mentioned above.

(c) Let $\mathcal{G}_3^{(t)}$ be the class of unicyclic graphs obtained from $H_1^{(t)}$ by joining atleast one P_m^+ $(m \geq 1)$ at atleast one of the two end vertices of above t consecutive vertices of C_n.

3. Main Results

Theorem 3.1. *Let G be a connected unicyclic graph with the cycle C_n $(n \geq 5)$ and be not a cycle. Then, $\gamma_{ctd}(G) = \gamma(G)$ if and only if $G \in \bigcup_{i=1}^{3} \mathcal{G}_i^{(n-3)}$.*

Proof. Let G be a connected unicyclic graph with the cycle C_n $(n \geq 5)$ and be not a cycle.
(a) If there exists a vertex in C_n which is a support of G and is adjacent to atleast two pendant vertices, then $\gamma(G) = \lceil \frac{n}{3} \rceil$ and $\gamma_{ctd}(G) \geq 2 + (n-3) = n-1$. Hence, $\gamma_{ctd}(G) > \gamma(G) + 1$, since $n \geq 5$. Therefore, each support v of G such that $v \in C_n$ is adjacent to exactly one pendant vertex. Similarly is the case, when $v \notin C_n$ and is a support of G.
(b) Let there exists a vertex $u \in G$ such that $u \notin C_n$ and be neither a support nor a pendant vertex. Then, G has a vertex in C_n, in which a path P of length atleast three is attached. A minimum dominating set of G will contain atleast one vertex from P and atleast two vertices of C_n, whereas a minimum ctd-set of G contains atleast two vertices from P and atleast three vertices of C_n. Therefore, $\gamma_{ctd}(G) > \gamma(G) + 1$. Hence, a vertex in $V(G) - V(C_n)$ is either a support or a pendant vertex of G. Therefore, G is the connected unicyclic graph obtained from C_n $(n \geq 5)$ by joining atleast one P_m^+ $(m \geq 1)$ or by attaching a pendant vertex (or) both at atleast one vertex of C_n. In this case, number of pendant vertices of G is the same as those of supports of G.
(c) If either G has s vertices $(0 \leq s \leq n,\ s \neq 3)$ in C_n, each is of degree 2 in G and these are the only vertices in $V(C_n) \cap V(G)$ of degree 2.
(or) G has three non consecutive vertices in C_n, each is of degree 2 in G, then also $\gamma_{ctd}(G) > \gamma(G)$, since in a dominating set support of G adjacent to a vertex of C_n dominates both its pendant vertices and a vertex of C_n, whereas in a ctd-set, pendant vertices dominate only its supports. Therefore, there exists exactly three consecutive vertices of C_n having degree

2 in G and the remaining $(n-3)$ vertices of C_n have degree atleast 3 in G.

(d) Let atleast one of the above $(n-3)$ vertices of C_n be not the supports of G. Then, atleast one P_m^+ $(m \geq 1)$ alone is joined at atleast one of the above $(n-3)$ vertices. Then, $\gamma(G) \geq$ (number of supports of G)+1 and $\gamma_{ctd}(G) \geq$ (number of pendant vertices) $+n-2$. That is, G is the connected graph obtained from C_n either by attaching a pendant edge (or) by attaching a pendant edge and then joining atleast one P_m^+ $(m \geq 1)$ at each of the $(n-3)$ consecutive vertices of C_n. Therefore, $G \in \mathcal{G}_1^{(n-3)} \cup \mathcal{G}_2^{(n-3)}$.

(e) Let w, x, y be the vertices in C_n each is of degree 2 in G such that x is adjacent to both w and y in C_n. If atleast one P_m^+ $(m \geq 1)$ is joined either at any two adjacent vertices of w, x, y or at x, then $\gamma_{ctd}(G) > \gamma(G)$. Therefore, atleast one P_m^+ $(m \geq 1)$ is joined at atleast one of w and y. Hence, $G \in \mathcal{G}_3^{(n-3)}$. In all the cases, $G \in \bigcup_{i=1}^{3} \mathcal{G}_i^{(n-3)}$.

Conversely, if $G \in \mathcal{G}_1^{(n-3)}$, then $\gamma(G) = \gamma_{ctd}(G) = n-2$ and if $G \in \mathcal{G}_2^{(n-3)} \cup \mathcal{G}_3^{(n-3)}$, then number of supports of G = number of pendant vertices of G and $\gamma(G) =$ (number of supports of G)+1 and $\gamma_{ctd}(G) =$ (number of pendant vertices) +1. Hence the theorem is proved. □

Example 3.1. *In the following graphs,* $G_1 \in \mathcal{G}_1^{(n-3)}$, $G_2 \in \mathcal{G}_2^{(n-3)}$, $G_3, G_4 \in \mathcal{G}_3^{(n-3)}$.

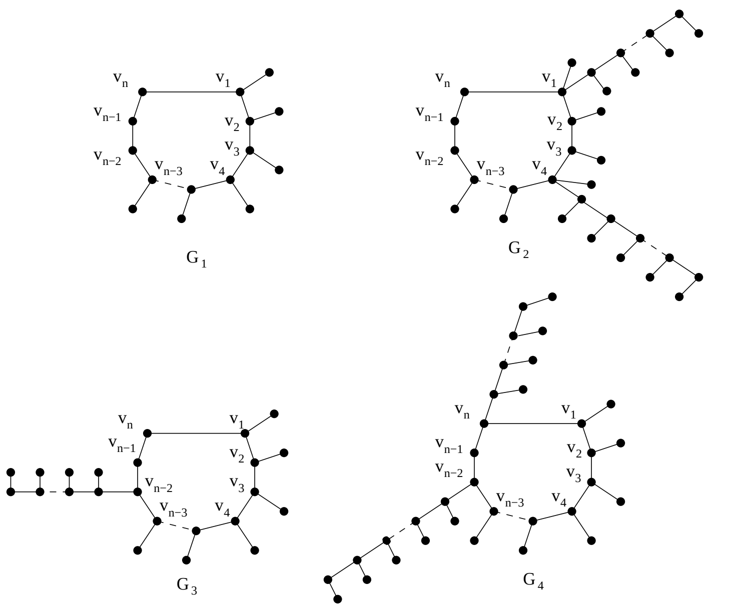

Figure 1.

In a similar manner, the following theorem can be proved.

Theorem 3.2. *Let G be a connected unicyclic graph with the cycle C_3 or C_4. Then, $\gamma(G) = \gamma_{ctd}(G)$ if and only if G is one of the following graphs.*

(a) G is obtained from C_3 by joining atleast one P_m^+ $(m \geq 1)$ at one or two vertices of C_n.

(b) G is obtained from C_4 by joining atleast one P_m^+ $(m \geq 1)$ at one or two adjacent vertices of C_4 and then attaching a pendant edge at exactly one of the above vertices.

(c) G is obtained from C_4 by joining atleast one P_m^+ $(m \geq 1)$ at a vertex, say v of C_4 and then attaching a pendant edge at a vertex of C_4 adjacent to v.

In the following, the connected unicyclic graphs, for which $\gamma_{ctd}(G) = \gamma(G) + 1$, are found.

Theorem 3.3. *Let G be a connected unicyclic graph with the cycle C_n, $n \geq 5$. Then, $\gamma_{ctd}(G) = \gamma(G) + 1$ if and only if*

(i) $G \in \{\mathcal{G}_1^{(t)}, n-4 \leq t \leq n, t \neq n-3\} \cup \{\mathcal{G}_2^{(t)}, n-4 \leq t \leq n-1, t \neq n-3\} \cup \{\mathcal{G}_3^{(t)}, t \neq n\}$ (or)

(ii) G is obtained from C_5 by joining atleast one P_m^+ $(m \geq 1)$ at a vertex of C_5 (or)

(iii) G is obtained from C_5 (or) C_6 by joining atleast one P_m^+ $(m \geq 1)$ at any two adjacent vertices of C_5 or C_6 and then attaching a pendant edge at one of the above two vertices.

Proof. Let G be a connected unicyclic graph with C_n $(n \geq 5)$ as the cycle. Assume $\gamma_{ctd}(G) = \gamma(G)+1$. From the proof of Theorem 3.1, G is a connected unicyclic graph obtained from C_n $(n \geq 5)$ by joining atleast one P_m^+ $(m \geq 1)$ or by attaching a pendant edge or both at atleast one vertex of C_n $(n \geq 5)$. In this case, number of pendant vertices of G is the same as those of supports of G. Let t be the number of supports of G in C_n.

(a) Let s consecutive vertices of C_n have degree 2 in G, where $s \geq 5$ and $s \leq n-1$ and $t+s = n$. Then, $\gamma(G) =$ (number of supports of G) + $\lceil \frac{s-2}{3} \rceil$ whereas, $\gamma_{ctd}(G) =$ (number of pendant vertices of G) + $(s-2)$. Hence, for $n \geq 5$, $\gamma_{ctd}(G) > \gamma(G) + 1$. Therefore, atmost four consecutive vertices of C_n have degree 2 in G. As in Theorem 3.1, G is a connected unicyclic graph obtained from C_n either by attaching a pendant edge (or) attaching a pendant edge and joining atleast one P_m^+ $(m \geq 1)$ at atleast $(n-4)$ consecutive vertices of C_n.

(b) If $(n-3)$ consecutive vertices of C_n are supports of G and the remaining three vertices of C_n have degree two, then $\gamma_{ctd}(G) = \gamma(G)$. Hence, s consecutive vertices of C_n have degree 2 in G, where $0 \leq s \leq 4$, $s \neq 3$. Therefore, t $(t \leq n)$ consecutive vertices of C_n are supports of G such that each support is adjacent to exactly one pendant vertex. At these support atleast one P_m^+ $(m \geq 1)$ may be or may not be joined. The remaining $(n-t)(= s)$ consecutive vertices of C_n have degree 2 in G, where $n-t \leq 4$ and $n-t \neq 3$. That is, $n-4 \leq t \leq n$, $t \neq n-3$. If both a pendant edge is attached and atleast one P_m^+ $(m \geq 1)$ is joined at each vertex of C_n in G, then $\gamma_{ctd}(G) > \gamma(G) + 1$. Therefore, the connected unicyclic graph G is such that

(i) t $(t \leq n)$ consecutive vertices of C_n are supports of G, each is adjacent to exactly one pendant vertex and the remaining $(n-t)$ consecutive vertices of C_n have degree 2 in G, where $n-4 \leq t \leq n$, $t \neq n-3$. That is, $G \in \mathcal{G}_1^{(t)}$, $n-4 \leq t \leq n$, $t \neq n-3$. (or)

(ii) G is obtained from the class of graphs $\mathcal{G}_1^{(t)}$, $n-4 \leq t \leq n-1$, $t \neq n-3$ by joining atleast one P_m^+ $(m \geq 1)$ at the above t vertices of C_n, where $n-4 \leq t \leq n-1$, $t \neq n-3$. That is, $G \in \mathcal{G}_2^{(t)}$, $n-4 \leq t \leq n-1$, $t \neq n-3$.

(c) Let $G \in \mathcal{G}_1^{(t)}$, $n-4 \leq t \leq n$, $t \neq n-3$ then $(n-t)$ consecutive vertices of C_n have degree 2 in G. If atleast one P_m^+ $(m \geq 1)$ is joined at atleast two of these $(n-t)$ consecutive vertices of C_n (or) at a vertex which is not adjacent to one of the end vertices of above $(n-t)$ $(n \neq t)$ consecutive vertices of C_n, then $\gamma_{ctd}(G) > \gamma(G) + 1$. Therefore, $G \in \mathcal{G}_3^{(t)}$, $t \neq n$. In a similar manner, it can also be proved that, if $\gamma_{ctd}(G) = \gamma(G) + 1$, then G can be one of the graphs mentioned in (ii) and (iii) in the theorem.

Conversely, if G is a connected unicyclic graph mentioned in (i), (ii) or (iii), then it can be verified that $\gamma_{ctd}(G) = \gamma(G)+1$. □

In a similar manner, the following Theorems 3.4 and 3.5 can be proved.

Theorem 3.4. *Let G be any connected unicyclic graph with C_3 as the unique cycle. Then, $\gamma_{ctd}(G) = \gamma(G) + 1$ if and only if G is one of the following graphs.*

(a) G is obtained from C_3 by attaching exactly one pendant edge at atleast one vertex of C_3.

(b) G is obtained from C_3 by attaching a path of length three (or) a path of length three and then joining atleast one P_m^+ $(m \geq 1)$ at exactly one vertex of C_3.

(c) G is obtained from C_3 by joining atleast one P_m^+ at one or two vertices of C_3 and then attaching a pendant edge at atleast one vertex of C_3.

Theorem 3.5. *Let G be any connected unicyclic graph with C_4 as the unique cycle. Then, $\gamma_{ctd}(G) = \gamma(G) + 1$ if and only if G is one of the following graphs.*

(a) G is obtained from C_4 by attaching exactly one pendant edge at atleast two vertices of C_4.

(b) G is obtained from C_4 by joining atleast one P_m^+ $(m \geq 1)$ at a vertex of C_4 and then attaching a pendant edge at t vertices of C_4, where $0 \leq t \leq 4$, $t \neq 1$.

(c) G is obtained from C_4 by attaching two pendant edges at a vertex of C_4 (or) by attaching two pendant edges at a vertex and joining atleast one P_m^+ $(m \geq 1)$ at this vertex or a vertex adjacent to it.

(d) G is obtained from C_4 by joining atleast one P_m^+ $(m \geq 1)$ at any two adjacent vertices, say u and v of C_4 and attaching a pendant edge at t vertices of C_4 where $0 \leq t \leq 4$, $t \neq 1$ and these t vertices include both u and v.

(e) G is obtained from C_4 by attaching a path of length 3 at a vertex of C_4.

(f) G is obtained from C_4 by attaching a path of length 3 at a vertex u and then attaching a pendant edge at u or at a vertex of C_4 adjacent to u.

(g) G is obtained from the graphs mentioned in (vi) by joining atleast one P_m^+ $(m \geq 1)$ at the vertex having the pendant edge.

Theorem 3.6. *For any integer $a \geq 2$, there exists a connected graph G with $\gamma_{ctd}(G) = \gamma(G) + a$.*

Proof. Consider the cycle C_{2a+3} on $(2a + 3)$ vertices. Attach exactly one pendant edge at each of any two consecutive vertices of C_{2a+3}. Let the resulting graph be G. For this G, $\gamma(G) = a + 1$, $\gamma_{ctd}(G) = 2a + 1$. Hence, $\gamma_{ctd}(G) = \gamma(G) + a$, $a \geq 2$. □

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