International Journal of Mathematics And its Applications
Volume 4, Issue 1–D (2016), 7–15.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Changing and Unchanging of Complementary Tree Domination Number in Graphs

Research Article

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Abstract: A set D of a graph $G = (V, E)$ is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set. A dominating set D is called a complementary tree dominating set if the induced subgraph $< V - D >$ is a tree. The minimum cardinality of a complementary tree dominating set is called the complementary tree domination number of G and is denoted by $\gamma_{ctd}(G)$. The concept of complementary tree domination number in graphs is studied in [**?**]. In this paper, we have studied the changing and unchanging of complementary tree domination number in graphs.

MSC: 05C69.

Keywords: Domination number, complementary tree dominating set, complementary tree domination number.

1. Introduction

The changing and unchanging terminology was first suggested by Harary [3]. It is useful to partition the vertex set or the edge set of a graph into sets according to how their addition or removal affects the domination number. This concept of changing and unchanging invariant of graphs is also studied in [1, 2, 4, 6, 8]. In this paper, a study of changing and unchanging of complementary tree domination number in connected graphs is initiated.

2. Prior Results

Definition 2.1. *A dominating set $D \subseteq V$ of a connected graph $G = (V, E)$ is said to be a* **complementary tree dominating set** *of a connected graph G, if the induced subgraph $< V - D >$ is a tree. The minimum cardinality of a complementary tree dominating set is called the* **complementary tree domination number** *of G and is denoted by $\gamma_{ctd}(G)$. A set corresponding to the complementary tree dominating number is called γ_{ctd}-set of G. A complementary tree dominating set is denoted as a ctd-set in brief.*

Here, it is assumed as K_1, the complete graph on a single vertex is connected. Therefore, a complementary tree dominating set can have atmost $(p - 1)$ vertices and hence, $\gamma_{ctd}(G) \leq p - 1$ and γ_{ctd}-set exists for all connected graphs. Since every ctd-set is a dominating set, $\gamma(G) \leq \gamma_{ctd}(G)$.

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A complementary tree dominating set D of G is said to be minimal, if no proper subset of D is a complementary tree dominating set of G.

Theorem 2.2. *A ctd-set D of a connected graph $G = (V, E)$ is minimal if and only if for each vertex v in D, one of the following conditions hold.*

(i) v is an isolated vertex of D.

(ii) there exists a vertex u in $V - D$ for which $N(u) \cap D = \{v\}$.

(iii) $N(v) \cap (V - D) = \phi$.

(iv) The subgraph $< (V - D) \cup \{v\} >$ induced by $(V - D) \cup \{v\}$, either contains a cycle or disconnected.

Proof. Suppose D is a minimal ctd-set. On the contrary, if there exists a vertex $v \in D$, such that v does not satisfy any of the given conditions. Then by (i) and (ii), $D' = D - \{v\}$ is a dominating set of G, by (iii), $< V - D' >$ is connected and by (iv), $< V - D' >$ is a tree. This implies that D' is a complementary tree dominating set of G, which is a contradiction. Therefore, for each $v \in D$, one of the conditions (i)-(iv) holds.

Conversely, suppose D is a ctd-set and for each vertex v in D, one of the four stated conditions holds. Now, D is a minimal ctd-set is to be proved. Suppose, D is not a minimal ctd-set, then there exists a vertex v in D, such that $D - \{v\}$ is a ctd-set. Thus, v is adjacent to atleast one vertex in $D - \{v\}$. Therefore, condition (i) does not hold. Also if $D - \{v\}$ is a dominating set, then any vertex in $V - (D - \{v\})$ is adjacent to atleast one vertex in $D - \{v\}$. Therefore, for v, the condition (ii) does not hold. Since $D - \{v\}$ is a ctd-set, $< V - (D - \{v\}) >$ is a tree, which contradicts the conditions (iii) and (iv). Therefore, there exists a vertex v in D such that v does not satisfy conditions (i), (ii), (iii) and (iv), a contradiction to the assumption. Hence, D is a minimal ctd-set. □

In the following, complementary tree domination number of some standard classes of graphs are given.

Observation 2.3.

(a) For any path P_n with n vertices, $\gamma_{ctd}(P_n) = n - 2$, $n \geq 4$.

(b) For any cycle C_n with n vertices, $\gamma_{ctd}(C_n) = n - 2$, $n \geq 3$. Let u, v be any two adjacent vertices of degree 2 in P_n (or C_n). Then $V(P_n) - \{u, v\}$ (or $V(C_n) - \{u, v\}$) is a γ_{ctd}-set of P_n (or C_n).

(c) For any complete graph K_n with n vertices, $\gamma_{ctd}(K_n) = n - 2$, $n \geq 3$. Here, a set having any $n - 2$ vertices of K_n is a γ_{ctd}-set of K_n, $n \geq 3$.

(d) For any star $K_{1,n}$, $\gamma_{ctd}(K_{1,n}) = n$, $n \geq 2$. Here, the set having all the vertices of $K_{1,n}$ except the central vertex forms a γ_{ctd}-set.

(e) For any complete bipartite graph $K_{m,n}$ with $m, n \geq 2$, $\gamma_{ctd}(K_{m,n}) = min\{m, n\}$. Let A, B be a bipartition of $K_{m,n}$ ($m, n \geq 2$ and $m \leq n$) with $|A| = m$ and $|B| = n$. Then, the set containing $(m - 1)$ vertices of A and a vertex of B forms a ctd-set of $K_{m,n}$.

(f) $\gamma_{ctd}(C_n \circ K_1) = n + 1$, $n \geq 3$, where $C_n \circ K_1$ is the Corona of C_n and K_1. Here, all the n-pendant vertices and a vertex of C_n forms a γ_{ctd}-set.

(g) For any wheel W_n with n vertices, $\gamma_{ctd}(W_n) = 2$, $n \geq 4$. Here, the central vertex and a vertex of C_{n-1} forms a γ_{ctd}-set.

(h) Let G be a subdivision of a star $K_{1,n}$, $n \geq 2$. Then $\gamma_{ctd}(G) = n + 1$. Here, all the n-pendant vertices and a vertex of degree 2 (other than the central vertex) forms a γ_{ctd}-set.

In the following, the graphs G for which $\gamma_{ctd}(G) = 1, 2, p-1$ and $p-2$ are found.

Proposition 2.4. *$\gamma_{ctd}(G) = 1$ if and only if $G \cong T + K_1$, where T is a tree.*

Proof. Assume $G \cong T + K_1$ and $V(K_1) = \{v\}$. Then, the set $\{v\}$ is a complementary tree dominating set of G. Conversely, if $\gamma_{ctd}(G) = 1$, then there exists a complementary tree dominating set D of G with $|D| = 1$ such that $< V - D >$ is a tree. Since, each vertex in $V - D$ is adjacent to the vertex in D, $G \cong T + K_1$, where $T =< V - D >$. □

Theorem 2.5. *Let G be a connected graph with $p \geq 4$. Then $\gamma_{ctd}(G) = p - 1$ if and only if G is a star on p vertices.*

Proof. If $G \cong K_{1,p-1}$, then the set of all pendant vertices of $K_{1,p-1}$ forms a minimum complementary tree dominating set for G. Hence, $\gamma_{ctd}(G) = p - 1$.

Conversely, assume $\gamma_{ctd}(G) = p - 1$. Then, there exists a complementary tree dominating set D containing $p - 1$ vertices. Let $V - D = \{v\}$. Since D is a dominating set of G, v is adjacent to atleast one of the vertices in D, say u. If u is adjacent to any of the vertices in D, then the vertex u must be in $V - D$. Since D is minimum, u is adjacent to none of the vertices in D. Hence, $G \cong K_{1,p-1}$. □

Theorem 2.6. *Let G be a connected graph containing a cycle. Then, $\gamma_{ctd}(G) = p - 2$ $(p \geq 5)$ if and only if G is isomorphic to one of the following graphs. C_p, K_p or G is the graph obtained from a complete graph by attaching pendant edges at atleast one of the vertices of the complete graph.*

Proof. For all graphs given in the theorem, $\gamma_{ctd}(G) = p - 2$ $(p \geq 5)$.

Conversely, let G be a connected graph with $\gamma_{ctd}(G) = p - 2$ and G contains a cycle. Let D be a complementary tree dominating set of G such that $|D| = p - 2$ and $V - D = \{w_1, w_2\}$ and $< V - D > \cong K_2$.

Case 1. $\delta(G) = 1$

By Proposition 2.4, all vertices of degree 1 are in D and any vertex of degree 1 in D is adjacent to atmost one vertex in $V - D$ since $< V - D > \cong K_2$. Also each vertex in $V - D$ is adjacent to atleast one vertex in D.

Let $D' = D - \{\text{pendant vertices}\}$. Then, $\{w_1, w_2\} \cup D'$ will be a complete graph. Otherwise, there exists a vertex $u \in D'$, such that u is not adjacent to atleast one of the vertices of $D' - \{u\}$ and hence, $D - \{u\}$ is a complementary tree dominating set. Therefore, G is the graph obtained from a complete graph by attaching pendant edges at atleast one of the vertices.

Case 2. $\delta(G) = 2$

Let w be vertex of degree atleast 3 in G and $w \in V - D$ and $w = w_1$. Let each vertex of D be adjacent to both w_1 and w_2. If $< D >$ is complete, then G is complete. Assume $< D >$ is not complete. Then, there exists atleast one pair of nonadjacent vertices in D, say $u, v \in D$ and $V - \{u, v, w_1\}$ is a complementary tree dominating set of G containing $(p - 3)$ vertices, which is a contradiction. Therefore, there exists a vertex in D which is adjacent to exactly one of w_1 and w_2 and again a complementary tree dominating set having $(p - 3)$ vertices is obtained and hence, $w \in D$. Since $deg(w) \geq 3$, there exists atleast one vertex, say $v \in D$, adjacent to w. Then, either $V - \{v, w, w_1\}$ or $V - \{v, w, w_2\}$ will be a complementary tree dominating set of G. Therefore, there exists no vertex of degree atleast 3 in G and hence, each vertex in G is of degree 2 and G is a cycle.

Case 3. $\delta(G) \geq 3$.

Let u, v be any two nonadjacent vertices in $< D >$. Then, either $V - \{u, v, w_1\}$ or $V - \{u, v, w_2\}$ will be a complementary tree dominating set, which is a contradiction. Therefore $< D \cup \{w_1, w_2\} >$ is complete. Hence, $G \cong K_p$. □

3. Main Results

Observation 3.1.

(a) If G is a cycle or a complete graph on atleast three vertices, then, $V(G) = VD^-$. Let $G \cong C_n$ or K_n, $n \geq 3$. By Observation 2.3(b) and 2.3(c) $\gamma_{ctd}(G) = n-2$. Let $v \in V(G)$. Then $G - v \cong P_{n-1}$ or K_{n-1} and $\gamma_{ctd}(G-v) = n-3 < \gamma_{ctd}(G)$. Therefore, $v \in VD^-$ and hence, $V(G) = VD^-$.

(b) If G is a path on atleast four vertices and if v is a pendant vertex of G, then $v \in VD^-$. Let $G \cong P_n$, $n \geq 4$. By Observation 2.3(a), $\gamma_{ctd}(P_n) = n-2$. Let v be a pendant vertex in P_n. Then, $G - v \cong P_{n-1}$ and $\gamma_{ctd}(G-v) = n-3 < \gamma_{ctd}(G)$. Therefore, $v \in VD^-$.

(c) If G is a complete bipartite graph $K_{m,n}$ $(m, n \geq 3)$, then, $V(G) = VD^- \cup VD^0$ and if G is $K_{2,n}$ $(n \geq 3)$, then $V(G) = VD^0 \cup VD^+$. Let G be a complete bipartite graph $K_{m,n}$, where $m \geq 2, n \geq 3$. Without loss of generality, let $m < n$. Therefore, $\gamma_{ctd}(G) = min(m,n) = m$ (by Observation 2.3(e)). Let $v \in V(G)$. If $G \cong K_{m,n}$ $(m, n \geq 3)$. Then, $G - v \cong K_{m-1,n}$ or $K_{m,n-1}$. Therefore, $\gamma_{ctd}(G-v) = m-1$ or m. Therefore, $v \in VD^- \cup VD^0$. Hence, $V(G) = VD^- \cup VD^0$. Similarly if $G \cong K_{2,n}$ $(n \geq 3)$, then $G - v \cong K_{1,n}$ or $K_{2,n-1}$. Therefore, $\gamma_{ctd}(G-v) = n$ or 2. Hence, $v \in VD^+ \cup VD^0$ and $V(G) = VD^+ \cup VD^0$.

(d) If G is a Corona $C_n \circ K_1$ $(n \geq 3)$ and if v is a pendant vertex of G, then $v \in VD^-$. Let G be the corona $C_n \circ K_1$ and let v be the pendant vertex of G. Then, $G - v$ is a graph obtained by attaching exactly one pendant edge at each of $(n-1)$ vertices of C_n. Then a minimum ctd-set of $G - v$ contains all the $(n-1)$ pendant vertices and a vertex of C_n and hence, $\gamma_{ctd}(G-v) = n$. But, $\gamma_{ctd}(G) = n+1 > \gamma_{ctd}(G-v)$. Therefore, $v \in VD^-$.

(e) If G is a wheel W_n on n $(n \geq 6)$ vertices, then $V(G) = VD^- \cup VD^+$. If $G \cong W_5$, then $V(G) = VD^0 \cup VD^+$. If $G \cong W_4$, then $V(G) = VD^-$. Let G be a wheel W_n on n $(n \geq 6)$ vertices, where $W_n = C_{n-1} + K_1$. Then, $\gamma_{ctd}(W_n) = 2$ (by Observation 2.3(g)). Let v be a vertex of W_n.

Case 1. *$v \in V(C_{n-1})$. Then, $G - v \cong K_1 + P_{n-2}$ and $\gamma_{ctd}(G-v) = 1 < \gamma_{ctd}(G)$. Hence, $v \in VD^-$.*

Case 2. *$v \in V(K_1)$. Then, $G - v \cong C_{n-1}$ and $\gamma_{ctd}(G-v) = n-3 > \gamma_{ctd}(G)$. Hence, $v \in VD^+$. Therefore, $V(G) = VD^- \cup VD^+$.*

Proposition 3.2. *Let G be a connected graph with p $(p \geq 4)$ vertices. If $\gamma_{ctd}(G) = 1$, then $V(G) = VD^0 \cup VD^+$.*

Proof. Assume $\gamma_{ctd}(G) = 1$. Then by the Proposition 2.4, $G \cong K_1 + T$, where T is a tree on $(p-1)$ vertices. Let $v \in V(G)$.

Case 1. T is a star. Then, $G \cong K_2 + (p-2)K_1$. If $v \in V(K_1)$, then $G - v \cong K_2 + (p-3)K_1$ and $\gamma_{ctd}(G-v) = 1 = \gamma_{ctd}(G)$. Therefore, $v \in VD^0$. If $v \in V(K_2)$, then $G - v \cong K_{1,p-2}$ and $\gamma_{ctd}(G-v) = p-2 > \gamma_{ctd}(G)$ and hence $v \in VD^+$

Case 2. T is not a star

Subcase 2.1. $v \in V(K_1)$. Then, $G - v \cong T$ and $\gamma_{ctd}(G-v) > 1 = \gamma_{ctd}(G)$. Hence, $v \in VD^+$.

Subcase 2.2. $v \in V(T)$ is such that $deg_T(v) = 1$. Then $G - v \cong K_1 + T'$, where $T' = T - v$ is a tree on $(p-2)$ vertices. Hence, $\gamma_{ctd}(G-v) = 1 = \gamma_{ctd}(G)$ and $v \in VD^0$.

Subcase 2.3. $v \in V(T)$ is such that $deg_T(v) \geq 2$. Then, $T - v$ is disconnected such that each component of $T - v$ is either a tree or an isolated vertex and $G - v \cong K_1 + (T - v)$. Hence, $\gamma_{ctd}(G-v) > 1 = \gamma_{ctd}(G)$ and $v \in VD^+$. From the above cases, it can be concluded that $v \in VD^0 \cup VD^+$, for all $v \in V(G)$ and hence, $V(G) = VD^0 \cup VD^+$. □

Proposition 3.3. *Let T be any tree. If G is a graph with atleast four vertices obtained from $K_1 + T$ with one pendant edge* *attached at the vertex of* K_1*, then* $V(G) = VD^- \cup VD^0 \cup VD^+$*, where*

$$VD^- = \{v \in V(G)/deg_G(v) = 1\}$$
$$VD^0 = \{v \in V(G)/v \in V(T) \ \ and \ \ deg_T(v) = 1\}$$
$$VD^+ = \{v \in V(G)/v \in V(T) \ \ and \ \ deg_T(v) \geq 2\}$$

Proof. Let G be a graph given above. Then by Theorem **??**, $\gamma_{ctd}(G) = 2$.

Case 1. $v \in V(G)$ is such that $deg_G(v) = 1$. Then, $G - v \cong K_1 + T$ and by Proposition 2.4, $\gamma_{ctd}(G - v) = 1$ and hence $v \in VD^-$.

Case 2. $v \in V(G) \cap V(T)$ is such that $deg_T(v) = 1$. Then, the set containing the pendant vertex of G and the vertex of K_1 forms a γ_{ctd}-set of $G - v$ and hence $\gamma_{ctd}(G - v) = 2 = \gamma_{ctd}(G)$. Therefore, $v \in VD^0$.

Case 3. $v \in V(G) \cap V(T)$ is such that $deg_T(v) \geq 2$. If v is a support of T, then $G - v$ has atleast two pendant vertices and the set containing pendant vertices of $G - v$ and the vertex of K_1 forms a γ_{ctd}-set of $G - v$. Hence, $\gamma_{ctd}(G - v) \geq 3$ and therefore, $v \in VD^+$. Let v be not a support of T and $deg_T(v) \geq 2$. Let $T_1, T_2, \ldots, T_n$ $(n \geq 2)$ be the components of $T - v$ and let T_i be a component of $T - v$ with maximum number of vertices. Then, $V(G) - V(T_i)$ is a ctd-set of $G - v$ having atleast three vertices. Choose a vertex from each component $T_1, T_2, \ldots, T_n$ $(n \geq 3)$. Let D be the set of these n vertices together with the vertex of K_1. Then, $< D >\cong K_{1,n}$ $(n \geq 3)$ and $V - D$ has atleast three vertices and is a ctd-set of $G - v$. Then, $\gamma_{ctd}(G - v) = min\{|V(G) - V(T_i)|, |V - D|\}$ and $\gamma_{ctd}(G - v) \geq 3$. Therefore, $v \in VD^+$. From the above cases, $V(G) = VD^- \cup VD^0 \cup VD^+$. □

Proposition 3.4. *Let* G *be a connected graph with* p $(p \geq 4)$ *vertices. If* $\gamma_{ctd}(G) = p-1$*, then* $VD^- = \{v \in V(G)/deg_G(v) = 1\}$.

Proof. Let $v \in V(G)$. Assume $\gamma_{ctd}(G) = p - 1$. Then, $G \cong K_{1,p-1}$. If $deg_G(v) = p - 1$, then $G - v$ is totally disconnected. If $deg_G(v) = 1$, then $G - v \cong K_{1,p-2}$ and $\gamma_{ctd}(G - v) = p - 2 < \gamma_{ctd}(G)$. Hence, $v \in VD^-$ and therefore, $VD^- = \{v \in V(G)/deg_G(v) = 1\}$. □

Proposition 3.5. *Let* G *be a connected graph with* p $(p \geq 5)$ *vertices. If* $\gamma_{ctd}(G) = p - 2$ *and if* S *be the set of cutvertices of* G*, then* $VD^- = V(G) - S$.

Proof. By Theorems 2.6 and **??**, $\gamma_{ctd}(G) = p - 2$ $(p \geq 5)$ if and only if G is one of the following graphs

(i) G is a cycle on p vertices

(ii) G is a complete graph on p vertices

(iii) G is a graph obtained from a complete graph by attaching pendant edges at atleast one of the vertices of the complete graph

(iv) G is a path on p vertices

(v) G is a tree obtained from a path by attaching pendant edges at atleast one of the end vertices of the path

Let $v \in V(G)$.

Case 1. G is a cycle on p vertices. Then, $G - v \cong P_{p-1}$ and $\gamma_{ctd}(G - v) = p - 3 < \gamma_{ctd}(G)$. Therefore, $v \in VD^-$

Case 2. G is a complete graph on p vertices. Then, $G - v \cong K_{p-1}$ and $\gamma_{ctd}(G - v) = p - 3 < \gamma_{ctd}(G)$. Therefore, $v \in VD^-$

Case 3. G is a graph obtained from a complete graph by attaching pendant edges at atleast one of the vertices of the complete graph.

(a) If $deg_G(v) = 1$ and if v is the only vertex of degree 1 in G, then $G - v \cong K_{p-1}$ and $\gamma_{ctd}(G - v) = p - 3 < \gamma_{ctd}(G)$. Hence, $v \in VD^-$.

(b) Let $deg_G(v) = 1$ and let there exists t $(t \geq 2)$ vertices of degree 1 in G. Then, $G - v$ is a graph with $(p - 1)$ vertices obtained from a complete graph by attaching $(t - 1)$ pendant edges at atleast one of the vertices of the complete graph. Then, $\gamma_{ctd}(G - v) = (p - 1) - 2 = p - 3 < \gamma_{ctd}(G)$. Hence, $v \in VD^-$.

(c) Let v be a vertex of the complete graph and be not a support of G. Then, $deg_G(v) = n - 1$, where n $(n < p)$ is the number of vertices of the complete graph and $G - v$ is the graph obtained by attaching pendant edges at atleast one of the vertices of the complete graph K_{n-1}. Since $G - v$ has $(p - 1)$ vertices, $\gamma_{ctd}(G - v) = p - 3$ and hence, $v \in VD^-$.

(d) If v is a support of G, then $G - v$ is disconnected.

Case 4. G is a path on p vertices (or) G is a tree obtained from a path by attaching pendant edges at atleast one of the vertices of the path. If v is a pendant vertex of G, then $\gamma_{ctd}(G-v) = p-3 < \gamma_{ctd}(G)$. Hence, $v \in VD^-$. If v is not a pendant vertex of G, then $G - v$ is disconnected. From Case 1-4, it can be seen that $v \in VD^-$ and therefore, $V(G) = VD^-$. □

Theorem 3.6. *Let G be a connected graph and let $v \in V(G)$ and D be a γ_{ctd}-set of G. Then, $v \in VD^-$ if either*

(i) vertices of $V - D$ adjacent to $v \in D$ are adjacent to atleast one vertex in D other than v (or)

(ii) v is a pendant vertex in $V - D$ and there exists a vertex $u \in N(v) \cap D$ such that $N(u) \cap D \neq \phi$ and u is adjacent to exactly one vertex, say w in $(V(G) - D) - \{v\}$ such that $N(w) \cap (D - \{u\}) \neq \phi$.

Proof. Let D be a γ_{ctd}-set of G and $v \in V(G)$. Assume (i).
Let $v \in D$ and let $D' = D - \{v\}$, $V - D' = V - (D - \{v\})$ and $D' \subseteq V - \{v\}$. Since $< V - D >$ is a tree and $v \in D$, $< V(G - v) - D' >$ is also a tree. Also, each vertex in $V(G - v) - D'$ is adjacent to atleast one vertex in D' and hence, $D' = D - \{v\}$ is a ctd-set of $G - v$. Therefore,

$$\begin{aligned}\gamma_{ctd}(G - v) &\leq |D - \{v\}| \\ &= \gamma_{ctd}(G) - 1 < \gamma_{vtd}(G)\end{aligned}$$

Hence, $v \in VD^-$. Assume (ii).
Let $v \in V - D$ and be a pendant vertex in $V - D$, $u \in N(v) \cap D$ be such that $N(u) \cap D \neq \phi$ and u be adjacent to exactly one vertex w in $(V - D) - \{v\}$ such that $N(w) \cap (D - \{u\}) \neq \phi$. Let $D' = D - \{u\}$. Then, $u \in V - D'$, $N(u) \cap D \neq \phi$ implies that u is adjacent to atleast one vertex in D'.
Similarly, $N(w) \cap (D - \{u\}) \neq \phi$ implies that w is also adjacent to atleast one vertex in D'. Since D is a dominating set of G, all the remaining vertices in $V - D'$ are adjacent to atleast one vertex in D'. Therefore, D' is a dominating set of $G - v$. Since $< V - D >$ is a tree and u is adjacent to exactly one vertex in $(V - D) - \{v\}$, $< V - D' >$ is also a tree. Hence, D' is a ctd-set of $G - v$ and $\gamma_{ctd}(G - v) \leq |D'| = |D| - 1 = \gamma_{ctd}(G) - 1 < \gamma_{ctd}(G)$. Therefore, $v \in VD^-$. □ □

Theorem 3.7. *Let G be a connected graph and let D be a γ_{ctd}-set of G. If $v \in V(G)$ is a pendant vertex in $V - D$ and for every $u \in D$, $< V - D) \cup \{u\} >$ either contains a cycle or is disconnected, then $v \in VD^0 \cup VD^-$.*

Proof. Let D be a γ_{ctd}-set of G and let v be a pendant vertex in $V - D$. If v satisfies the conditions given in the theorem, then D is also a ctd-set of $G - v$. Therefore, $\gamma_{ctd}(G - v) \leq |D| = \gamma_{ctd}(G)$ and hence $v \in VD^0 \cup VD^-$. □

Observation 3.8. *Let G be a connected graph and let $v \in V(G)$*

(i) Let $G - v$ be a connected graph such that each vertex of degree atleast two is a support. Let t be the number of pendant vertices of G. Then,

(a) $v \in VD^0$, if $t = \gamma_{ctd}(G)$

(b) $v \in VD^-$, if $t < \gamma_{ctd}(G)$

(c) $v \in VD^+$, if $t > \gamma_{ctd}(G)$

(ii) If $G - v$ is a connected graph with $\gamma_{ctd}(G)$ pendant vertices and if there exists atleast one nonsupport vertex of degree atleast two, then $v \in VD^+$.

(iii) Let $G - v$ be a complete graph, a cycle or a path on n vertices, then

(a) $v \in VD^0$, if $\gamma_{ctd}(G) = n - 2$

(b) $v \in VD^-$, if $\gamma_{ctd}(G) > n - 2$

(c) $v \in VD^+$, if $\gamma_{ctd}(G) < n - 2$

(iv) Let $G - v$ be a graph which is the one point union of t triangles. Then

(a) $v \in VD^0$, if $t = \gamma_{ctd}(G)$

(b) $v \in VD^-$, if $t < \gamma_{ctd}(G)$

(c) $v \in VD^+$, if $t > \gamma_{ctd}(G)$

Proposition 3.9. *If G is a connected graph having atleast four vertices with $\gamma_{ctd}(G) = 1$, then $E(G) = ED^+$.*

Proof. Let G be a connected graph with p $(p \geq 4)$ vertices. $\gamma_{ctd}(G) = 1$ implies that $G \cong K_1 + T$, where T is a tree on $(p-1)$ vertices (by Proposition 2.4). Let $e = (u, v) \in E(G)$ and let D be a γ_{ctd}-set of G. Therefore, $|D| = 1$.

Case 1. $u \in D$ and $v \in V - D$. Then, $u \in V(K_1)$ and $v \in V(T)$.

Subcase 1.1. v is a pendant vertex in T, then $G - e$ is a graph obtained by attaching a pendant edge at a vertex of the graph $K_1 + T'$, where T' is a tree on $(p-2)$ vertices. $\gamma_{ctd}(G - e) = \gamma_{ctd}(K_1 + T') = 2 > \gamma_{ctd}(G)$. Hence, $e \in ED^+$.

Subcase 1.2. v is a vertex of degree atleast two in T. Then, $G - e$ is not isomorphic to $K_1 + T''$, for any tree T''. Therefore, $\gamma_{ctd}(G - e) \geq 2 > \gamma_{ctd}(G)$. Hence, $e \in ED^+$.

Case 2. $u, v \in V - D$. Then, $G - e$ is a graph $K_1 + (T_1 \cup T_2)$, where T_1 and T_2 are any two disjoint trees and the number of vertices in $T_1 \cup T_2$ is $p - 1$. $\gamma_{ctd}(G - e) = 1 + min(|T_1|, |T_2|) > \gamma_{ctd}(G)$ and hence, $e \in ED^+$. From Case 1 and Case 2, it can be concluded that $E(G) = ED^+$. □

Proposition 3.10. *Let T be any tree. Let G be the graph with atleast four vertices, obtained from $K_1 + T$ with one pendant edge attached at the vertex of K_1. If e is not a pendant edge of G, then $e \in ED^0 \cup ED^+$.*

Proof. Let G be the graph with atleast four vertices obtained from $K_1 + T$ with one pendant edge attached at the vertex of K_1, where T is any tree. Let D be a γ_{ctd}-set of G. D contains the vertices of the pendant edge. By Theorem **??**, $\gamma_{ctd}(G) = 2$. Let $e = (u, v) \in E(G)$.

Case 1. $u, v \in D$. Then, $e = (u, v)$ is the pendant edge and $G - e$ is disconnected with one isolated vertex.

Case 2. $u \in D$, $v \in V - D$ and $deg_G(v) = 2$. Then, v is a pendant vertex in T and $G - e$ has two pendant vertices. Therefore, $\gamma_{ctd}(G - e) \geq 2 = \gamma_{ctd}(G)$. Hence, $e \in ED^0 \cup ED^+$.

Case 3. $u \in D$, $v \in V - D$ and $deg_G(v) > 2$. If T is a path on three vertices, then $\gamma_{ctd}(G - e) = \gamma_{ctd}(G) = 2$. Therefore, $e \in ED^0$. Let T be not a path on three vertices. If v is a support of T, then the set $\{u, w, x\}$ is a γ_{ctd}-set of $G - e$, where w is the pendant vertex of G and $x \in N(v)$ is a pendant vertex of T. Therefore, $\gamma_{ctd}(G - e) = 3 > \gamma_{ctd}(G)$ and hence, $e \in ED^+$. If v is not a support of T, then the set containing u, pendant vertex of T and atleast two vertices of T forms a γ_{ctd}-set of $G - e$. Therefore, $e \in ED^+$.

Case 4. $u, v \in V - D$.

Subcase 4.1. $deg_G(u) = 2$ and $deg_G(v) \geq 2$. Then, u is a pendant vertex of T and v is a support of T adjacent to u in T, and $G-e$ contains two pendant vertices. Since $deg_G(v) \geq 2$, T contains atleast three vertices and hence, $\gamma_{ctd}(G-e) \geq 3 > \gamma_{ctd}(G)$. Therefore, $e \in ED^+$.

Subcase 4.2. $deg_G(u) \geq 2$ and $deg_G(v) \geq 2$. Then, $G - e$ is a graph $K_1 + (T_1 \cup T_2)$ with a pendant edge attached at the vertex of K_1, where T_1 and T_2 are any two trees. Therefore, γ_{ctd}-set of $G - e$ contains a pendant vertex and atleast one vertex from each of T_1 and T_2. Hence, $\gamma_{ctd}(G - e) \geq 3$ and $e \in ED^+$. From the above cases, it can be concluded that $e \in ED^+$, if e is not a pendant edge of G. □

Proposition 3.11. *Let G be a connected graph obtained from a tree by joining each of the vertices of the tree to the vertices of K_2 such that for all $v \in V(K_2)$, $deg_G(v) \geq 2$ and let $e = (u, v) \in E(G)$. If D is a γ_{ctd}-set of G and if atleast one of u and v is an element of D, then $e \in ED^0 \cup ED^+$.*

Proof. Let G be a connected graph as given in the proposition. Then by Theorem **??**, $\gamma_{ctd}(G) = 2$. Let $e = (u, v) \in E(G)$ and let D be a γ_{ctd}-set of G. Assume $u \in D$.

Case 1. $v \in D$. Then $G - e$ is a graph obtained from a tree by joining each of the vertices of the tree to the vertices of $2K_1$ such that $deg_G(w) \geq 1$, for all $w \in V(2K_1)$. By Theorem **??**, $\gamma_{ctd}(G - e) = 2$. Therefore, $\gamma_{ctd}(G - e) = \gamma_{ctd}(G)$ and hence, $e \in ED^0$.

Case 2. $v \in V - D$.

Subcase 2.1. $deg_G(u) = deg_G(v) = 2$. Then, $G - e$ is a graph with two pendant vertices. If $|D| = |V - D| = 2$, then G is a path on four vertices. Therefore, $\gamma_{ctd}(G - e) = \gamma_{ctd}(G) = 2$. Otherwise, $G - e$ contains a cycle with two pendant vertices and hence, $\gamma_{ctd}(G - e) \geq 3 > \gamma_{ctd}(G)$. Therefore, $e \in ED^+$.

Subcase 2.2. $deg_G(u) \geq 3$. If $deg_G(u) = 3$, and $w \in N(u) \cap D$ is adjacent to all the vertices of the tree, then $\{w\}$ is a ctd-set of $G - e$ and hence, $\gamma_{ctd}(G - e) = 1 < \gamma_{ctd}(G)$. Therefore, $e \in ED^0$. If $deg_G(u) \geq 3$ and if $G - e$ is a graph obtained from a tree by joining each of the vertices of the tree to the vertices of $2K_2$ such that $deg_{G-e}(x) \geq 2$, for all $x \in V(2K_2)$, then $\gamma_{ctd}(G - e) = 2$. Otherwise, $\gamma_{ctd}(G - e) \geq 2$. Hence, $e \in ED^0 \cup ED^+$. From the above cases, $e \in ED^0 \cup ED^+$. □

In analogous to Proposition **??**, the following proposition is stated without proof.

Proposition 3.12. *Let G be a connected graph obtained from a tree by joining each of the vertices of the tree to the vertices of $2K_1$ such that $deg_G(v) \geq 1$ for all $v \in V(2K_1)$ and let $e = (u, v) \in E(G)$. If G is a γ_{ctd}-set of G and if atleast one of u and v is a member of $V - D$, then $e \in ED^- \cup ED^0 \cup ED^+$.*

Proposition 3.13. *Let G be a connected graph with p $(p \geq 5)$ vertices and let $\gamma_{ctd}(G) = p - 2$. If e is not a cut edge of G, then $e \in ED^0 \cup ED^-$.*

Proof. By Theorem 2.6 and Theorem **??**, $\gamma_{ctd}(G) = p - 2$ if and only if G is one of the following graphs

(i) G is a cycle on p vertices.

(ii) G is a complete graph on p vertices.

(iii) G is a graph obtained from a complete graph by attaching pendant edges at atleast one of the vertics of the complete graph.

(iv) G is a path on p vertices.

(v) G is a tree obtained from a path by attaching pendant edges at atleast one of the end vertices of the path.

If G is a graph as in (i) or (ii), then $e \in ED^0$ or $e \in ED^-$. If G is a graph as in (iv) and (v), then each edge of G is a cut edge. Let G be a graph given as in (iii). Let the complete graph be K_n, where $n < p$. Since e is not a cut edge of G, $e \in E(K_n)$. Then $G - e$ is a graph obtained from $K_n - e$ by attaching pendant edges at atleast one of the vertices of $K_n - e$. Therefore, $\gamma_{ctd}(G - e) = p - 3 < p - 2 = \gamma_{ctd}(G)$ and $e \in ED^-$. Hence, $e \in ED^0 \cup ED^-$. □ □

Observation 3.14. *Let G be a connected graph and let $e = (u, v) \in E(G)$. Let D be a γ_{ctd}-set of G, then*

i) $e \in ED^0$, if either

(a) both $u, v \in D$ (or)

(b) $u \in D$, $v \in V - D$ and v is adjacent to atleast two vertices in D

ii) $e \in ED^-$, if $u, v \in V - D$ and there exists a vertex $w \in D$ such that $N(w) \cap D \neq \phi$ and $N(w) \cap (V - D) = \{u, v\}$

iii) $e \in ED^0 \cup ED^+$, if either

(a) $u, v \in V - D$ (or)

(b) $u \in D$, $v \in V - D$ and v is adjacent to exactly one vertex in D

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