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More Results on Complementary Tree Domination Number of Graphs

Research Article

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Abstract: A set D of a graph $G = (V, E)$ is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set. A dominating set D is called a complementary tree dominating set if the induced subgraph $< V - D >$ is a tree. The minimum cardinality of a complementary tree dominating set is called the complementary tree domination number of G and is denoted by $\gamma_{ctd}(G)$. In this paper, some results on complementary tree domination established.

MSC: 05C69.

Keywords: Domination number, graph.

1. Introduction

Graphs discussed in this paper are undirected and simple graph $G(V, E)$. The graph $C_n^{(t)}$ is the one point union of t cycles of length n. If $n = 3$, it is called the Dutch t-windmill or friendship graph. The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 are defined as the graph G obtained by taking one copy of G_1 of order p_1 and p_1 copies of G_2 and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2. The corona has $p_1(1 + p_2)$ vertices and $q_1 + p_1q_2 + p_1p_2$ edges. A set of vertices in G is independent, if no two of them are adjacent. The largest number of vertices in such a set is called the independence number of G and is denoted by $\beta_0(G)$. Any undefined terms in this paper may be found in Harary [1].

The concept of domination in graphs was introduced by Ore [2]. A set $D \subseteq V(G)$ is said to be a dominating set of G. If every vertex in $V - D$ is adjacent to some vertex in D. D is said to be a minimal dominating set if $D - \{u\}$ is not a dominating set, for any $u \in D$. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set.

In this paper, more results on complementary tree domination number of graphs are found.

2. Prior Results

Definition 2.1 ([1]). *A set $D \subseteq V(G)$ is said to be a complementary tree dominating set (ctd-set) if the induced subgraph $< V(G) - D >$ is a tree. The minimum cardinality of a ctd-set is called the complementary tree domination number of G and is denoted by $\gamma_{ctd}(G)$.*

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Observation 2.2 ([1]).

(a). For any connected graph G, $\gamma(G) \leq \gamma_{ctd}(G)$.

(b). For any spanning subgraph H of G, $\gamma_{ctd}(G) \leq \gamma_{ctd}(H)$.

Theorem 2.3. *A ctd-set D of G is minimal if and only if for each vertex v in D one of the following conditions hold.*

(a). v is an isolated vertex of D.

(b). there exists a vertex u in $V - D$ for which $N(u) \cap D = \{v\}$.

(c). $N(v) \cap (V - D) = \phi$

(d). The subgraph $< (V - D) \cup \{v\} >$ induced by $(V - D) \cup \{v\}$, either contains a cycle or disconnected.

Proposition 2.4. *Let $C_n^{(t)}$, $t \geq 2$ be the one point union of t cycles of length n $(n \geq 3)$. Then*

$$\gamma_{ctd}(C_n^{(t)}) = \begin{cases} t, & n = 3 \\ (n-3)t + 1, & n \geq 4 \end{cases}$$

Proof. Let $G = C_n^{(t)}$ and u be the point of union of t cycles of length n. G has $t(n-1)+1$ vertices. Let the vertex set of k^{th} cycle in $C_n^{(t)}$ be $V_k = \{u, u_{k1}, u_{k2}, ..., u_{k,n-1}\}$, $k = 1, 2, \ldots, t$.

Case 1. $n = 3$

Let $D_k = \{u_{k1}\}$, $k = 1, ..., t$ and $D = \bigcup_{k=1}^{t} D_k \subseteq V(G)$. Then $< V - D > \cong K_{1,t}$. Let $v \in D$. Then $< V - (D - \{v\}) >$ either contains a cycle or is disconnected and hence, D is a minimum ctd-set of G and $\gamma_{ctd}(G) = |D| = t$.

Case 2. $n \geq 4$

Let $D_k = \{u_{k2}, u_{k3}, ..., u_{k,n-2}\}$, $k = 1, ..., t$ and $D = \bigcup_{k=1}^{t} D_k \cup \{u_{11}\} \subseteq V(G)$. Then $< V - D > \cong K_{1,2t-1}$. As in Case 1, D is a minimum ctd-set of G and hence, $\gamma_{ctd}(G) = |D| = (n-3)t + 1$. □

In the following, upper bound of $\gamma_{ctd}(G_1 \circ G_2)$ is given.

Theorem 2.5. *Let $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ be two connected graphs of order atleast two. Let T be an induced subgraph of G_1 having maximum number of vertices such that T is a tree. If β_0 is the independence number of G_2, then $\gamma_{ctd}(G_1 \circ G_2) \leq p_1(p_2 + 1) - t(1 + \beta_0)$, where $t = |T|$.*

Proof. Let T be an induced subgraph of G_1 having maximum number of vertices such that T is a tree. Let $|T| = t$. Let S be a maximum independent set of G_2 such that $|S| = \beta_0$. Let D' be the set of vertices of S in copies of G_2 which are adjacent to vertices of T. Then $|D'| = t\beta_0$. Let $D = V(G_1 \circ G_2) - (V(T) \cup D')$. Then $V(G_1 \circ G_2) - D = V(T) \cup D'$ and each vertex in $V(T)$ is adjacent to $(p_2 - \beta_0)$ vertices in a copy of G_2. Also, each vertex in D' is adjacent to atleast one of $(p_2 - \beta_0)$ vertices in a copy of G_2. Therefore, D is a dominating set of $G_1 \circ G_2$ and $< V(G_1 \circ G_2) - D >$ is the tree obtained from T by attaching β_0 pendant edges at each vertex of T. Therefore, D is a ctd-set of $G_1 \circ G_2$.

$$\begin{aligned} \gamma_{ctd}(G_1 \circ G_2) &\leq |D| \\ &= |V(G_1 \circ G_2) - (V(T) \cup D')| \\ &= p_1 + p_1 p_2 - (t + t\beta_0) \\ &= p_1(1 + p_2) - t(1 + \beta_0) \end{aligned}$$

Equality holds, if $G_1 \circ G_2 \cong C_n \circ C_3$, $n \geq 4$. □

Replacing t by p_1 in Theorem 2.5, the corollary follows.

Corollary 2.6. *Let $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ be two connected graphs of order atleast two. If G_1 is a tree and if β_0 is the independence number of G_2, then $\gamma_{ctd}(G) \leq p_1(p_2 - \beta_0)$.*
Equality holds, if $G_1 \circ G_2 \cong T \circ K_n$, $n \geq 3$, where T is any tree.

Theorem 2.7. *Let T_1 and T_2 be two trees with orders p_1 and p_2, respectively. Then $\gamma_{ctd}(T_1 \circ T_2) \leq p_1(p_2 - 1)$.*

Proof. Let $S \subseteq V(T_1 \circ T_2)$ be the set containing vertices of T_1 and one vertex from each copy of T_2. Let $D = V(T_1 \circ T_2) - S$. Then D is a dominating set of $T_1 \circ T_2$ and $< V(T_1 \circ T_2) - D > = < S > = T_1 \circ K_1$. Therefore, D is a ctd-set of $T_1 \circ T_2$ and $|D| = |V(T_1 \circ T_2)| - 2|T_1| = p_1 + p_1p_2 - 2p_1 = p_1(p_2 - 1)$. Hence, $\gamma_{ctd}(T_1 \circ T_2) \leq |D| = p_1(p_2 - 1)$. Equality holds, if $T_1 \cong P_4$, $T_2 \cong K_2$. □

Notation 2.8.

(a). By attaching a pendant edge at a vertex v of a graph G, it is meant that merging a vertex of the pendant edge with v.

(b). By attaching a graph H at a vertex v of a graph G, it is meant that merging a vertex of H to v.

Theorem 2.9. *Let G be a connected graph. Then $\gamma_{ctd}(G) = 2$ if and only if G is one of the following graphs*

(i) G is the graph obtained from $K_1 + T$ with one pendant edge attached at the vertex of K_1, where T is any tree.

(ii) G is the graph obtained from a tree by joining each of the vertices of the tree to the vertices of K_2 such that $deg_G v \geq 2$, for all $v \in V(K_2)$.

(iii) G is the graph obtained from a tree by joining each of the vertices of the tree to the vertices of $2K_1$ such that $deg_G v \geq 1$, for all $v \in V(2K_1)$.

Proof. Let G be one of the graph mentioned in (i), (ii) and (iii). Since G is not isomorphic to $K_1 + T$, for any tree T, $\gamma_{ctd}(G) \geq 2$. If G is the graph as in (i), the subset of $V(G)$ consisting of the vertex of K_1 and the pendant vertex of G forms a ctd-set of G. Therefore, $\gamma_{ctd}(G) \leq 2$ and hence $\gamma_{ctd}(G) = 2$.
Conversely, assume $\gamma_{ctd}(G) = 2$. Then, there exists a ctd-set D such that $|D| = 2$.
Case 1. Let $D = \{u, v\}$.
(a) If u or v is a pendant vertex in G, then all the vertices of $V - D$ are adjacent to v or u. Therefore, G is the graph mentioned in (i).
(b) Let $deg_G(u) \geq 2$ and $deg_G(v) \geq 2$. Since $< V - D >$ is a tree and D is a dominating set of G, each vertex in $V - D$ is adjacent to atleast one vertex in D. Hence, G is the graph as in (ii).
Case 2 $< D > \cong 2K_1$.
Then G is the graph mentioned in (iii). □

Theorem 2.10. *Let p $(p \geq 4)$ be an integer. For each k satisfying*
$2 \leq k \leq p - 2$, there is a connected graph G with $\gamma_{ctd}(G) = k$.

Proof. Let G be a graph obtained from $K_1 + T$ either (i) by attaching a complete graph on k $(k \geq 2)$ vertices at the vertex of K_1 or (ii) by attaching $k - 1$ pendant edges at the vertex of K_1, then the set containing either vertices of the complete graph on k vertices or $(k - 1)$ pendant vertices with the vertex of K_1 forms a γ_{ctd}-set and hence, $\gamma_{ctd}(G) = k$. □

Example 2.11. *Let $T \cong P_5$ and $k = 4$. Then, G is atleast one of G_1 and G_2 given in Figure 1 and $\gamma_{ctd}(G_1) = \gamma_{ctd}(G_2) = 4$.*

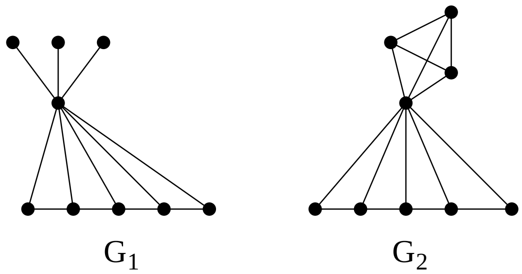

Figure 1.

Theorem 2.12. *Given two integers a and b with $2 \leq a \leq b$, there exists a graph G with $a+b+1$ vertices, such that $\gamma(G) = a$ and $\gamma_{ctd}(G) = b$.*

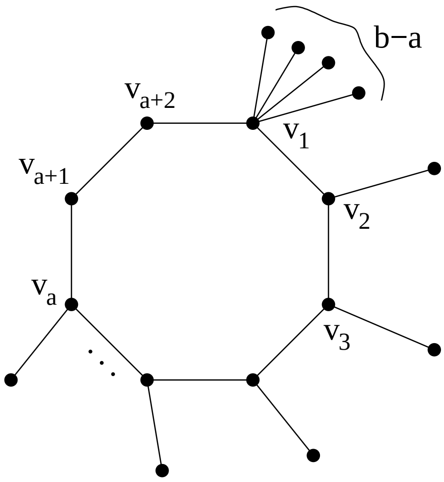

Figure 2.

Proof. In the cycle C_{a+2} $(a \geq 2)$ of length $a+2$, consider a path P of length a. In this path, attach $(b-a)$ pendant edges at exactly one vertex and attach one pendant edge at each of the remaining $(a-1)$ vertices. Let the graph thus obtained be denoted by G and G has $a+b+1$ vertices. The set of supports of the graph G forms a minimum dominating set of G and the set consisting of all the pendant vertices of G and a vertex of degree 2 in G, forms a minimum ctd-set of G. Therefore, $\gamma(G) = a$ and $\gamma_{ctd}(G) = b - a + a - 1 + 1 = b$. Hence, there exists a graph G with $\gamma(G) = a$, $\gamma_{ctd}(G) = b$. □

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