International Journal of Mathematics And its Applications
Volume 4, Issue 1–D (2016), 21–33.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Cordial Labeling of Generalized Jahangir Graph

Research Article

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Abstract: In this paper we have proved that the Generalized Jahangir graph $J_{m,n}$ is cordial for all $m \geqslant 1$ and $n \geqslant 3$, except $J_{1,4n-1}$ for all $n \geqslant 1$.

MSC: 05C78

Keywords: Graph Labeling, Cordial Labeling, Generalized Jahangir graph.

1. Introduction

We consider only simple, finite, undirected and non-trivial graph $G = (V, E)$ with the vertex set V (often denoted as $V(G)$) and the edge set E (often denoted as $E(G)$). The number of elements of V, denoted as $|V|$ is called the order of the graph G while the number of elements of E, denoted as $|E|$ is called the size of the graph G. $J_{m,n}$ denotes the Generalized Jahangir graph. For various graph theoretic notations and terminology we follow Gross and Yellen [1] whereas for number theory we follow D. M. Burton [2]. We will give brief summary of definitions and other information which are useful for the present investigations.

1.1. Preliminaries

Definition 1.1. *If the vertices of the graph are assigned values subject to certain conditions then it is known as graph labeling.*

For latest survey on graph labeling we refer to J. A. Gallian [3]. Vast amount of literature is available on different types of graph labeling and more than 2000 research papers have been published so far in last four decades according to the above mentioned survey article. The aim of the present work is to discuss one such labeling known as cordial labeling.

Definition 1.2. *A mapping $f : V(G) \to \{0, 1\}$ is called binary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f.*

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Definition 1.3. *For an edge $e = uv$, the induced edge labeling $f^* : E(G) \to \{0,1\}$ is given by $f^*(e) = |f(u) - f(v)|$. We introduce the following notations:*

$$\left.\begin{array}{rl} v_f(i) = & \text{number of vertices of } G \text{ having label } i \text{ under } f \\ e_f(i) = & \text{number of edges of } G \text{ having label } i \text{ under } f^* \end{array}\right\} \text{where } i = 0 \text{ or } 1$$

Definition 1.4. *A binary vertex labeling of a graph G is called a cordial labeling if $|v_f(0) - v_f(1)| \leqslant 1$ and $|e_f(0) - e_f(1)| \leqslant 1$. A graph G is cordial if it admits cordial labeling.*

The concept of cordial labeling was introduced by Cahit[4]. In the same paper the author proved that tree is cordial, K_n is cordial if and only if $n \leqslant 3$ and W_n is cordial if and only if $n \not\equiv 3 (\text{mod } 4)$. Ho et al.[5] proved that unicyclic graph is cordial unless it is C_{4k+2}. Andar et al.[6] have discussed cordiality of multiple shells. Vaidya et al.[7–9] have also discussed the cordiality of various graphs.

Definition 1.5. *The Generalized Jahangir graph $J_{m,n}$ for $m \geqslant 3$ is a graph on $mn + 1$ vertices, consisting of a cycle C_{mn} with one additional vertex that is adjacent to n vertices of C_{mn} at distance m to each other on C_{mn}.*

Let for a Generalized Jahangir graph $J_{m,n}$, v_0 be an apex vertex and $v_1, v_2, v_3, \ldots, v_{mn}$ be the rim vertices. Set of edges $E(J_{m,n}) = \{v_i v_{i+1} : i = 1, 2, 3, \ldots, mn-1\} \cup \{v_{mn} v_1\} \cup \{v_0 v_{1+m(i-1)} : i = 1, 2, 3, \ldots, n\}$. So for a Generalized Jahangir graph $J_{m,n}$, $|V| = mn + 1$ and $|E| = (m+1)n$. $J_{1,n}$ is a wheel graph W_n. Therefore as mentioned above, $J_{1,4n-1}$ is not cordial for all $n \geqslant 1$.

2. Cordial labeling of Generalized Jahangir graph

Theorem 2.1. *$J_{2m-1,4n}$ is cordial for $m \geqslant 1, n \geqslant 1$.*

Proof. Let v_0 be an apex vertex and $v_1, v_2, v_3, \ldots, v_{8mn-4n}$ be the rim vertices of Generalized Jahangir graph $J_{2m-1,4n}$. So for the graph $J_{2m-1,4n}$, $|V(J_{2m-1,4n})| = 8mn - 4n + 1$ and $|E(J_{2m-1,4n})| = 8mn$. Define a binary vertex labeling $f : V(J_{2m-1,4n}) \to \{0,1\}$ as follows:

$$f(v) = \begin{cases} 0 & \text{for } v = v_0; \\ 1 & \text{for } v = v_{4i-3}, v_{4i-2} \text{ where } i = 1, 2, 3, \ldots, (2m-1)n; \\ 0 & \text{for } v = v_{4i-1}, v_{4i} \text{ where } i = 1, 2, 3, \ldots, (2m-1)n. \end{cases}$$

From the above labeling, we can easily check that $v_f(0) = 4mn - 2n + 1$ and $v_f(1) = 4mn - 2n$. So $|v_f(0) - v_f(1)| = 1$. Let e be an arbitrary edge of $J_{2m-1,4n}$, then we can easily check that:

- If $e = v_{4i-3} v_{4i-2}$ for $i = 1, 2, \ldots, (2m-1)n$, then $f^*(e) = 0$.
- If $e = v_{4i-1} v_{4i}$ for $i = 1, 2, \ldots, (2m-1)n$, then $f^*(e) = 0$.
- If $e = v_0 v_{1+(2m-1)(4i-3)}$ for $i = 1, 2, \ldots, n$, then $f^*(e) = 0$.
- If $e = v_0 v_{1+(2m-1)(4i-2)}$ for $i = 1, 2, \ldots, n$, then $f^*(e) = 0$.

Thus $e_f(0) = (2m-1)n + (2m-1)n + n + n = 4mn$. For the remaining edges, $f^*(e) = 1$. So we have $e_f(1) = 8mn - 4mn = 4mn$. Hence $|e_f(0) - e_f(1)| = |4mn - 4mn| = 0$. Therefore $J_{2m-1,4n}$ is cordial graph for $m \geqslant 1, n \geqslant 1$. □

Example 2.2. *Cordial labeling of $J_{3,8}$.*

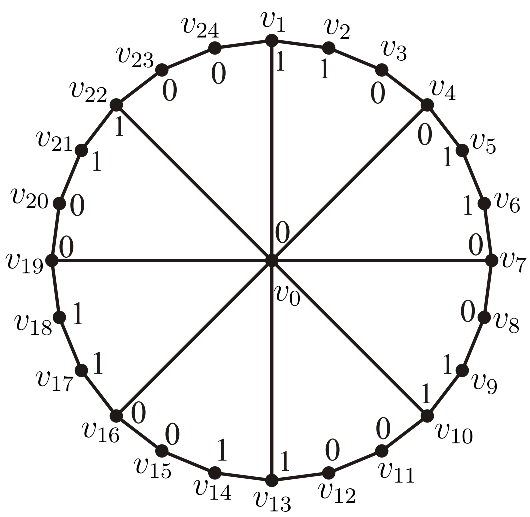

Figure 1: Cordial labeling of $J_{3,8}$.

Here we can easily check from the Figure 1, that $v_f(0) = 13$ and $v_f(1) = 12$. So $|v_f(0)-v_f(1)| = 1$. Also $e_f(0) = e_f(1) = 16$, so $|e_f(0) - e_f(1)| = 0$. Thus $J_{3,8}$ is cordial graph.

Theorem 2.3. *$J_{4m-1,4n+2}$ is cordial for $m \geqslant 1, n \geqslant 1$.*

Proof. Let v_0 be an apex vertex and $v_1, v_2, v_3, \ldots, v_{16mn+8m-4n-2}$ be the rim vertices of Generalized Jahangir graph $J_{4m-1,4n+2}$. So for the graph $J_{4m-1,4n+2}$, $|V(J_{4m-1,4n+2})| = 16mn + 8m - 4n - 1$ and $|E(J_{4m-1,4n+2})| = 16mn + 8m$. Define a binary vertex labeling $f : V(J_{4m-1,4n+2}) \to \{0, 1\}$ as follows:

$$f(v) = \begin{cases} 0 & \text{for } v = v_0, v_1; \\ 1 & \text{for } v = v_{4i-3} \text{ where } i = 2, 3, \ldots, (4m-1)n + 2m; \\ 1 & \text{for } v = v_{4i-2} \text{ where } i = 1, 2, 3, \ldots, (4m-1)n + 2m; \\ 0 & \text{for } v = v_{4i-1}, v_{4i} \text{ where } i = 1, 2, 3, \ldots, (4m-1)n + 2m - 1. \end{cases}$$

From the above labeling, we can easily check that $v_f(0) = 8mn+4m-2n$ and $v_f(1) = 8mn+4m-2n-1$. So $|v_f(0)-v_f(1)| = 1$. Let e be an arbitrary edge of $J_{4m-1,4n+2}$, then we can easily check that:

- If $e = v_{4i-3}v_{4i-2}$ for $i = 2, \ldots, (4m-1)n + 2m$, then $f^*(e) = 0$.
- If $e = v_{4i-1}v_{4i}$ for $i = 1, 2, \ldots, (4m-1)n + 2m - 1$, then $f^*(e) = 0$.
- If $e = v_0v_1$, then $f^*(e) = 0$.
- If $e = v_0v_{1+(4m-1)(4i-3)}$ for $i = 1, 2, \ldots, n + 1$, then $f^*(e) = 0$.
- If $e = v_0v_{1+(4m-1)(4i-2)}$ for $i = 1, 2, \ldots, n$, then $f^*(e) = 0$.

Thus $e_f(0) = (4m-1)n + 2m - 1 + (4m-1)n + 2m - 1 + 1 + n + 1 + n = 8mn + 4m$. For the remaining edges, $f^*(e) = 1$. So we have $e_f(1) = 16mn + 8m - 8mn - 4m = 8mn + 4m$. Thus $|e_f(0) - e_f(1)| = 0$.
Therefore $J_{4m-1,4n+2}$ is cordial graph for $m \geqslant 1, n \geqslant 1$. □

Example 2.4. *Cordial labeling of* $J_{3,6}$.

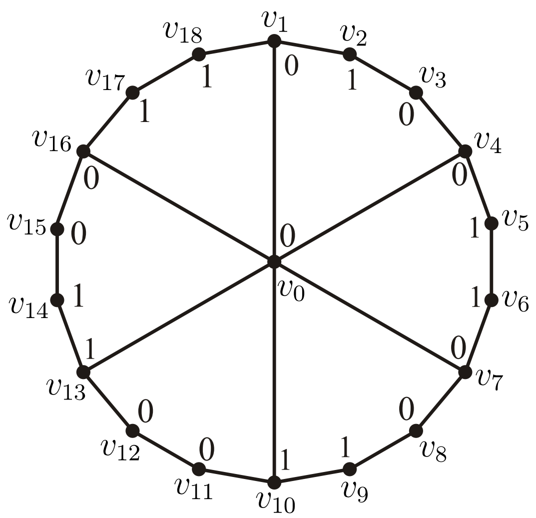

Figure 2: Cordial labeling of $J_{3,6}$.

Here we can easily check from the Figure 2, that $v_f(0) = 10$ and $v_f(1) = 9$. So $|v_f(0) - v_f(1)| = 1$. Also $e_f(0) = e_f(1) = 12$, so $|e_f(0) - e_f(1)| = 0$. Thus $J_{3,6}$ is cordial graph.

Theorem 2.5. $J_{4m-3,4n+2}$ *is cordial for* $m \geqslant 1, n \geqslant 1$.

Proof. Let v_0 be an apex vertex and $v_1, v_2, v_3, \ldots, v_{16mn+8m-12n-6}$ be the rim vertices of Generalized Jahangir graph $J_{4m-3,4n+2}$. So for the graph $J_{4m-3,4n+2}$, $|V(J_{4m-3,4n+2})| = 16mn+8m-12n-5$ and $|E(J_{4m-3,4n+2})| = 16mn+8m-8n-4$. Define a binary vertex labeling $f : V(J_{4m-3,4n+2}) \to \{0, 1\}$ as follows:

$$f(v) = \begin{cases} 0 \text{ for } v = v_0; \\ 1 \text{ for } v = v_{4i-3}, v_{4i-2} \text{ where } i = 1, 2, 3, \ldots, (4m-3)n + 2m - 1; \\ 0 \text{ for } v = v_{4i-1}, v_{4i} \text{ where } i = 1, 2, 3, \ldots, (4m-3)n + 2m - 2. \end{cases}$$

From the above labeling, we can easily check that $v_f(0) = 8mn + 4m - 6n - 3$ and $v_f(1) = 8mn + 4m - 6n - 2$. So $|v_f(0) - v_f(1)| = 1$. Let e be an arbitrary edge of $J_{4m-3,4n+2}$, then we can easily check that:

- If $e = v_{4i-3}v_{4i-2}$ for $i = 1, 2, \ldots, (4m-3)n + 2m - 1$, then $f^*(e) = 0$.
- If $e = v_{4i-1}v_{4i}$ for $i = 1, 2, \ldots, (4m-3)n + 2m - 2$, then $f^*(e) = 0$.
- If $e = v_{16mn+8m-12n-6}v_1$, then $f^*(e) = 0$.
- If $e = v_0v_{1+(4m-3)(4i-3)}$ for $i = 1, 2, \ldots, n$, then $f^*(e) = 0$.
- If $e = v_0v_{1+(4m-1)(4i-2)}$ for $i = 1, 2, \ldots, n$, then $f^*(e) = 0$.

Thus $e_f(0) = (4m-3)n + 2m - 1 + (4m-3)n + 2m - 2 + 1 + n + n = 8mn + 4m - 4n - 2$. For the remaining edges, $f^*(e) = 1$. So we have $e_f(1) = 16mn + 8m - 8n - 4 - 8mn - 4m + 4n + 2 = 8mn + 4m - 4n - 2$. Hence $|e_f(0) - e_f(1)| = 0$. Therefore $J_{4m-3,4n+2}$ is cordial graph for $m \geqslant 1, n \geqslant 1$. □

Example 2.6. *Cordial labeling of* $J_{5,6}$.

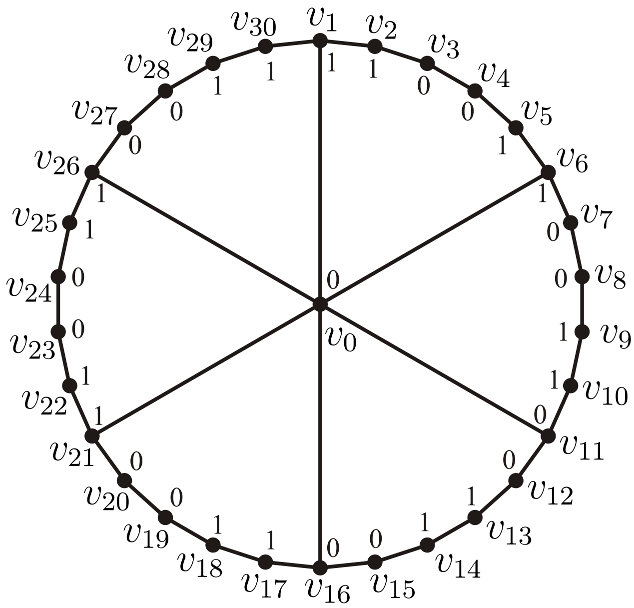

Figure 3: Cordial labeling of $J_{5,6}$.

Here we can easily check from the Figure 3, that $v_f(0) = 15$ and $v_f(1) = 16$. So $|v_f(0)-v_f(1)| = 1$. Also $e_f(0) = e_f(1) = 18$, so $|e_f(0) - e_f(1)| = 0$. Thus $J_{5,6}$ is cordial graph.

Theorem 2.7. $J_{4m-1,4n-1}$ *is cordial for* $m \geqslant 1, n \geqslant 1$.

Proof. Let v_0 be an apex vertex and $v_1, v_2, v_3, \ldots, v_{16mn-4m-4n+1}$ be the rim vertices of Generalized Jahangir graph $J_{4m-1,4n-1}$. So for the graph $J_{4m-1,4n-1}$, $|V(J_{4m-1,4n-1})| = 16mn - 4m - 4n + 2$ and $|E(J_{4m-1,4n-1})| = 16mn - 4m$. Define a binary vertex labeling $f : V(J_{4m-1,4n-1}) \to \{0,1\}$ as follows:

$$f(v) = \begin{cases} 1 & \text{for } v = v_0; \\ 1 & \text{for } v = v_{4i-3}, v_{4i-2} \text{ where } i = 1,2,3,\ldots,(4m-1)n-m; \\ 0 & \text{for } v = v_{4i-1}, v_{4i} \text{ where } i = 1,2,3,\ldots,(4m-1)n-m; \\ 0 & \text{for } v = v_{16mn-4m-4n+1}. \end{cases}$$

From the above labeling, we can easily check that $v_f(0) = 8mn - 2m - 2n + 1$ and $v_f(1) = 8mn - 2m - 2n + 1$. So $|v_f(0) - v_f(1)| = 0$. Let e be an arbitrary edge of $J_{4m-1,4n-1}$, then we can easily check that:

- If $e = v_{4i-3}v_{4i-2}$ for $i = 1, 2, \ldots, (4m-1)n - m$, then $f^*(e) = 0$.
- If $e = v_{4i-1}v_{4i}$ for $i = 1, 2, \ldots, (4m-1)n - m$, then $f^*(e) = 0$.
- If $e = v_{16mn-4m-4n}v_{16mn-4m-4n+1}$, then $f^*(e) = 0$.
- If $e = v_0v_{1+(4m-1)(4i-1)}$ for $i = 1, 2, \ldots, n-1$, then $f^*(e) = 0$.
- If $e = v_0v_{1+(4m-1)(4i)}$ for $i = 1, 2, \ldots, n-1$, then $f^*(e) = 0$.
- If $e = v_0v_1$, then $f^*(e) = 0$.

Thus $e_f(0) = (4m-1)n - m + (4m-1)n - m + 1 + n - 1 + n - 1 + 1 = 8mn - 2m$. For the remaining edges, $f^*(e) = 1$. So we have $e_f(1) = 16mn - 4m - 8mn + 2m = 8mn - 2m$. Hence $|e_f(0) - e_f(1)| = 0$.

Therefore $J_{4m-1,4n-1}$ is cordial graph for $m \geqslant 1, n \geqslant 1$. □

Example 2.8. *Cordial labeling of $J_{3,7}$.*

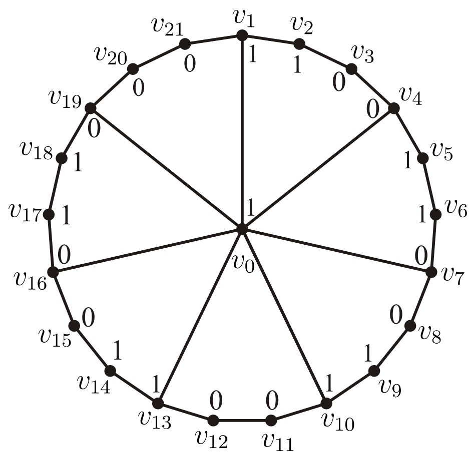

Figure 4: Cordial labeling of $J_{3,7}$.

Here we can easily check from the Figure 4, that $v_f(0) = v_f(1) = 11$. So $|v_f(0) - v_f(1)| = 0$. Also $e_f(0) = e_f(1) = 14$, so $|e_f(0) - e_f(1)| = 0$. Thus $J_{3,7}$ is cordial graph.

Theorem 2.9. *$J_{4m-1,4n+1}$ is cordial for $m \geqslant 1, n \geqslant 1$.*

Proof. Let v_0 be an apex vertex and $v_1, v_2, v_3, \ldots, v_{16mn+4m-4n-1}$ be the rim vertices of Generalized Jahangir graph $J_{4m-1,4n+1}$. So for the graph $J_{4m-1,4n+1}$, $|V(J_{4m-1,4n+1})| = 16mn + 4m - 4n$ and $|E(J_{4m-1,4n+1})| = 16mn + 4m$. Define a binary vertex labeling $f : V(J_{4m-1,4n+1}) \to \{0,1\}$ as follows:

$$f(v) = \begin{cases} 1 & \text{for } v = v_0; \\ 1 & \text{for } v = v_{4i-3} \text{ where } i = 1,2,3,\ldots,(4m-1)n+m; \\ 1 & \text{for } v = v_{4i-2} \text{ where } i = 1,2,3,\ldots,(4m-1)n+m-1; \\ 0 & \text{for } v = v_{4i-1} \text{ where } i = 1,2,3,\ldots,(4m-1)n+m; \\ 0 & \text{for } v = v_{4i} \text{ where } i = 1,2,3,\ldots,(4m-1)n+m-1; \\ 0 & \text{for } v = v_{16mn+4m-4n-2}. \end{cases}$$

From the above labeling, we can easily check that $v_f(0) = v_f(1) = 8mn + 2m - 2n$. So $|v_f(0) - v_f(1)| = 0$. Let e be an arbitrary edge of $J_{4m-1,4n+1}$, then we can easily check that:

- If $e = v_{4i-3}v_{4i-2}$ for $i = 1,2,\ldots,(4m-1)n+m-1$, then $f^*(e) = 0$.
- If $e = v_{4i-1}v_{4i}$ for $i = 1,2,\ldots,(4m-1)n+m-1$, then $f^*(e) = 0$.
- If $e = v_{16mn+4m-4n-2}v_{16mn+4m-4n-1}$, then $f^*(e) = 0$.
- If $e = v_0v_{1+(4m-1)(4i-3)}$ for $i = 1,2,\ldots,n$, then $f^*(e) = 0$.
- If $e = v_0v_{1+(4m-1)(4i-2)}$ for $i = 1,2,\ldots,n$, then $f^*(e) = 0$.

- If $e = v_0v_1$, then $f^*(e) = 0$.

Thus $e_f(0) = (4m-1)n + m - 1 + (4m-1)n + m - 1 + 1 + n + n + 1 = 8mn + 2m$. For the remaining edges, $f^*(e) = 1$. So we have $e_f(1) = 16mn + 4m - 8mn - 2m = 8mn + 2m$. Hence $|e_f(0) - e_f(1)| = 0$.
Therefore $J_{4m-1,4n+1}$ is cordial graph for $m \geqslant 1, n \geqslant 1$. □

Example 2.10. *Cordial labeling of* $J_{3,5}$.

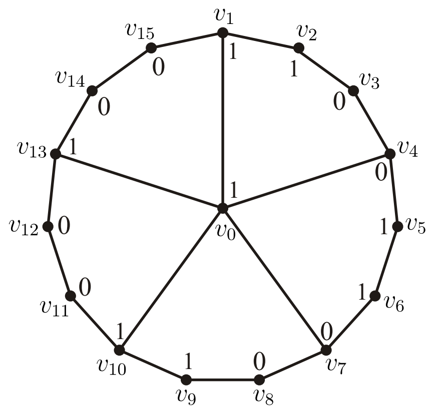

Figure 5: Cordial labeling of $J_{3,5}$.

Here we can easily check from the Figure 5, that $v_f(0) = v_f(1) = 8$. So $|v_f(0) - v_f(1)| = 0$. Also $e_f(0) = e_f(1) = 10$, so $|e_f(0) - e_f(1)| = 0$. Thus $J_{3,5}$ is cordial graph.

Theorem 2.11. $J_{4m-3,4n-1}$ *is cordial for* $m > 1, n \geqslant 1$.

Proof. Let v_0 be an apex vertex and $v_1, v_2, v_3, \ldots, v_{16mn-4m-12n+3}$ be the rim vertices of Generalized Jahangir graph $J_{4m-3,4n-1}$. So for the graph $J_{4m-3,4n-1}$, $|V(J_{4m-3,4n-1})| = 16mn-4m-12n+4$ and $|E(J_{4m-3,4n-1})| = 16mn-4m-8n+2$. Define a binary vertex labeling $f : V(J_{4m-3,4n-1}) \to \{0,1\}$ as follows:

$$f(v) = \begin{cases} 1 & \text{for } v = v_0; \\ 1 & \text{for } v = v_{4i-3} \text{ where } i = 1,2,3,\ldots,(4m-3)n-m+1; \\ 1 & \text{for } v = v_{4i-2} \text{ where } i = 1,2,3,\ldots,(4m-3)n-m; \\ 0 & \text{for } v = v_{4i-1} \text{ where } i = 1,2,3,\ldots,(4m-3)n-m+1; \\ 0 & \text{for } v = v_{4i} \text{ where } i = 1,2,3,\ldots,(4m-3)n-m; \\ 0 & \text{for } v = v_{16mn-4m-12n+2}. \end{cases}$$

From the above labeling, we can easily check that $v_f(0) = v_f(1) = 8mn - 2m - 6n + 2$. So $|v_f(0) - v_f(1)| = 0$. Let e be an arbitrary edge of $J_{4m-3,4n-1}$, then we can easily check that:

- If $e = v_{4i-3}v_{4i-2}$ for $i = 1,2,\ldots,(4m-3)n-m$, then $f^*(e) = 0$.
- If $e = v_{4i-1}v_{4i}$ for $i = 1,2,\ldots,(4m-3)n-m$, then $f^*(e) = 0$.
- If $e = v_{16mn-4m-12n+2}v_{16mn-4m-12n+3}$, then $f^*(e) = 0$.
- If $e = v_0v_{1+(4m-3)(4i-4)}$ for $i = 1,2,\ldots,n$, then $f^*(e) = 0$.
- If $e = v_0v_{1+(4m-3)(4i-3)}$ for $i = 1,2,\ldots,n$, then $f^*(e) = 0$.

Thus $e_f(0) = (4m-3)n - m + (4m-3)n - m + 1 + n + n = 8mn - 2m - 4n + 1$. For the remaining edges, $f^*(e) = 1$. So we have $e_f(1) = 16mn - 4m - 8n + 2 - 8mn + 2m + 4n - 1 = 8mn - 2m - 4n + 1$. Hence $|e_f(0) - e_f(1)| = 0$. Therefore $J_{4m-3,4n-1}$ is cordial graph for $m > 1, n \geqslant 1$. □

Example 2.12. *Cordial labeling of* $J_{5,3}$.

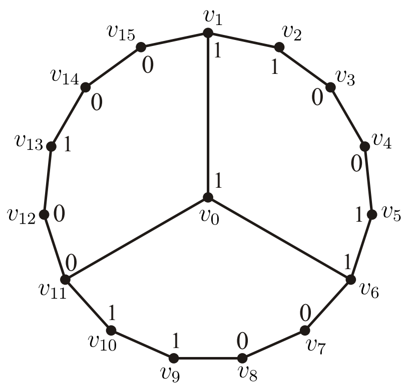

Figure 6: Cordial labeling of $J_{5,3}$.

Here we can easily check from the Figure 6, that $v_f(0) = v_f(1) = 8$. So $|v_f(0) - v_f(1)| = 0$. Also $e_f(0) = e_f(1) = 9$, so $|e_f(0) - e_f(1)| = 0$. Thus $J_{5,3}$ is cordial graph.

Theorem 2.13. $J_{4m-3,4n+1}$ *is cordial for* $m \geqslant 1, n \geqslant 1$.

Proof. Let v_0 be an apex vertex and $v_1, v_2, v_3, \ldots, v_{16mn+4m-12n-3}$ be the rim vertices of generalized Jahangir graph $J_{4m-3,4n+1}$. So for the graph $J_{4m-3,4n+1}$, $|V(J_{4m-3,4n+1})| = 16mn+4m-12n-2$ and $|E(J_{4m-3,4n+1})| = 16mn+4m-8n-2$. Define a binary vertex labeling $f : V(J_{4m-3,4n+1}) \to \{0,1\}$ as follows:

$$f(v) = \begin{cases} 0 & \text{for} \quad v = v_0; \\ 1 & \text{for} \quad v = v_{4i-3}, v_{4i-2} \quad \text{where} \quad i = 1, 2, 3, \ldots, (4m-3)n + m - 1; \\ 0 & \text{for} \quad v = v_{4i-1}, v_{4i} \quad \text{where} \quad i = 1, 2, 3, \ldots, (4m-3)n + m - 1; \\ 1 & \text{for} \quad v = v_{16mn+4m-12n-3}. \end{cases}$$

From the above labeling, we can easily check that $v_f(0) = v_f(1) = 8mn + 2m - 6n - 1$. So $|v_f(0) - v_f(1)| = 0$. Let e be an arbitrary edge of $J_{4m-3,4n+1}$, then we can easily check that:

- If $e = v_{4i-3}v_{4i-2}$ for $i = 1, 2, \ldots, (4m-3)n + m - 1$, then $f^*(e) = 0$.
- If $e = v_{4i-1}v_{4i}$ for $i = 1, 2, \ldots, (4m-3)n + m - 1$, then $f^*(e) = 0$.
- If $e = v_{16mn+4m-12n-3}v_1$, then $f^*(e) = 0$.
- If $e = v_0 v_{1+(4m-1)(4i-2)}$ for $i = 1, 2, \ldots, n$, then $f^*(e) = 0$.
- If $e = v_0 v_{1+(4m-1)(4i-1)}$ for $i = 1, 2, \ldots, n$, then $f^*(e) = 0$.

Thus $e_f(0) = (4m-3)n + m - 1 + (4m-3)n + m - 1 + 1 + n + n = 8mn + 2m - 4n - 1$. For the remaining edges, $f^*(e) = 1$. So we have $e_f(1) = 16mn + 4m - 8n - 2 - 8mn - 2m + 4n + 1 = 8mn + 2m - 4n - 1$. Hence $|e_f(0) - e_f(1)| = 0$. Therefore $J_{4m-3,4n+1}$ is cordial graph for $m \geqslant 1, n \geqslant 1$. □

Example 2.14. *Cordial labeling of $J_{5,5}$.*

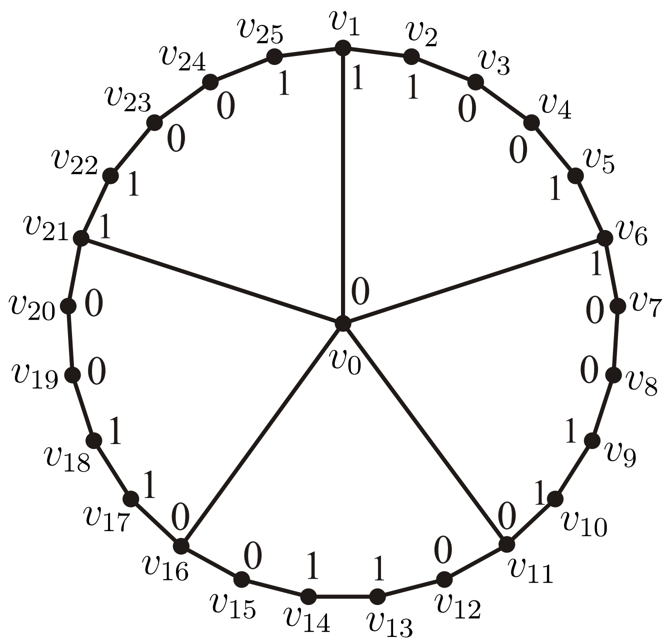

Figure 7: Cordial labeling of $J_{5,5}$.

Here we can easily check from the Figure 7, that $v_f(0) = v_f(1) = 13$. So $|v_f(0) - v_f(1)| = 0$. Also $e_f(0) = e_f(1) = 15$, so $|e_f(0) - e_f(1)| = 0$. Thus $J_{5,5}$ is cordial graph.

Theorem 2.15. *$J_{4m-2,2n}$ is cordial for $m \geqslant 1, n \geqslant 2$.*

Proof. Let v_0 be an apex vertex and $v_1, v_2, v_3, \dots, v_{8mn-4n}$ be the rim vertices of Generalized Jahangir graph $J_{4m-2,2n}$. So for the graph $J_{4m-2,2n}$, $|V(J_{4m-2,2n})| = 8mn - 4n + 1$ and $|E(J_{4m-2,2n})| = 8mn - 2n$. Define a binary vertex labeling $f : V(J_{4m-2,2n}) \to \{0, 1\}$ as follows:

$$f(v) = \begin{cases} 0 \text{ for } v = v_0; \\ 1 \text{ for } v = v_{4i-3}, v_{4i-2} \text{ where } i = 1, 2, 3, \dots, (2m-1)n; \\ 0 \text{ for } v = v_{4i-1}, v_{4i} \text{ where } i = 1, 2, 3, \dots, (2m-1)n. \end{cases}$$

From the above labeling, we can easily check that $v_f(0) = 4mn - 2n + 1$ and $v_f(1) = 4mn - 2n$. So $|v_f(0) - v_f(1)| = 1$. Let e be an arbitrary edge of $J_{4m-2,2n}$, then we can easily check that:

- If $e = v_{4i-3}v_{4i-2}$ for $i = 1, 2, \dots, (2m-1)n$, then $f^*(e) = 0$.
- If $e = v_{4i-1}v_{4i}$ for $i = 1, 2, \dots, (2m-1)n$, then $f^*(e) = 0$.
- If $e = v_0 v_{1+(4m-2)(2i-1)}$ for $i = 1, 2, \dots, n$, then $f^*(e) = 0$.

Thus $e_f(0) = (2m-1)n + (2m-1)n + n = 4mn - n$. For the remaining edges, $f^*(e) = 1$. So we have $e_f(1) = 8mn - 2n - 4mn + n = 4mn - n$. Hence $|e_f(0) - e_f(1)| = 0$.

Therefore $J_{4m-2,2n}$ is cordial graph for $m \geqslant 1, n \geqslant 2$. □

Example 2.16. *Cordial labeling of $J_{2,6}$.*

Here we can easily check from the Figure 8, that $v_f(0) = 7$ and $v_f(1) = 6$. So $|v_f(0) - v_f(1)| = 1$. Also $e_f(0) = e_f(1) = 9$, so $|e_f(0) - e_f(1)| = 0$. Thus $J_{2,6}$ is cordial graph.

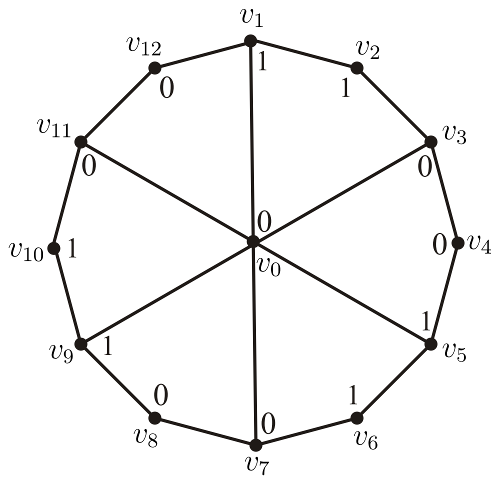

Figure 8: Cordial labeling of $J_{2,6}$.

Theorem 2.17. *$J_{4m-2,2n+1}$ is cordial for $m \geqslant 1, n \geqslant 1$.*

Proof. Let v_0 be an apex vertex and $v_1, v_2, v_3, \ldots, v_{8mn+4m-4n-2}$ be the rim vertices of Generalized Jahangir graph $J_{4m-2,2n+1}$. So for the graph $J_{4m-2,2n+1}$, $|V(J_{4m-2,2n+1})| = 8mn+4m-4n-1$ and $|E(J_{4m-2,2n+1})| = 8mn+4m-2n-1$. Define a binary vertex labeling $f : V(J_{4m-2,2n+1}) \to \{0,1\}$ as follows:

$$f(v) = \begin{cases} 0 \text{ for } v = v_0; \\ 1 \text{ for } v = v_{4i-3}, v_{4i-2} \text{ where } i = 1,2,3,\ldots,(2m-1)n+m; \\ 0 \text{ for } v = v_{4i-1}, v_{4i} \text{ where } i = 1,2,3,\ldots,(2m-1)n+m-1. \end{cases}$$

From the above labeling, we can easily check that $v_f(0) = 4mn+2m-2n-1$ and $v_f(1) = 4mn+2m-2n$. So $|v_f(0)-v_f(1)| = 1$. Let e be an arbitrary edge of $J_{4m-2,2n+1}$, then we can easily check that:

- If $e = v_{4i-3}v_{4i-2}$ for $i = 1,2,\ldots,(2m-1)n+m$, then $f^*(e) = 0$.
- If $e = v_{4i-1}v_{4i}$ for $i = 1,2,\ldots,(2m-1)n+m-1$, then $f^*(e) = 0$.
- If $e = v_0v_{1+(4m-2)(2i-1)}$ for $i = 1,2,\ldots,n$, then $f^*(e) = 0$.
- If $e = v_{8mn+4m-4n-2}v_1$, then $f^*(e) = 0$.

Thus $e_f(0) = (2m-1)n+m+(2m-1)n+m-1+n+1 = 4mn+2m-n$. For the remaining edges, $f^*(e) = 1$. So we have $e_f(1) = 8mn+4m-2n-1-4mn-2m+n = 4mn+2m-n-1$. Hence $|e_f(0)-e_f(1)| = 1$.
Therefore $J_{4m-2,2n+1}$ is cordial graph for $m \geqslant 1, n \geqslant 1$. □

Example 2.18. *Cordial labeling of $J_{6,3}$.*

Here we can easily check from the Figure 9, that $v_f(0) = 9$ and $v_f(1) = 10$. So $|v_f(0)-v_f(1)| = 1$. Also $e_f(0) = 11$ and $e_f(1) = 10$, so $|e_f(0)-e_f(1)| = 1$. Thus $J_{6,3}$ is cordial graph.

Theorem 2.19. *$J_{4,n}$ is cordial for $n \geqslant 3$.*

Proof. **Case:1** $n = 2k, k \geqslant 2$

Let v_0 be an apex vertex and $v_1, v_2, v_3, \ldots, v_{8k}$ be the rim vertices of Generalized Jahangir graph $J_{4,2k}$. So for the graph $J_{4,2k}$, $|V(J_{4,2k})| = 8k+1$ and $|E(J_{4,2k})| = 10k$. Define a binary vertex labeling $f : V(J_{4,2k}) \to \{0,1\}$ as follows:

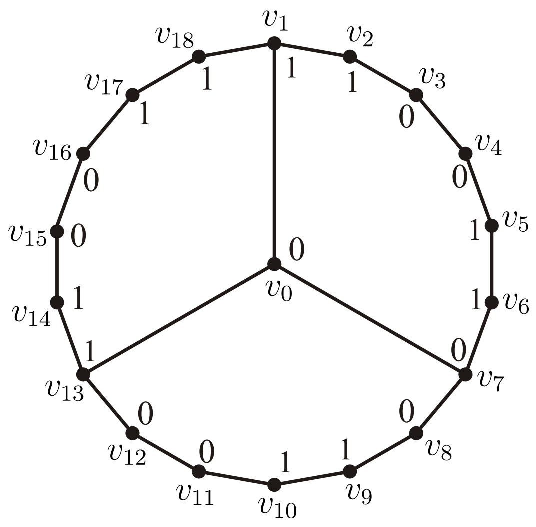

Figure 9: Cordial labeling of $J_{6,3}$.

$$f(v) = \begin{cases} 0 \text{ for } v = v_0; \\ 0 \text{ for } v = v_{8i-7}, v_{8i-6}, v_{8i-5}, v_{8i-1} \text{ where } i = 1, 2, 3, \ldots, k; \\ 1 \text{ for } v = v_{8i-4}, v_{8i-3}, v_{8i-2}, v_{8i} \text{ where } i = 1, 2, 3, \ldots, k. \end{cases}$$

From the above labeling, we can easily check that $v_f(0) = 4k + 1$ and $v_f(1) = 4k$. So $|v_f(0) - v_f(1)| = 1$. Let e be an arbitrary edge of $J_{4,2k}$, then we can easily check that:

- If $e = v_{8i-7}v_{8i-6}$ for $i = 1, 2, \ldots, k$, then $f^*(e) = 0$.
- If $e = v_{8i-6}v_{8i-5}$ for $i = 1, 2, \ldots, k$, then $f^*(e) = 0$.
- If $e = v_{8i-4}v_{8i-3}$ for $i = 1, 2, \ldots, k$, then $f^*(e) = 0$.
- If $e = v_{8i-3}v_{8i-2}$ for $i = 1, 2, \ldots, k$, then $f^*(e) = 0$.
- If $e = v_0v_{8i-7}$ for $i = 1, 2, \ldots, k$, then $f^*(e) = 0$.

Thus $e_f(0) = k + k + k + k + k = 5k$. For the remaining edges, $f^*(e) = 1$. So we have $e_f(1) = 10k - 5k = 5k$. Hence $|e_f(0) - e_f(1)| = 0$.

Therefore $J_{4,2k}$ is cordial graph for $k \geqslant 2$.

Case:2 $n = 2k + 1, k \geqslant 1$

Let v_0 be an apex vertex and $v_1, v_2, v_3, \ldots, v_{8k+4}$ be the rim vertices of Generalized Jahangir graph $J_{4,2k+1}$. So for the graph $J_{4,2k+1}$, $|V(J_{4,2k+1})| = 8k+5$ and $|E(J_{4,2k+1})| = 10k+5$. Define a binary vertex labeling $f : V(J_{4,2k+1}) \to \{0, 1\}$ as follows:

$$f(v) = \begin{cases} 1 \text{ for } v = v_0; \\ 0 \text{ for } v = v_{8i-7}, v_{8i-6}, v_{8i-5} \text{ where } i = 1, 2, 3, \ldots, k+1; \\ 0 \text{ for } v = v_{8i-1} \text{ where } i = 1, 2, 3, \ldots, k; \\ 1 \text{ for } v = v_{8i-4}, v_{8i-3}, v_{8i-2} \text{ where } i = 1, 2, 3, \ldots, k; \\ 1 \text{ for } v = v_{8n+4}; \\ 1 \text{ for } v = v_{8i} \text{ where } i = 1, 2, 3, \ldots, k; \end{cases}$$

From the above labeling, we can easily check that $v_f(0) = 4k+3$ and $v_f(1) = 4k+2$. So $|v_f(0) - v_f(1)| = 1$. Let e be an arbitrary edge of $J_{4,2k+1}$, then we can easily check that:

- If $e = v_{8i-7}v_{8i-6}$ for $i = 1, 2, \ldots, k+1$, then $f^*(e) = 0$.
- If $e = v_{8i-6}v_{8i-5}$ for $i = 1, 2, \ldots, k+1$, then $f^*(e) = 0$.
- If $e = v_{8i-4}v_{8i-3}$ for $i = 1, 2, \ldots, k$, then $f^*(e) = 0$.
- If $e = v_{8i-3}v_{8i-2}$ for $i = 1, 2, \ldots, k$, then $f^*(e) = 0$.
- If $e = v_0v_{8i-3}$ for $i = 1, 2, \ldots, k$, then $f^*(e) = 0$.

Thus $e_f(0) = k+1+k+1+k+k+k = 5k+2$. For the remaining edges, $f^*(e) = 1$. So we have $e_f(1) = 10k+5-5k-2 = 5k+3$. Hence $|e_f(0) - e_f(1)| = 1$.

Therefore $J_{4,2k+1}$ is cordial graph for $k \geqslant 1$.

Thus from Case: 1 and Case: 2 we can say that $J_{4,n}$ is cordial for $n \geqslant 3$. □

Theorem 2.20. *$J_{4m,n}$ is cordial graph for $m \geqslant 1, n \geqslant 3$.*

Proof. In Theorem 2.19, we have proved that $J_{4m,n}$ is cordial graph for $m = 1, n \geqslant 3$.
Now for $m = 2$, to prove that the graph $J_{8,n}$ is cordial, first of all take the graph $J_{4,n}$ and label the vertices as shown in the theorem 2.19. From this labeling we can easily check that, out of the 4 edges between the two consecutive vertices adjacent to the apex vertex v_0, at least one edge has label 0.
Now choose exactly one edge with 0 label between the two consecutive vertices adjacent to the apex vertex v_0. Thus we have chosen exactly n edges on the rim with label 0. Now add 4 vertices on these n chosen edges and label them as shown in the Figure 10.

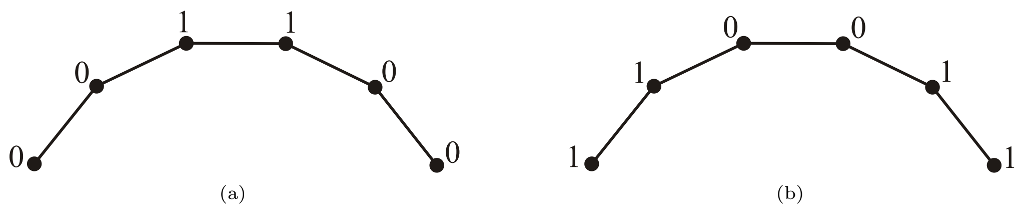

Figure 10:

If the labels of the end vertices of the chosen edge are 0 then label the inserted new four vertices between them as shown in Figure 10a and if the labels of the end vertices of the chosen edge are 1 then label the inserted new four vertices between them as shown in Figure 10b. Thus out of new added $4n$ vertices, $2n$ have label 0 and $2n$ have label 1. So the difference of $v_f(1)$ and $v_f(0)$ remains same as in $J_{4,n}$. Now by adding four vertices on an edge, we are adding four new edges. Out of these four edges, we can check from the Figure 10 that 2 edges have label 0 and 2 edges have label 1. So the difference of $e_f(1)$ and $e_f(0)$ remains same as in $J_{4,n}$. Thus $J_{8,n}$ is cordial graph.
By repeating the above process $m-1$ times on the graph $J_{4,n}$, we can get the cordial labeling of the graph $J_{4m,n}$. Thus $J_{4m,n}$ is a cordial graph. □

3. Concluding Remarks

Here we have proved that Generalized Jahangir graph $J_{m,n}$ is cordial for all $m \geqslant 1$ and for all $n \geqslant 3$ except $J_{1,4n-1}$ for all $n \geqslant 1$. Extending the study to other results about cordial labeling of other families of graph is an open area of research.

References

[1] J.Gross and J.Yellen, *Handbook of Graph Theory*, CRC Press, (2004).

[2] D.M.Burton, *Elementary Number Theory. 6th ed*, Brown Publishers, (2007).

[3] J.A.Gallian, *A Dynamic Survey of Graph Labeling*, The Electronic Journal of Combinatorics, 17(2014), # DS6

[4] I.Cahit, *Cordial Graphs: A weaker version of graceful and harmonious Graphs*, Ars Combinatoria, 23(1987), 201-207.

[5] Y.S.Ho, S.M.Lee and S.C.Shee, *Cordial labeling of unicyclic graphs and generalized Petersen graphs*, Congress. Numer., 68(1989), 109-122.

[6] M.Andar, S.Boxwala and N.B.Limaye, *A Note on cordial labeling of multiple shells*, Trends Math., (2002), 77-80.

[7] S.K.Vaidya, G.V.Ghodasara, Sweta Srivastav and V.J.Kaneria, *Cordial labeling for two cycle related graphs*, The Math.Student, J. of Indian Math. Society, 76(2007), 237-246.

[8] S.K.Vaidya, G.V.Ghodasara, Sweta Srivastav and V.J.Kaneria, *Some new cordial graphs*, Int. J. of Sci. Comp., 2(1)(2008), 81-92.

[9] S.K.Vaidya, Sweta Srivastav, G.V.Ghodasara and V.J.Kaneria, *Cordial labeling for cycle with one chord and its related graphs*, Indian J. of Math. and Math. Sciences, 4(2)(2008), 145-156.