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Stability of Quintic Functional Equation in 2-Banach Space

Research Article

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Abstract: In this paper, we investigate the Hyers-Ulam stability of the functional equation

$$2g(2x+y)+2g(2x-y)+g(x+2y)+g(x-2y)=20\,[g(x+y)+g(x-y)]+90g(x) \tag{1}$$

in 2-Banach space.

MSC: 39B82, 46B86, 17C65.

Keywords: Hyers-Ulam stability, 2-Banach space, Quintic functional equation.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [8] concerning the stability of group homomorphisms. Hyers [4] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyer's theorem was generalized by Aoki [1] for additive mappings and by Rassias [5] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias has provided a lot of influence in the development of what we call Hyers-Ulam-Rassias stability of functional equations. In 1990, Rassias [6] asked whether such a theorem can also be proved for $p \geq 1$. In 1991,Gajda [2] gave an affirmative solution to this question when $p > 1$, but it was proved by Gajda [2] and Rassias and Semrl [7] that one cannot prove an analogous theorem when p=1. In 1994, a generalization was obtained by Gavruta [3], who replaced the bound $\epsilon\left(\|x\|^p+\|y\|^p\right)$ by a general control function $\phi(x,y)$. Beginning around 1980, the stability problems of several functional equations and approximate homomorphisms have been extensively investigated by a number of authors, and there are many interesting results concerning this problem. In the 1960s, S. Gahler and A. White [9] introduced the concept of 2-normed spaces. We introduced 2-normed space and topology on it.

Definition 1.1. *Let X be a linear space over $\mathbb{R}$ with $dimX > 1$ and let $\|.,.\| : X \times X \to \mathbb{R}$ be a function satisfying the following properties:*

(1). $\|x,y\| = 0$ if and only if x and y are linearly dependent

(2). $\|x,y\| = \|y,x\|$

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(3). $\|\lambda x, y\| = |\lambda| \, \|x, y\|$

(4). $\|x, y+z\| \leq \|x, y\| + \|x, z\|$

for each $x, y, z \in X$ *and* $a \in \mathbb{R}$*. Then the function* $\|.,.\|$ *is called a 2-norm on* X *and* $(X, \|.,.\|)$ *is a called 2-normed space. We introduce a basic property of 2-normed spaces as follows. Let* $(X, \|.,.\|)$ *be a linear 2-normed space,* $x \in X$ *and* $\|x, y\| = 0$ *for each* $y \in X$*. Suppose* $x \neq 0$*, since* $dimX > 1$*, choose* $y \in X$ *such that x,y is linearly independent so we have* $\|x, y\| \neq 0$*, which is a contradiction. Therefore we have the following Lemma.*

Lemma 1.2. *Let* $(X, \|.,.\|)$ *be a 2-normed space. If* $x \in X$ *and* $\|x, y\| = 0$*, for each* $y \in X$*, then* $x = 0$*.*

Proof. Let $(X, \|.,.\|)$ be a 2-normed space. For $x, z \in X$, let $p_z(x) = \|x, z\|$, $x \in X$. Then for each $z \in X$, p_z is a real-valued function on X such that $p_z(x) = \|x, z\| \geq 0$, $p_z(\alpha x) = |\alpha| \, \|x, z\|$ and $p_z(x+y) = \|x+y, z\| = \|z, x+y\| \leq \|z, x\| + \|z, y\| = \|x, z\| + \|y, z\| = p_z(x) + p_z(y)$, for each $\alpha \in \mathbb{R}$ and all $x, y \in X$. Thus p_z is a semi-norm for each $z \in X$. For $x \in X$, let $\|x, z\| = 0$ for each $z \in X$. By Lemma 1.2, $x = 0$. Thus for $0 \neq x \in X$, there is $z \in X$ such that $P_z(x) = \|x, z\| \neq 0$. Hence the family $\{p_z(x) : z \in X\}$ is a separating family of semi-norms. Let $x_0 \in X$, for $\epsilon > 0$, $z \in X$, let $U_{z,\epsilon}(x_0) := \{x \in X : p_z(x - x_0) < \epsilon\} = \{x \in X : \|x - x_0, z\| < \epsilon\}$. Let

$$S(x_0) := \{U_{z,\epsilon}(x_0) : \epsilon > 0, z \in X\}$$

and

$$\beta(x_0) := \{\mathcal{F} : \mathcal{F} \text{is a finite subcollection of } S(x_0)\}.$$

Define a topology τ on X by saying that a set U is open if for every $x \in U$, there is some $N \in \beta(x)$ such that $N \subset U$. That is, τ is the topology on X that has subbase $\{U_{z,\epsilon}(x_0) : \epsilon > 0, x_0 \in X, z \in X\}$. The topology τ on X makes X a topological vector space. Since for $x \in X$, collection $\beta(x)$ is a local base whose members are convex, X is locally convex. □

In the 1960 s, S. Gahler and A. White [9] introduced the concept of 2-Banach spaces.

Definition 1.3. *A sequence* $\{x_n\}$ *in a 2-Banach space* X *is called a 2-Cauchy sequence if* $\lim_{m,n\to\infty} \|x_n - x_m, x\| = 0$ *for each* $x \in X$*.*

Definition 1.4. *A sequence* $\{x_n\}$ *in a 2-normed space* X *is called a 2-convergent sequence if there is an* $x \in X$ *such that* $\lim_{x\to x_0} \|x_n - x, y\| = 0$ *for each* $y \in X$*. If* $\{x_n\}$ *converges to* x*, we write* $\lim_{n\to\infty} x_n = x$*.*

Definition 1.5. *We say that a 2-normed space* $(X, \|.,.\|)$ *is a 2-Banach space if every 2-Cauchy sequence in X is 2-convergent in X. By using (2) and (4) of definition (1.1) one can see that* $\|.,.\|$ *is continuous in each component. For a convergent sequence* x_n *in a 2-normed space X,* $\lim_{x\to x_0} \|x_n, y\| = \left\| \lim_{n\to\infty} x_n, y \right\|$ *for each* $y \in X$*.*

2. Stability of a Functional Equation for Functions $g : (X, \|.\|) \to (X, \|.,.\|)$

Throughout this section, consider X a real normed linear space. We also consider that there is a 2-norm on X which makes $(X, \|.,.\|)$ a 2-Banach space. For a function $g : (X, \|.\|) \to (X, \|.,.\|)$, define $D_g : X \times X \to X$ by

$$D_g(x, y) = 2g(2x+y) + 2g(2x-y) + g(x+2y) + g(x-2y) - 20\left[g(x+y) + g(x-y)\right] - 90g(x)$$

for each $x, y \in X$.

Theorem 2.1. *Let $\theta \geq 0, u > 0, 0 < s, t < 5$. If $g : X \to X$ is a function such that*

$$\|D_g(x,y), z\| \leq \theta(\|x\|^s + \|y\|^t)\,\|z\|^u \tag{2}$$

for each $x, y, z \in X$. Then there exists a unique quintic function $Q : X \to X$ satisfying (1) and

$$\|Q(x) - g(x), z\| \leq \frac{\theta\,\|x\|^s\,\|z\|^u}{4(32 - 2^s)} \tag{3}$$

for each $x, z \in X$.

Proof. Let x=y=0 in (2), we have $\|124g(0), z\| = 0$ for each $z \in X$, so we have g(0)=0. Put y=0 in (2), we have

$$\left\|\frac{g(2x)}{32} - g(x), z\right\| \leq \frac{\theta}{128}\,\|x\|^s\,\|z\|^u \tag{4}$$

for each $x, z \in X$. Replacing x by 2x and dividing by 32 in (4), we get

$$\left\|\frac{g(4x)}{32^2} - \frac{g(2x)}{32}, z\right\| \leq \frac{1}{32}\frac{\theta}{128}2^s\,\|x\|^s\,\|z\|^u \tag{5}$$

for each $x, z \in X$. Combine (4) and (5), we get

$$\begin{aligned}\left\|\frac{g(4x)}{32^2} - g(x), z\right\| &\leq \frac{\theta}{128}\,\|x\|^s\,\|z\|^u + \frac{\theta}{128}\frac{2^s}{32}\,\|x\|^s\,\|z\|^u \\ &\leq \frac{\theta}{128}\,\|x\|^s\,\|z\|^u\left[1 + \frac{2^s}{32}\right]\end{aligned} \tag{6}$$

for each $x, z \in X$. By using induction on n, we can show that

$$\begin{aligned}\left\|\frac{g(2^n x)}{32^n} - g(x), z\right\| &= \frac{\theta}{128}\,\|x\|^s\,\|z\|^u\sum_{j=0}^{n-1}2^{(s-5)j} \\ &\leq \frac{\theta}{128}\,\|x\|^s\,\|z\|^u\left(\frac{1 - 2^{(s-5)n}}{1 - 2^{s-5}}\right)\end{aligned} \tag{7}$$

for each $x, z \in X$. Dividing by 32^m and replacing x by $2^m x$ in (7), we get

$$\begin{aligned}\left\|\frac{g(2^{m+n}x)}{32^{m+n}} - \frac{g(2^m x)}{32^m}, z\right\| &\leq \frac{1}{32^m}\frac{\theta}{128}\,\|2^m x\|^s\,\|z\|^u\sum_{j=0}^{n-1}2^{(s-5)j} \\ &\leq \frac{\theta}{128}\,\|x\|^s\,\|z\|^u\left(\frac{2^{(s-5)m}(1 - 2^{(s-5)n})}{1 - 2^{(s-5)}}\right) \\ &\longrightarrow 0 \ as\ m, n \to \infty.\end{aligned}$$

for each $z \in X$. This shows that $\left\{\frac{g(2^n x)}{32^n}\right\}$ is a 2-Cauchy sequence in X, for each $x \in X$. Since X is a 2-Banach space, the sequence $\left\{\frac{g(2^n x)}{32^n}\right\}$ 2-converges in X, for each $x \in X$. Define $Q : X \to X$ as

$$Q(x) = \lim_{n\to\infty}\frac{g(2^n x)}{32^n}$$

for each $z \in X$. By (7), we have

$$\lim_{n\to\infty}\left\|\frac{g(2^n x)}{32^n} - g(x), z\right\| = \frac{\theta}{128}\,\|x\|^s\,\|z\|^u\left(\frac{1}{1 - 2^{s-5}}\right)$$

$$\begin{aligned}\|Q(x)-g(x),z\| &\le \frac{\theta}{128}\|x\|^s\|z\|^u\left(\frac{1}{1-\frac{2^s}{2^5}}\right)\\ &= \frac{\theta\|x\|^s\|z\|^u}{4(32-2^s)}\end{aligned}$$

for each $x, z \in X$. Next we show that Q satisfies (1). For $x \in X$

$$\begin{aligned}\|D_Q(x,y),z\| &= \lim_{n\to\infty}\frac{1}{32^n}\|D_g(2^nx,2^ny),z\|\\ &\le \lim_{n\to\infty}\theta\left[2^{(s-5)n}\|x\|^s+2^{(t-5)n}\|y\|^t\right]\|z\|^u\\ &= 0.\end{aligned}$$

for each $x, z \in X$. Therefore $D_Q(x,y)=0$ for each $x, y \in X$. To show that Q is unique. Suppose there exists another quintic function $Q':X\to X$ which satisfies (1) and (3). Since Q and Q' are quintic. $Q(2^nx)=32^nQ(x), Q'(2^nx)=32^nQ'(x)$ for each $x \in X$. It follows that

$$\begin{aligned}\left\|Q'(x)-Q(x),z\right\| &= \frac{1}{32^n}\left\|Q'(2^nx)-Q(2^nx),z\right\|\\ &\le \frac{1}{32^n}\left[\left\|Q'(2^nx)-g(2^nx),z\right\|+\|g(2^nx)-Q(2^nx),z\|\right]\\ &\le \frac{1}{32^n}\frac{2\theta}{4(32-2^s)}\|2^nx\|^s\|z\|^u\\ &= \frac{2\theta\|x\|^s\|z\|^u2^{(s-5)n}}{4(32-2^s)}\\ \left\|Q'(x)-Q(x),z\right\| &= 0 \ as\ n\to\infty\ for\ each\ z\in X. \qquad (8)\end{aligned}$$

Hence $Q'(x)=Q(x)$ for each $x \in X$. □

Theorem 2.2. *Let $\theta \ge 0, u > 0$ with $s, t > 5$. If $g : X \to X$ is a function such that*

$$\|D_g(x,y),z\| \le \theta(\|x\|^s+\|y\|^t)\|z\|^u \qquad (9)$$

for each $x, y, z \in X$. Then there exists a unique quintic function $Q : X \to X$ satisfying (1) and

$$\|g(x)-Q(x),z\| \le \frac{\theta\|x\|^s\|z\|^u}{4(2^s-32)} \qquad (10)$$

for each $x, z \in X$

Proof. Put y=0 in (9) , we have

$$\|4g(2x)-128g(x),z\| \le \theta\|x\|^s\|z\|^u \qquad (11)$$

for each $x, z \in X$. Therefore

$$\left\|32g\left(\frac{x}{2}\right)-g(x),z\right\| \le \frac{\theta 2^{-s}}{4}\|x\|^s\|z\|^u \qquad (12)$$

for each $x, z \in X$. By using induction on n, we have

$$\left\|32^ng\left(\frac{x}{2^n}\right)-g(x),z\right\| = \frac{\theta 2^{-s}}{4}\|x\|^s\|z\|^u\left(\frac{1-2^{(5-s)n}}{1-2^{5-s}}\right) \qquad (13)$$

for each $x, z \in X$.

We can shows that $\left\{32^ng\left(\frac{x}{2^n}\right)\right\}$ is a 2-Cauchy sequence in X, for each $x \in X$. Since X is a 2-Banach space, the sequence $\left\{32^ng\left(\frac{x}{2^n}\right)\right\}$ 2-converges in X, for each $x \in X$. Define $Q : X \to X$ as

$$Q(x) = \lim_{n\to\infty}32^ng(2^{-n}x)$$

for each $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

3. Stability of a Functional Equation for Functions $g : (X, \|.,.\|) \to (X, \|.,.\|)$

In this section we study problems which we have studied in section 2 for functions $g : X \to X$, where $(X, \|.,.\|)$ is a 2-Banach space.

Theorem 3.1. *Let $\theta \geq 0, 0 < s, t < 5$. If $g : X \to X$ is a function such that*

$$\|D_g(x,y), z\| \leq \theta(\|x,z\|^s + \|y,z\|^t) \tag{14}$$

for each $x, y, z \in X$. Then there exists a unique quintic function $Q : X \to X$ satisfying (1) and

$$\|g(x) - Q(x), z\| \leq \frac{\theta \|x,z\|^s}{4(32 - 2^s)} \tag{15}$$

for each $x, z \in X$.

Proof. Let $x = y = 0$ in (14), we have $\|124g(0), z\| = 0$ for each $z \in X$, so we have g(0)=0. Put y=0 in (14), we have

$$\|4g(2x) - 128g(x), z\| \leq \theta \|x,z\|^s \tag{16}$$

for each $x, z \in X$. By using induction on n, we can show that

$$\left\| \frac{g(2^n x)}{32^n} - g(x), z \right\| \leq \frac{\theta}{128} \|x,z\|^s \left(\frac{1 - 2^{(s-5)n}}{1 - 2^{s-5}} \right) \tag{17}$$

for each $x, z \in X$. Dividing by 32^m and replacing x by $2^m x$ in (17), we get

$$\begin{aligned} \left\| \frac{g(2^{m+n}x)}{32^{m+n}} - \frac{g(2^m x)}{32^m}, z \right\| &\leq \frac{\theta}{128} \|x,z\|^s \left(\frac{2^{(s-5)m}(1 - 2^{(s-5)n})}{1 - 2^{(s-5)}} \right) \\ &\longrightarrow 0 \text{ as } m, n \to \infty. \end{aligned}$$

for each $z \in X$. This shows that $\left\{ \frac{g(2^n x)}{32^n} \right\}$ is a 2-Cauchy sequence in X, for each $x \in X$. Since X is a 2-Banach space, the sequence $\left\{ \frac{g(2^n x)}{32^n} \right\}$ 2-converges in X, for each $x \in X$. Define $Q : X \to X$ as

$$Q(x) = \lim_{n \to \infty} \frac{g(2^n x)}{32^n}$$

for each $z \in X$. The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 3.2. *Let $\theta \geq 0$ with $s, t > 5$. If $g : X \to X$ is a function such that*

$$\|D_g(x,y), z\| \leq \theta(\|x,z\|^s + \|y,z\|^t) \tag{18}$$

for each $x, y, z \in X$. Then there exists a unique quintic function $Q : X \to X$ satisfying (1) and

$$\|g(x) - Q(x), z\| \leq \frac{\theta \|x,z\|^s}{4(2^s - 32)} \tag{19}$$

for each $x, z \in X$.

Proof. Put y=0 in (18), we have

$$\|4g(2x) - 128g(x), z\| \leq \theta \|x, z\|^s \tag{20}$$

for each $x, z \in X$. By using induction on n, we have

$$\left\|32^n g\left(\frac{x}{2^n}\right) - g(x), z\right\| = \frac{\theta \|x, z\|^s 2^{-s}}{4}\left(\frac{1 - 2^{(5-s)n}}{1 - 2^{5-s}}\right) \tag{21}$$

for each $x, z \in X$. We can shows that $\left\{32^n g\left(\frac{x}{2^n}\right)\right\}$ is a 2-Cauchy sequence in X, for each $x \in X$. Since X is a 2-Banach space, the sequence $\left\{32^n g\left(\frac{x}{2^n}\right)\right\}$ 2-converges in X, for each $x \in X$. Define $Q : X \to X$ as

$$Q(x) = \lim_{n\to\infty} 32^n g(2^{-n}x)$$

for each $x \in X$. The rest of the proof is similar to the proof of Theorem 2.1. □

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