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Invariant and Some Reductions of (2+1)- Dimensional Burger's Equation with damping

Research Article

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Abstract: We establish the invariant and reductions of (2+1)- Dimensional Burger's Equation with damping term, $u_t + u + uu_x + uu_y = u_{xx} + u_{yy}$ is subjected to the Lie's classical method. Classification of its symmetry algebra into one- and two-dimensional subalgebras are carried out in order to facilitate its reduction systematically to (1+1)-dimensional PDEs and then to first or second-order ODEs.

Keywords: Burger's Equation, Symmetry algebra,Invariant.

1. Introduction

It is well known that the symmetry group method plays an important role in the analysis of differential equations. The history of group classification methods goes back to Sophus Lie. The first paper on this subject is [1], where Lie proves that a linear two-dimensional second-order PDE may admit at most a three-parameter invariance group (apart from the trivial infinite parameter symmetry group, which is due to linearity). He computed the maximal invariance group of the one-dimensional heat conductivity equation and utilized this symmetry to construct its explicit solutions. Saying it the modern way, he performed symmetry reduction of the heat equation. Nowadays, symmetry reduction is one of the most powerful tools for solving nonlinear partial differential equations (PDEs). Recently, there have been several generalizations of the classical Lie group method for symmetry reductions. Ovsiannikov [2] developed the method of partially invariant solutions. His approach is based on the concept of an equivalence group, which is a Lie transformation group acting in the extended space of independent variables, functions and their derivatives, and preserving the class of partial differential equations under study [3–5]. The investigation of the exact solutions plays an important role in the study of nonlinear physical systems. A wealth of methods have been developed to find these exact physically significant solutions of a PDE though it is rather difficult. Some of the most important methods are the inverse scattering method [6], Darboux and Bäcklund transformations [7], Hirota bilinear method [7, 8], Lie symmetry analysis [9–11], etc.

In this paper, we discuss the symmetry analysis of the (2+1)-dimensional Burger's equation with damping term

$$u_t + u + uu_x + uu_y = u_{xx} + u_{yy}. \tag{1}$$

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Our intention is to show that equation (1) admits a four-dimensional symmetry group and determine the corresponding Lie algebra, classify the one- and two-dimensional subalgebras of the symmetry algebra of (1) in order to reduce (1) to (1+1)-dimensional PDEs and then to ODEs. We shall establish that the symmetry generators form a closed Lie algebra and this allowed us to use the recent method due to Ahmad, Bokhari, Kara and Zaman [?] to successively reduce (1) to (1+1)-dimensional PDEs and ODEs with the help of two-dimensional Abelian and non-Abelian solvable subalgebras. This paper is organised as follows: In section 2, we determine the symmetry group of (1) and write down the associated Lie algebra. In section 3, we consider all one-dimensional subalgebras and obtain the corresponding reductions to (1+1)-dimensional PDEs. In section 4, we show that the generators form a closed Lie algebra and use this fact to reduce (1) successively to (1+1)-dimensional PDEs and ODEs. In section 5, we summarises the conclusions of the present work.

2. The Symmetry Group and Lie Algebra of $u_t+u+uu_x+uu_y = u_{xx}+u_{yy}$

If (1) is invariant under a one parameter Lie group of point transformations (Bluman and Kumei [3], Olver [11])

$$x^* = x+\epsilon\,\xi(x,y,t;u)+O(\epsilon^2)\ , \tag{2}$$

$$y^* = y+\epsilon\,\eta(x,y,t;u)+O(\epsilon^2)\ , \tag{3}$$

$$t^* = t+\epsilon\,\tau(x,y,t;u)+O(\epsilon^2)\ , \tag{4}$$

$$u^* = u+\epsilon\,\phi(x,y,t;u)+O(\epsilon^2)\ , \tag{5}$$

with infinitesimal generator

$$X = \xi(x,y,t;u)\frac{\partial}{\partial x}+\eta(x,y,t;u)\frac{\partial}{\partial y}+\tau(x,y,t;u)\frac{\partial}{\partial t}+\phi(x,y,t;u)\frac{\partial}{\partial u}. \tag{6}$$

In order to determine the four infinitesimals $\xi,\ \eta,\ \tau$ and ϕ, we prolong V to fourth order. This prolongation is given by the formula

$$\begin{aligned} V^{(4)} = {} & V+\phi^x\frac{\partial}{\partial u_x}+\phi^y\frac{\partial}{\partial u_y}+\phi^t\frac{\partial}{\partial u_t}+\phi^{xx}\frac{\partial}{\partial u_{xx}}+\phi^{xy}\frac{\partial}{\partial u_{xy}}+\phi^{xt}\frac{\partial}{\partial u_{xt}} \\ & +\phi^{yy}\frac{\partial}{\partial u_{yy}}+\phi^{yt}\frac{\partial}{\partial u_{yt}}+\phi^{tt}\frac{\partial}{\partial u_{tt}}+\phi^{xxx}\frac{\partial}{\partial u_{xxx}}+\phi^{xyy}\frac{\partial}{\partial u_{xyy}} \\ & +\phi^{xxy}\frac{\partial}{\partial u_{xxy}}+\phi^{xtt}\frac{\partial}{\partial u_{xtt}}+\phi^{xyt}\frac{\partial}{\partial u_{xyt}}+\phi^{yyy}\frac{\partial}{\partial u_{yyy}}+\phi^{ttt}\frac{\partial}{\partial u_{ttt}} \\ & +\phi^{xxt}\frac{\partial}{\partial u_{xxt}}+\phi^{yyt}\frac{\partial}{\partial u_{yyt}}+\phi^{ytt}\frac{\partial}{\partial u_{ytt}}+\phi^{xxxx}\frac{\partial}{\partial u_{xxxx}}+\cdots+ \\ & \phi^{tttt}\frac{\partial}{\partial u_{tttt}}\ . \end{aligned} \tag{7}$$

In the above expression every coefficient of the prolonged generator is a function of x, y, t and u can be determined by the formulae,

$$\phi^i = D_i(\phi-\xi u_x-\eta u_y-\tau u_t)+\xi u_{x,i}+\eta u_{y,i}+\tau u_{t,i}\ , \tag{8}$$

$$\phi^{ij} = D_iD_j(\phi-\xi u_x-\eta u_y-\tau u_t)+\xi u_{x,ij}+\eta u_{y,ij}+\tau u_{t,ij}\ , \tag{9}$$

$$\phi^{ijkl} = D_iD_jD_kD_l(\phi-\xi u_x-\eta u_y-\tau u_t)+\xi u_{x,ijkl}+\eta u_{y,ijkl}+\tau u_{t,ijkl}\ , \tag{10}$$

where D_i represents total derivative and subscripts of u derivative with respect to the respective coordinates. To proceed with reductions of Eq.(1) we now use symmetry criterion for PDEs. The determining equations are obtained from Eq.(6)

are as follows:

$$(\xi_1)_u = 0 \tag{11}$$

$$(\xi_2)_u = 0 \tag{12}$$

$$(\xi_3)_u = 0 \tag{13}$$

$$(\phi_1)_{u,u} = 0 \tag{14}$$

$$(\xi_3)_y = 0 \tag{15}$$

$$(\xi_3)_x = 0 \tag{16}$$

$$\phi_1 - (\xi_2)_t - u(\xi_2)_x - u(\xi_2)_y + u(\phi_1)_u + (\xi_2)_{x,x} + (\xi_2)_{y,y} - 2(\phi_1)_{y,u} = 0 \tag{17}$$

$$\phi_1 - (\xi_1)_t - u(\xi_1)_x - u(\xi_1)_y + u(\phi_1)_u + (\xi_1)_{x,x} + (\xi_1)_{y,y} - 2(\phi_1)_{x,u} = 0 \tag{18}$$

$$\phi_1 + (\phi_1)_t + u(\phi_1)_x + u(\phi_1)_y - (\phi_1)_{x,x} - (\phi_1)_{y,y} = 0 \tag{19}$$

$$(\xi_1)_y + (\xi_2)_x = 0 \tag{20}$$

$$2(\xi_1)_x - (\phi_1)_u = 0 \tag{21}$$

$$2(\xi_2)_y - (\phi_1)_u = 0 \tag{22}$$

$$-(\xi_3)_t + (\phi_1)_u = 0 \tag{23}$$

Using the above equations and some more manipulations, we get,

$$\xi = -k_3 + e^{-t}k_4 \ , \tag{24}$$

$$\eta = -k_2 - k_3 + e^{-t}k_4 \ , \tag{25}$$

$$\tau = k_1 \ , \tag{26}$$

$$\phi = -e^{-t}k_4 \ . \tag{27}$$

Now. we write down the four symmetry generators corresponding to each of the constants $k_i, i = 1, 2, 3, 4$ involved in the infintesimals, viz.,

$$\begin{aligned} V_1 &= \partial t, \\ V_2 &= -\partial y, \\ V_3 &= -\partial x - \partial y \ , \\ V_4 &= e^{-t}\partial x + e^{-t}\partial y - e^{-t}\partial u \ . \end{aligned} \tag{28}$$

The symmetry generators found in Eq.(28) form a closed Lie Algebra whose commutation table is shown below.

$[V_i, V_j]$	V_1	V_2	V_3	V_4
V_1	0	0	0	$-V_4$
V_2	0	0	0	0
V_3	0	0	0	0
V_4	V_4	0	0	0

Table 1. Commutation relations satisfied by above generators is

The commutation relations of the Lie algebra, determined by V_1, V_2, V_3 and V_4 are shown in the above table. These vector fields form a Lie algebra L by:

$$[V_1, V_4] = -V_4.$$

For this four-dimensional Lie algebra the commutator table for V_i is a $(4 \otimes 4)$ table whose $(i,j)^{th}$ entry expresses the Lie Bracket $[V_i, V_j]$ given by the above Lie algebra L. The table is skew-symmetric and the diagonal elements all vanish. The coefficient $C_{i,j,k}$ is the coefficient of V_i of the $(i,j)^{th}$ entry of the commutator table and the related structure constants can be easily calculated from above table are as follows:

$$C_{1,4,4} = -1.$$

The Lie algebra L is solvable. The radical of G is ,

$$R =< V_2, V_3 > \oplus < V_1, V_4 >.$$

In the next section, we derive the reduction of (1) to PDEs with two independent variables and ODEs. These are four one-dimensional Lie subalgebras

$$L_{s,1} = \{V_1\}\ ,\ L_{s,2} = \{V_2\}\ ,\ L_{s,3} = \{V_3\}\ ,\ L_{s,4} = \{V_4\}$$

and corresponding to each one-dimensional subalgebras we may reduce (1) to a PDE with two independent variables. Further reductions to ODEs are associated with two-dimensional subalgebras. It is evident from the commutator table that there are one two-dimensional solvable non-abelian subalgebras. And there are five two-dimensional Abelian subalgebras, namely,

$$L_{A,1} = \{V_1, V_2\}\ ,\ L_{A,2} = \{V_1, V_3\}\ ,\ L_{A,3} = \{V_2, V_3\}\ ,\ L_{A,4} = \{V_2, V_4\}\ \ L_{A,5} = \{V_3, V_4\}\ .$$

The Non-abelian subalgebra is

$$L_{nA,1} = \{V_1, V_4\}.$$

3. Reductions of $u_t + u + uu_x + uu_y = u_{xx} + u_{yy}$ by One-Dimensional Subalgebras

Case 1 : $V_1 = \partial_t$.

The characteristic equation associated with this generator is

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{1} = \frac{du}{0}\ .$$

We integrate the characteristic equation to get three similarity variables,

$$y = p, \quad x = q \ \text{ and } \ u = W(p,q)\ . \tag{29}$$

Using these similarity variables in Eq.(1) can be recast in the form

$$WW_q + WW_p + W = W_{pp} + W_{qq} = 0\ . \tag{30}$$

Case 2 : $V_2 = -\partial_y$.

The characteristic equation associated with this generator is

$$\frac{dx}{0} = \frac{dy}{-1} = \frac{dt}{0} = \frac{du}{0} \ .$$

Following the standard procedure we integrate the characteristic equation to get three similarity variables,

$$x = p, \quad t = q \quad \text{and} \quad u = W(p,q) \ . \tag{31}$$

Using these similarity variables in Eq.(1) can be recast in the form

$$W_q + WW_p + W = W_{pp}. \tag{32}$$

Case 3 : $V_3 = -\partial_x - \partial_y$.

The characteristic equation associated with this generator is

$$\frac{dx}{-1} = \frac{dy}{-1} = \frac{dt}{0} = \frac{du}{0} \ .$$

We integrate the above characteristic equation to get three similarity variables,

$$t = p, \quad x - y = q \quad \text{and} \quad u = W(p,q) \ . \tag{33}$$

Using these similarity variables in Eq.(1) can be recast in the form

$$W + W_p = 2W_{qq}. \tag{34}$$

Case 4 : $V_4 = e^{-t}\partial x + e^{-t}\partial y - e^{-t}\partial u.$

The characteristic equation associated with this generator is

$$\frac{dx}{e^{-t}} = \frac{dy}{e^{-t}} = \frac{dt}{0} = \frac{du}{e^{-t}} \ .$$

On integrating the above characteristic equation to get three similarity variables,

$$x - y = p, \quad t = q \quad \text{and} \quad u = W(p,q) \ . \tag{35}$$

Using these similarity variables in Eq.(1) can be recast in the form

$$W + W_q = 2W_{pp}. \tag{36}$$

4. Reductions of $u_t + u + uu_x + uu_y = u_{xx} + u_{yy}$ by Two-Dimensional Abelian Subalgebras

Case I : Reduction under V_1 and V_2 .

From Table 1, we find that the given generators commute $[V_1, V_2] = 0$. Thus either of V_1 or V_2 can be used to start the reduction with. For our purpose we begin reduction with V_1. Therefore we get Eq.(29) and Eq.(30).

At this stage, we express V_2 in terms of the similarity variables defined in (29). The transformed V_2 is

$$\tilde{V_2} = -\partial_p \ .$$

The characteristic equation for $\tilde{V_2}$ is

$$\frac{dp}{-1} = \frac{dq}{0} = \frac{dW}{0} \ .$$

Integrating this equation as before leads to new variables

$$q = \zeta \quad \text{and} \quad W = R(\zeta),$$

which reduce Eq.(30) to a second order ODE

$$RR_\zeta + R = R_{\zeta\zeta} \ . \tag{37}$$

Case II : Reduction under V_1 and V_3 .

From Table 1, we find that the given generators commute $[V_1, V_3] = 0$. Thus either of V_1 or V_3 can be used to start the reduction with. For our convenience we begin reduction with V_1. Therefore we get Eq.(29) and Eq.(30).

At this stage, we express V_3 in terms of the similarity variables defined in Eq.(29). The transformed V_3 is

$$\tilde{V_3} = -\partial_p - \partial_q \ .$$

The characteristic equation for $\tilde{V_3}$ is

$$\frac{dp}{-1} = \frac{dq}{-1} = \frac{dW}{0} \ .$$

Integrating this equation as before leads to the new variables

$$p - q = \zeta \quad \text{and} \quad W = R(\zeta),$$

which reduce Eq.(30) to a second order ODE

$$R_\zeta = 2R_{\zeta\zeta} \ . \tag{38}$$

Case III : Reduction under V_2 and V_3 .

In this case, the two symmetry generators V_2 and V_3 satisfy the commutation relation $[V_2, V_3] = 0$. We begin with V_2. We express V_3 in terms of the similarity variables defined in Eq.(33). The transformed V_3 is

$$\tilde{V_3} = -\partial_p.$$

The similarity variables are

$$q = \zeta \ \text{ and } W = R(\zeta).$$

The corresponding reduced PDE is

$$R + R_\zeta = 0. \tag{39}$$

Case IV : Reduction under V_2 and V_4 .

In this case, the two symmetry generators V_2 and V_4 satisfy the commutation relation $[V_2, V_4] = 0$. This suggests, that reduction in this case should start with V_2. Therefore, we get Eq.(31) and Eq.(32). The transformed V_4 is

$$\tilde{V_4} = e^{-q}\partial_p - e^{-q}\partial_W.$$

The invariants of $\tilde{V_4}$ are $q = \zeta$ and $W = R(\zeta)$. The corresponding reduced PDE is

$$R + R_\zeta = 0 \ . \tag{40}$$

Case V : Reduction under V_3 and V_4 .

In this case, the two symmetry generators V_3 and V_4 satisfy the commutation relation $[V_3, V_4] = 0$. This suggests, that reduction in this case should start with V_3. Therefore, we get Eq.(33) and Eq.(34). Similarly, the corresponding reduced PDE is

$$R + R_\zeta = 0 \ . \tag{41}$$

5. Reductions of $u_t + u + uu_x + uu_y = u_{xx} + u_{yy}$ by Two-Dimensional Non-Abelian Subalgebras

Case a : Reduction under V_1 and V_4 .

From Table 1, we find that the given generators commute $[V_1, V_4] = -V_4$. Thus either of V_1 or V_4 can be used to start the reduction with. For our purpose, we begin reduction with V_4. Therefore we get Eq.(29) and Eq.(30).
At this stage, we express V_1 in terms of the similarity variables defined in (29). The transformed V_1 is

$$\tilde{V_1} = \partial_q \ .$$

The characteristic equation for $\tilde{V_1}$ is

$$\frac{dp}{0} = \frac{dq}{1} = \frac{dW}{0} \ .$$

Integrating this equation as before leads to new variables

$$p = \zeta \ \text{ and } \ W = R(\zeta),$$

which reduce Eq.(30) to a second order ODE

$$R = R_{\zeta\zeta} \ . \tag{42}$$

6. Conclusions

In this Paper,

(1). A (2+1)-dimensional Burgers equation with damping term $u_t+u+uu_x+uu_y=u_{xx}+u_{yy}$, is subjected to Lie's classical method.

(2). Equation (1) admits a four-dimensional symmetry group.

(3). It is established that the symmetry generators form a closed Lie algebra.

(4). Classification of symmetry algebra of (1) into one- and two-dimensional abelian and non-abelian subalgebras are carried out.

(5). Systematic reduction to (1+1)-dimensional PDE and then to first- or second order ODEs are performed using one-dimensional and two-dimensional solvable Abelian subalgebras.

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