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Optimization Strategies Applied to Discrete Epidemic Models with Specific Nonlinear Incidence Rate

Research Article

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Abstract: The aim of this paper is to study discrete epidemic models with specific non linear incidence rate, and to investigate, in discrete time, optimal control strategies in which the controls are: vaccination and/or treatment. So, we make use of Pontryagin's maximum principle in order to compare and choose the best medical policy and economical strategy can be adopted and implemented. To end, numerical simulations are carried out to confirm the validity of the models and to prove the performance of the optimization strategies.

Keywords: Optimal control, Discrete epidemic model, Vaccination, Pontryagin's maximum principle.

1. Introduction

Infectious diseases have caused several epidemics, leaving behind them not only millions of dead and infected individuals but also severe socioeconomic consequences. In fact, mathematical modeling of infectious diseases is one of the most important research areas [1]. Mathematical modeling of biological processes aims to better understand complex or often misunderstood phenomena of these bio-processes. A mathematical model is a set of mathematical equations that links; on one side, a set of variables which are states of the system studied for example body temperature, viral load, on the other hand, a set of parameters that are constants or variables specific to the system, for example the mass body, the life of the virus. In addition, the mathematical model takes also into consideration a set of constraints [2].

Enermous budgets and depth scientific researches wether mathematical, medical or others have a noble and ultimate goal, not limited to studies of diseases and prediction of their evolution. In other words, the main objective is to fight the spread of epidemics through developpement of appropriate economical and medical policies. So, we make use of optimal control theory as a powerful mathematical tool that can help the intervention of public health authorities. In fact, we investigate an effective strategy to control the spread of infectious diseases by setting an optimal control problem. An optimal control problem consists of finding a control function u^* which minimizes a given functional cost (performance index) while satisfying the system state equations and constraints.

It has successful applications in many disciplines, namely, economics, environment, management, engineering, etc [3]. In the literature of mathematical epidemiology, multi-group epidemic models have been proposed to describe the spread of many

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infectious diseases in heterogeneous populations, such as measles, mumps, gonorrhea, and HIV/AIDS [2, 4]. A heterogeneous host population can be divided into several homogeneous groups according to modes of transmission [4]. O.balatif et al [5] investigate the optimal control strategy of a simple SIR epidemic model in discrete-time. They implement just one control and don't shed light on stability. K.hattaf et al [6] analyse the effect of two different discretizations forward and backward Euler methods, however, they don't introduce controls to limit the widespread of the disease. A.Jihad et al [7] investigate continuous SIR epidemic model with specific non linear incidence rate. They study the effects of the environmental fluctuations on dynamical behavior.

This paper is organized as follows. In the next section, we present a discrete SIR epidemic model with specific non linear incidence rate. After taking vaccination and treatment into consideration, a new epidemic model is formulated and developped in section 3. The analysis of optimization problem by the biais of the optimal control theory is presented in section 4. In section 5, the models are simulated, compared, commented and interpreted. Finally, the conclusions are summarized in section 6.

2. Discrete SIR Epidemic Models with Specific Non Linear Incidence Rate

We consider a discrete SIR epidemic model. The population is divided into three disease-state compartments: susceptible individuals (S), people who can catch the disease; infectious (infective) individuals (I), people who have the disease and can transmit the disease; recovered individuals (R), people who have recovered from the disease. We assume that an individual can be infected only through contacts with infectious individuals and that immunity is permanent. Our model is described as follows

$$\begin{cases} S_{k+1} = \Lambda - (\mu - 1)\, S_k - \frac{\beta S_k I_k}{1+\alpha_1 S_k+\alpha_2 I_k+\alpha_3 S_k I_k} \\ I_{k+1} = \frac{\beta S_k I_k}{1+\alpha_1 S_k+\alpha_2 I_k+\alpha_3 S_k I_k} - (\mu + d + r - 1)\, I_k \\ R_{k+1} = -(\mu - 1)\, R_k + r I_k \end{cases} \tag{1}$$

with $S_0 \geq 0$, $I_0 \geq 0$ and $R_0 \geq 0$ are given. Note that $k = 0, 1, 2, .., T-1$, is the index for the time steps. The transitions between different states are described by the following parameters:

Λ	is the recruitment rate of susceptibles;
β	is the effective contact rate;
μ	is the natural mortality rate;
d	is the disease induced death rate;
r	is the recovery rate;
α_1,α_2 and α_3	are positive constants.

3. The Epidemic Model with Vaccination and/or Treatment

Taking in account treatment and/or vaccination, the problem is converted to the model below

$$\begin{cases} S_{k+1} = \Lambda - (\mu - 1)\, S_k - \frac{\beta S_k I_k}{1+\alpha_1 S_k+\alpha_2 I_k+\alpha_3 S_k I_k} - \epsilon_1 u_{1,k} S_k \\ I_{k+1} = \frac{\beta S_k I_k}{1+\alpha_1 S_k+\alpha_2 I_k+\alpha_3 S_k I_k} - (\mu + d + r - 1)\, I_k - \epsilon_2 u_{2,k} I_k \\ R_{k+1} = \epsilon_1 u_{1,k} S_k + (r + \epsilon_2 u_{2,k})\, I_k - (\mu - 1)\, R_k \end{cases} \tag{2}$$

Where

$$\epsilon_i = \begin{cases} 1 \\ \quad for\ i = 1, 2. \\ 0 \end{cases} \tag{3}$$

There are two controls $u_i = (u_{i,0}, u_{i,1}, .., u_{i,T-1})$ with $i = 1, 2$. On the one hand, u_1 is the percentage of susceptible individuals being vaccinated per time unit, on the other hand, u_2 is the percentage of infected individuals being treated per time unit. The first control can be interpreted as the proportion to be vaccinated, so we note that $u_{1,k}S_k$ individuals move from the susceptible class to the removed class at time step k. The second control can be also interpreted as the proportion to be treated, so we note that $u_{2,k}I_k$ individuals move from the infected class to the removed class at time step k. Indeed, the system above (2) presents four different models as the table 1 explains.

ϵ_1	ϵ_2	Interpretations
0	0	Discrete epidemic model (without controls)
1	0	Discrete epidemic model with vaccination
0	1	Discrete epidemic model with treatment
1	1	Discrete epidemic model with vaccination and treatment

Table 1. **Interpretations according to the values of epsilons**

4. The Optimal Control Problem

Our goal is reducing the number of infected individulas during the time steps $k = 0$ to T and also minimizing the cost of treatment and the cost of vaccination. To simplify, we assume that the costs of administering the controls are quadratic. Then, the objective functional is presented as follows

$$J(u_1, u_2) = A_T I_T + \sum_{k=0}^{T-1} \left(A_k I_k + \frac{1}{2} B_{1,k} \epsilon_1 u_{1,k}^2 + \frac{1}{2} B_{2,k} \epsilon_2 u_{2,k}^2 \right) \tag{4}$$

Where A_k, $B_{1,k}$ and $B_{2,k}$ are the cost coefficients. The goal is to find an opptimal control $u^* = (u_1^*, u_2^*)$ which minimize the objective functional

$$J(u_1^*, u_2^*) = \min_{(u_1, u_2) \in U_{ad}} J(u) \tag{5}$$

Where U_{ad} is the set of admissible controls defined by

$$U_{ad} = \{u = (u_1, u_2) \,|0 \leq u_{i,k} \leq u_i^{max} \prec 1; k = 0, 1.., T-1; i = 1, 2\} \tag{6}$$

4.1. Existence of an Optimal Control

The existence of the optimal control pair can be obtained using a result by Fleming and Rishel [8] and by Lukes [9].

Theorem 4.1. *There exists controls functions* $u^* = (u_1^*, u_2^*)$ *so that*

$$J(u_1^*, u_2^*) = \min_{(u_1, u_2) \in U_{ad}} J(u) \tag{7}$$

Proof. To prove the existence of an optimal control pair it is suffisicient to verify that

(1). The set of controls and corresponding state variables is nonempty.

(2). The admissible set U_{ad} is convex and closed.

(3). The right hand side of the state system (2) is bounded by a linear function in the state and control variables.

(4). The integrand of the objective functional is convex on U_{ad} .

(5). There exists constants $c_1 \succ 0$, $c_2 \succ 0$ and $\rho \succ 1$ such that the integrand $L(S, I, u_1, u_2)$ of the objective functional satisfies

$$L(S, I, u_1, u_2) \succ c_2 + c_1 \left(|u_1|^2 + |u_2|^2\right)^{\rho/2}$$

The result follows directly from (Fleming and Rishel 1975). □

4.2. Characterization of the Optimal Control

We use the Pontryagin's maximum principle in discrete time, given in [10]. Then, we have the Hamiltonian at each time step k, where our adjoint function is

$$\lambda_j = (\lambda_{j,1}, \lambda_{j,2}, .., \lambda_{j,T}), \ j = 1, 2, 3. \tag{8}$$

$$H_k = A_k I_k + \frac{1}{2} B_{1,k} \epsilon_1 u_{1,k}^2 + \frac{1}{2} B_{2,k} \epsilon_2 u_{2,k}^2 + \sum_{j=1}^{3} \lambda_{j,k+1} g_{j,k} \tag{9}$$

Note that $g_{j,k}$ is the right side of the difference equation of the j^{th} state variable at time step $k+1$.

Theorem 4.2. *Given an optimal control $u_k^* = \left(u_{1,k}^*, u_{2,k}^*\right) \in U_{ad}$, and solutions S_k^*, I_k^* and R_k^* of the corresponding state system (2), there exists adjoint functions $\lambda_{1,k}, \lambda_{2,k}$ and $\lambda_{3,k}$ which satisfy*

$$\begin{aligned}
\lambda_{1,k} &= \lambda_{1,k+1}\left[-(\mu-1) - \frac{\beta I_k (1+\alpha_2 I_k)}{(1+\alpha_1 S_k + \alpha_2 I_k + \alpha_3 S_k I_k)^2} - \epsilon_1 u_{1,k}\right] + \lambda_{2,k+1}\left[\frac{\beta I_k (1+\alpha_2 I_k)}{(1+\alpha_1 S_k + \alpha_2 I_k + \alpha_3 S_k I_k)^2}\right] + \lambda_{3,k+1}\epsilon_1 u_{1,k} \\
\lambda_{2,k} &= A_k + \lambda_{1,k+1}\left[-\frac{\beta S_k (1+\alpha_1 S_k)}{(1+\alpha_1 S_k + \alpha_2 I_k + \alpha_3 S_k I_k)^2}\right] + \lambda_{2,k+1}\left[\frac{\beta S_k (1+\alpha_1 S_k)}{(1+\alpha_1 S_k + \alpha_2 I_k + \alpha_3 S_k I_k)^2} - (\mu + d + r - 1 + \epsilon_2 u_{2,k})\right] \\
&\quad + \lambda_{3,k+1}(r + \epsilon_2 u_{2,k}) \qquad (10) \\
\lambda_{3,k} &= \lambda_{3,k+1}(1-\mu)
\end{aligned}$$

with the following transversality conditions at time T

$$\lambda_{1,T} = \lambda_{3,T} = 0 \quad \text{and} \quad \lambda_{2,T} = A_T. \tag{11}$$

In addition to this, the optimal control (u_1^*, u_2^*) for $k = 0, 1, \ldots, T-1$ is given by

- *if $\epsilon_1 = \epsilon_2 = 1$*

$$u_{1,k}^* = \min\left[u_1^{max}, \max\left(0; (\lambda_{1,k+1} - \lambda_{3,k+1}) S_k / B_1\right)\right] \tag{12}$$

$$u_{2,k}^* = \min\left[u_2^{max}, \max\left(0; (\lambda_{2,k+1} - \lambda_{3,k+1}) I_k / B_2\right)\right] \tag{13}$$

- *if $\epsilon_1 = 1$ and $\epsilon_2 = 0$*

$$u_{1,k}^* = \min\left[u_1^{max}, \max\left(0; (\lambda_{1,k+1} - \lambda_{3,k+1}) S_k / B_1\right)\right] \tag{14}$$

- *if* $\epsilon_1 = 0$ *and* $\epsilon_2 = 1$

$$u_{2,k}^{*} = \min\left[u_2^{max}, \max\left(0; \left(\lambda_{2,k+1} - \lambda_{3,k+1}\right) I_k / B_2\right)\right] \tag{15}$$

Proof. The hamiltonian is given by

$$\begin{aligned} H_k = {} & A_k I_k + \frac{1}{2} B_1 \epsilon_1 u_{1,k}^2 + \frac{1}{2} B_2 \epsilon_2 u_{2,k}^2 + \lambda_{1,k+1} \left\{ \Lambda - (\mu - 1) S_k - \frac{\beta S_k I_k}{1 + \alpha_1 S_k + \alpha_2 I_k + \alpha_3 S_k I_k} - \epsilon_1 u_{1,k} S_k \right\} \\ & + \lambda_{2,k+1} \left\{ \frac{\beta S_k I_k}{1 + \alpha_1 S_k + \alpha_2 I_k + \alpha_3 S_k I_k} - (\mu + d + r - 1) I_k - \epsilon_2 u_{2,k} I_k \right\} \\ & + \lambda_{3,k+1} \left\{ \epsilon_1 u_{1,k} S_k + (r + \epsilon_2 u_{2,k}) I_k - (\mu - 1) R_k \right\} \end{aligned} \tag{16}$$

By the bias of Pontryagin's Maximum Principle, in discrete time, the adjoint equations and corresponding final time conditions (transversality conditions) are given

$$\begin{cases} \lambda_{1,k} & = \frac{\partial H_k}{\partial S_k}, \lambda_{1,T} = 0 \\ \lambda_{2,k} & = \frac{\partial H_k}{\partial I_k}, \lambda_{2,T} = A_T \\ \lambda_{3,k} & = \frac{\partial H_k}{\partial R_k}, \lambda_{3,T} = 0 \end{cases} \tag{17}$$

for $k = 0, 1, \ldots, T-1$; the optimal control $u^{*} = \left(u_{1,k}^{*}, u_{2,k}^{*}\right)$ is obtained as well

$$\frac{H_k}{\partial u_{i,k}} = 0 \ \text{ for } \ k = 0, 1, 2, \ldots, T-1 \ \text{ and } \ i = 1, 2. \tag{18}$$

Subject to the lower and upper bounds for (u_1, u_2) and for $\epsilon_1 = \epsilon_2 = 1$, the characterizations become:

$$u_{1,k}^{*} = \min\left[u_1^{max}, \max\left(0; \left(\lambda_{1,k+1} - \lambda_{3,k+1}\right) S_k / B_1\right)\right] \tag{19}$$

$$u_{2,k}^{*} = \min\left[u_2^{max}, \max\left(0; \left(\lambda_{2,k+1} - \lambda_{3,k+1}\right) I_k / B_2\right)\right] \tag{20}$$

However, if $\epsilon_1 = 1$ and $\epsilon_2 = 0$ the second control $u_{2,k}$ will be eliminated and removed, moreover if $\epsilon_1 = 0$ and $\epsilon_2 = 1$, the first control $u_{1,k}$ will be suppressed. □

5. Numerical Simulation and Interpretation

In this section,the optimality system will be solved numerically by an iterative method, so the results are obtained and commented. Now, we have the optimality system

$$\begin{cases} S_{k+1} & = \Lambda - (\mu - 1) S_k - \frac{\beta S_k I_k}{1 + \alpha_1 S_k + \alpha_2 I_k + \alpha_3 S_k I_k} - \epsilon_1 \min\left[u_1^{max}, \max\left(0; \left(\lambda_{1,k+1} - \lambda_{3,k+1}\right) S_k / B_1\right)\right] S_k \\ I_{k+1} & = \frac{\beta S_k I_k}{1 + \alpha_1 S_k + \alpha_2 I_k + \alpha_3 S_k I_k} - (\mu + d + r - 1) I_k - \epsilon_2 \min\left[u_2^{max}, \max\left(0; \left(\lambda_{2,k+1} - \lambda_{3,k+1}\right) I_k / B_2\right)\right] I_k \\ R_{k+1} & = \epsilon_1 \min\left[u_1^{max}, \max\left(0; \left(\lambda_{1,k+1} - \lambda_{3,k+1}\right) S_k / B_1\right)\right] S_k - (\mu - 1) R_k \\ & \quad + \left(r + \epsilon_2 \min\left[u_2^{max}, \max\left(0; \left(\lambda_{2,k+1} - \lambda_{3,k+1}\right) I_k / B_2\right)\right]\right) I_k \\ \lambda_{1,k} & = \lambda_{1,k+1} \left[- (\mu - 1) - \frac{\beta I_k (1 + \alpha_2 I_k)}{(1 + \alpha_1 S_k + \alpha_2 I_k + \alpha_3 S_k I_k)^2} - \epsilon_1 \min\left[u_1^{max}, \max\left(0; \left(\lambda_{1,k+1} - \lambda_{3,k+1}\right) S_k / B_1\right)\right] \right] \\ & \quad + \lambda_{2,k+1} \left[\frac{\beta I_k (1 + \alpha_2 I_k)}{(1 + \alpha_1 S_k + \alpha_2 I_k + \alpha_3 S_k I_k)^2} \right] + \lambda_{3,k+1} \epsilon_1 \min\left[u_1^{max}, \max\left(0; \left(\lambda_{1,k+1} - \lambda_{3,k+1}\right) S_k / B_1\right)\right] \\ \lambda_{2,k} & = A_k + \lambda_{1,k+1} \left[- \frac{\beta S_k (1 + \alpha_1 S_k)}{(1 + \alpha_1 S_k + \alpha_2 I_k + \alpha_3 S_k I_k)^2} \right] \\ & \quad + \lambda_{2,k+1} \left[\frac{\beta S_k (1 + \alpha_1 S_k)}{(1 + \alpha_1 S_k + \alpha_2 I_k + \alpha_3 S_k I_k)^2} - (\mu + d + r - 1 + \epsilon_2 \min\left[u_2^{max}, \max\left(0; \left(\lambda_{2,k+1} - \lambda_{3,k+1}\right) I_k / B_2\right)\right]) \right] \\ & \quad + \lambda_{3,k+1} \left(r + \epsilon_2 \min\left[u_2^{max}, \max\left(0; \left(\lambda_{2,k+1} - \lambda_{3,k+1}\right) I_k / B_2\right)\right]\right) \\ \lambda_{3,k} & = \lambda_{3,k+1} (1 - \mu) \end{cases} \tag{21}$$

with $S(0) = S_0$, $I(0) = I_0$, $R(0) = R_0$, $\lambda_{1,T} = \lambda_{3,T} = 0$ and $\lambda_{2,T} = A_T$. This optimality system is a two-point boundary value problem with seperated boundary conditions : there are initial conditions for the state variables and terminal conditions for the adjoint. So, we make use of an iterative method with forward solving of the state system followed by backward solving of the adjoint system. To materialize this research, we apply it to the influenza A(H1N1) in Morocco and we bank on the references [11] and [12] in order to obtain the initial conditions and parameters of the system (2) as the table 2 describes.

Initial conditions and parameters	Value
S_0	30×10^6
R_0	28
Λ	1174.17
β	0.3095
μ	3.9139×10^{-5}
d	0.0063
r	0.2
α_1	0.5
α_2	0.5
α_3	0
u_1^{max}	0.8
u_2^{max}	0.9

Table 2. **Initial conditions and parameters**

The figures below give us the opportunity to compare easily the behaviors of the four discrete epidemic models, see table 1. The first Figure shows that the number of infected (without controls) reaches a maximum of 12.5 million and in case of vaccination (only) reaches 36 thousand, consequently, vaccination contributes greatly to the fight against the epidemic. In the opposite side, treatment only is able to reduce more and more the number of infected, while the use of two controls at a same time (vaccination and treatment) don't bring an extra supply. Otherwise, the infected curve (in case of treatment only) and infected curve (in case of two controls) are almost similar; this fact can be explained by the poor impact of vaccination compared to treatment. To conclude, vaccination has an important positive effect, however its effect is still weak compared to treatment. Another point to raise the curves in figure 1 have different scales, so at the first glance, one can be mistaken for thinking the number of infected (without controls) is higher than the number of infected (with controls) in the last days of treatment. For this reason we add another table 3 showing the number of infected in the last day of the four curves.

Infected	without controls	with vaccination only	with treatment only	with two controls
Color	red	green	blue	yellow
ϵ_1	0	1	0	1
ϵ_2	0	0	1	1
$t = 0$	30	30	30	30
$t = 200$ days	5690	1935	40	40

Table 3. **Infected in the four models**

The second figure gives optimal controls adopted (vaccination and treatment) which allow to minimize the objective functional and reach our purpose. We observe that treatment curve in case of treatment only ($\epsilon_1 = 0$; $\epsilon_2 = 1$) and treatment curve in case of two controls: vaccination and treatment ($\epsilon_1 = \epsilon_2 = 1$) are almot similar. In fact there is very little difference will be revealed in the third figure. On the whole, these results could clearly compare four epidemiological models and predict the evolution of each one. Treatment is more effective than vaccination while the role of vaccination can be neglected compared the powerful role of the treatment. To end, these results prove the validity of the mathematical models.

6. Conclusion

In this work, vaccination and treatment are introduced to our model so new discrete epidemic models with two controls: vaccination and treatment are developed and validated. Furthermore, Pontryagin's maximum principle is applied to these discrete models in order to reduce the number of infected individuals and also minimize the cost of treatment and the cost of vaccination. The optimality systems are solved by an iterative method with forward solving of the state system followed by backward solving of the adjoint system, as a consequence the number of infected decreases enormously as the first figure reveals and optimal controls are obtained in figure 2 for the purpose of minimizing the costs of administering controls. Further, We show the utility of vaccination and its weakness compared to treatment. Finally, the work in this paper contributes to a growing literature on applying stability and optimal control techniques to epidemiology.

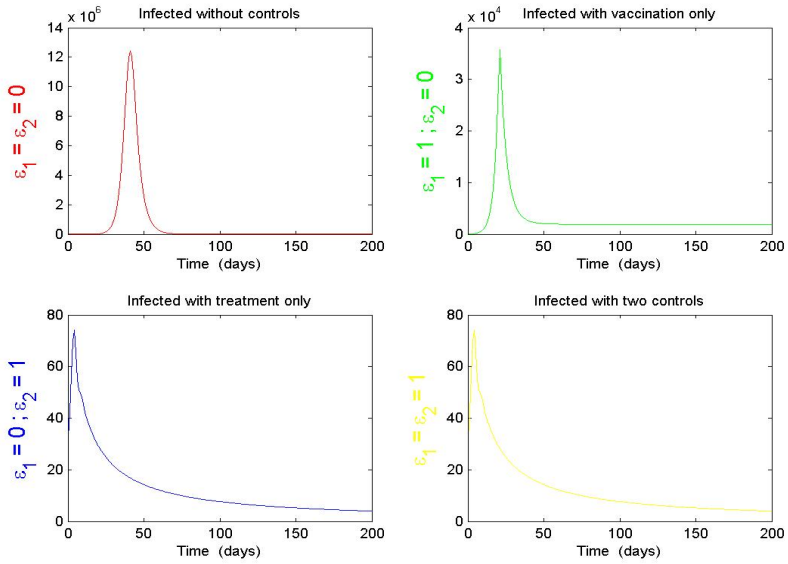

Figure 1. **Infected people with and without controls**

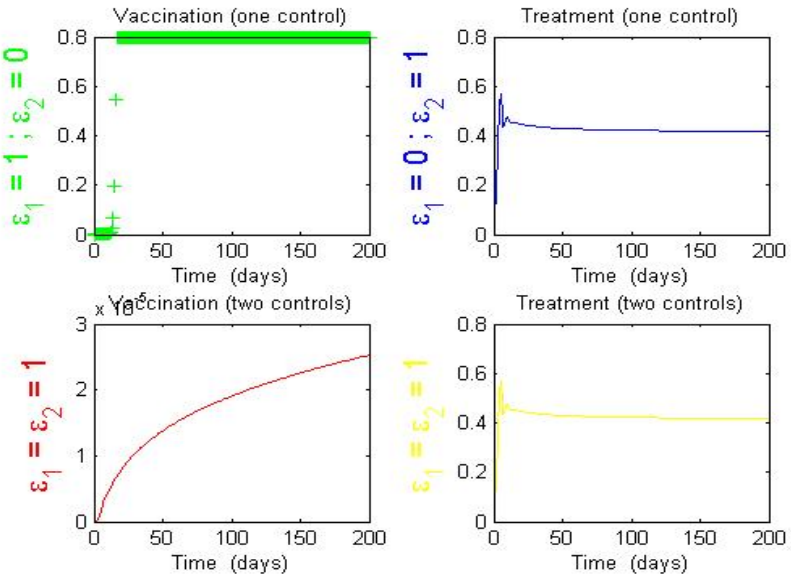

Figure 2. **The optimal controls**

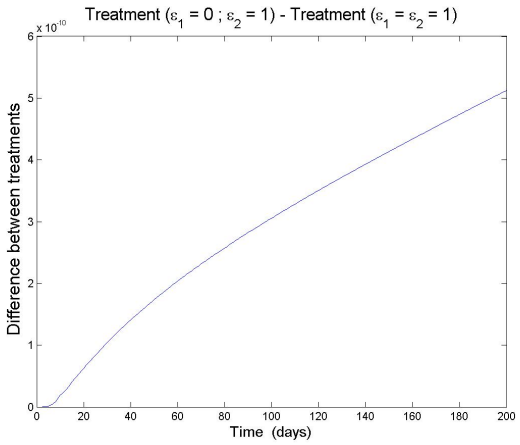

Figure 3. **Difference between treatments**

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