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Relation Between Dominating and Total Dominating Color Transversal number of Graphs

Research Article

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Abstract: Total Dominating Color Transversal Set is the combination of three very well known concepts of graph theory, viz., Total Dominating Set, Transversal and Proper Coloring of vertices of a graph. It is defined as a Total Dominating Set which is also Transversal of Some χ-Partition of vertices of G. Here χ is the Chromatic number of the graph G. Total Dominating Color Transversal number of a graph is the cardinality of a Total Dominating Color Transversal Set which has minimum cardinality among all such Sets that the graph admits. Analogously, R.L.J.Manoharan defined Dominating Color Transversal number of a graph in [2]. In this paper we investigate some relations between these two numbers. We give sufficient examples to justify our results.

MSC: 05C15, 05C69.

Keywords: Total Dominating Color Transversal number, Dominating Color Transversal number, χ-Partition of a graph.

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1. Introduction

In [2], R. L. J. Manoharan defined Dominating Color Transversal number of a graph. Motivated by this concept, we defined Total Dominating Color Transversal number of Graph in [1]. In this Paper we determine some relations between these two numbers. We begin with simple, finite, connected and undirected graph without isolated vertices (unless otherwise stated). We know that proper coloring of vertices of graph G partitions the vertex set V of G into equivalence classes (also called the color classes of G). Using minimum number of colors to properly color all the vertices of G yields χ equivalence classes. Transversal of a χ - Partition of G is a collection of vertices of G that meets all the color classes of the χ-Partition. That is, if T is a subset of V(the vertex set of G) and $\{V_1, V_2, \ldots, V_\chi\}$ is a χ-Partition of G then T is called a Transversal of this χ-Partition if $T \cap V_i \neq \emptyset, \forall\ i \in \{1, 2, \ldots, \chi\}$. Total Dominating Color Transversal Set of graph G is a Total Dominating Set with the extra property that it is also Transversal of some such χ-Partition of G. We first of all mention some definitions.

2. Definitions

Definition 2.1 ([4] Dominating Set)**.** *Let $G = (V, E)$ be a graph. Then a subset S of V (the vertex set of G) is said to be a Dominating set of G if for each $v \in V$ either $v \in S$ or v is adjacent to some vertex in S.*

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Definition 2.2 ([4] Minimum Dominating Set/ Domination number). *Let $G = (V, E)$ be a graph. Then a dominating set S is called the Minimum Dominating set of G if $|S| = \min\{D : D \quad \text{is a Dominating Set of } G\}$. In such case S is called a γ-Set of G and the cardinality of S is called Domination number of the graph G denoted by $\gamma(G)$ or just by γ.*

Definition 2.3 ([3] Total Dominating Set). *Let $G = (V, E)$ be a graph. Then a subset S of V (the vertex set of G) is said to be a Total Dominating Set of G if for each $v \in V$, v is adjacent to some vertex in S.*

Definition 2.4 ([3] Minimum Total Dominating Set/Total Domination number). *Let $G = (V, E)$ be a graph. Then a Total Dominating set S is said to be a Minimum Total Dominating set of G if $|S| = \min\{D : D \quad \text{is a Total Dominating set of } G\}$. Here S is called γ_t-set and its cardinality, denoted by $\gamma_t(G)$ or just by γ_t, is called the Total Domination number of G.*

Definition 2.5 ([1] χ-partition of a graph). *Proper coloring of vertices of a graph G, by using minimum number of colors, yields minimum number of independent subsets of vertex set of G called equivalence classes (also called color classes of G). Such a partition of a vertex set of G is called a χ-Partition of the graph G.*

Definition 2.6 ([1] Transversal of a χ-Partition of a graph). *Let $G = (V, E)$ be a graph with χ-Partition $\{V_1, V_2, \ldots, V_\chi\}$. Then a set $S \subset V$ is called a Transversal of this χ-Partition if $S \cap V_i \neq \emptyset$, $\forall \ i \in \{1, 2, 3, \ldots, \chi\}$.*

Definition 2.7 ([2] Dominating Color Transversal Set). *A dominating set D ? V is called a Dominating Color Transversal Set (std-set) of a graph G if D is a Transversal of at least one χ- Partition of G.*

Definition 2.8 ([2] Minimum Dominating Color Transversal Set/Dominating Color Transversal number). *Let $G = (V, E)$ be a graph. Then $S \subset V$ is called a Minimum Dominating Color Transversal Set of G if $|S| = \min\{D : D \quad \text{is a Minimum Dominating Color Transversal Set of } G\}$. Here S is denoted by $\gamma_{sdt}-$ Set and its cardinality is called the Total Dominating Color Transversal number denoted by by $\gamma_{std}(G)$ or just by γ_{std}.*

Definition 2.9 ([4] Total Dominating Color Transversal Set). *Let $G = (V, E)$ be a graph. Then a Total Dominating Set $S \subset V$ is called a Total Dominating Color Transversal Set of G if it is Transversal of at least one χ-partition of G.*

Definition 2.10 ([1] Minimum Total Dominating Color Transversal Set). *Let $G = (V, E)$ be a graph. Then $S \subset V$ is called a Minimum Total Dominating Color Transversal Set of G if $|S| = \min\{D : D \quad \text{is a Minimum Total Dominating Color Transversal Set of } G\}$. Here S is called γ_{tstd}-Set and its cardinality, denoted by by $\gamma_{tstd}(G)$ or just by γ_{tstd}, is called the Total Dominating Color Transversal number of G.*

3. Main Results

Result 3.1. *For any graph G, if $\gamma = \gamma_t$ then $\gamma_{std} = \gamma_{tstd}$.*

Result 3.2. *For any graph G, $1 \leq \gamma \leq \gamma_t \leq \gamma_{tstd}$.*

Result 3.3. *For any graph G, $\gamma_{std}(G) \leq \gamma_{tstd}(G) \leq 2\gamma_{std}(G)$.*

Proof. (1) $\gamma_{std}(G) \leq \gamma_{tstd}(G)$ as Total Dominating Set is always a Dominating Set.

(2) If Dominating Color Transversal Set S is not a Total Dominating Color Transversal Set then there exists isolates in S. At most $|S|$ number of vertices in S can be isolates. As G is a graph without isolated vertices, each vertex in S has adjacent vertex in G and hence by adding at most $|S|$ vertices to S from $V \setminus S$, we obtain a Total Dominating Color Transversal Set. Hence $\gamma_{tstd}(G) \leq 2\gamma_{std}(G)$. ◻

Remark 3.4. *Result 3.3 indicates that the upper bound of γ_{tstd} number of any graph is $2\gamma_{std}$. We provide Example 3.5 and Example 3.6 indicating that this bound is sharp.*

Example 3.5. *Graph G is disconnected.*

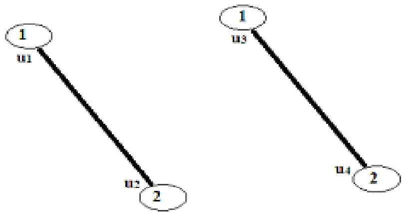

Figure 1. $\gamma_{tstd}(G) = 4$ **and** $\gamma_{std}(G) = 2$.

Example 3.6. *Graph G is Connected. γ_{tstd}- Set of G is $\{u_2, u_3, u_6, u_8, u_9, u_{10}\}$ and γ_{std}-Set of G is $\{u_2, u_6, u_{10}\}$. So*

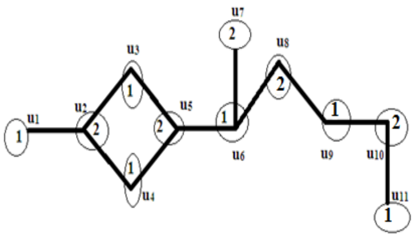

Figure 2. G

$\gamma_{tstd}(G) = 6$ *and* $\gamma_{std}(G) = 3$. *Hence* $\gamma_{tstd}(G) = 2\gamma_{std}(G)$. *Note that graph G is connected*

Theorem 3.7 ([2]). *If $\gamma(G) = 1$ then $\gamma_{std}(G) = \chi(G)$.*

Theorem 3.8 ([1]). *If $\gamma_t(G) = 2$ then $\gamma_{tstd}(G) = \chi(G)$. (G may be disconnected graph.)*

Theorem 3.9. *Let G be a graph. If $\gamma(G) = 1$ then $\gamma_{tstd}(G) = \gamma_{std}(G) = \chi(G)$.*

Proof. We know that $\gamma(G) = 1$ implies that $\gamma_t(G) = 2$. By Theorem 3.7 and 3.8, $\gamma_{std}(G) = \gamma_{tstd}(G) = \chi(G)$. □

Corollary 3.10. *Let G be a graph. If $\gamma_t(G) = 2$ then $\gamma_{tstd}(G)\gamma =_{std} (G) = \chi(G)$.*

Remark 3.11. *Converse of above Corollary 3.10 is not true in general. Following Example 3.12 and Example 3.13 justifies this.*

Example 3.12. *Consider the graph G given in Fig.3, where $\gamma_{tstd}(G) = \gamma_{std}(G) = \chi(G) = 4$ but $\gamma(G) = \gamma_t(G) = 3$.*

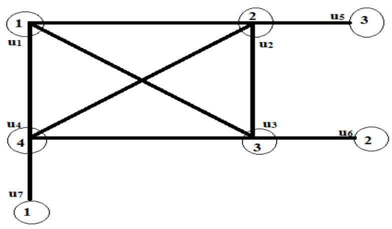

Figure 3. G

Example 3.13. *Consider the graph G given in Fig.4, where $\gamma_{tstd}(G) = \gamma_{std}(G) = \chi(G) = 3$ but $\gamma(G) = 2 \neq \gamma_t(G) = 3$.*

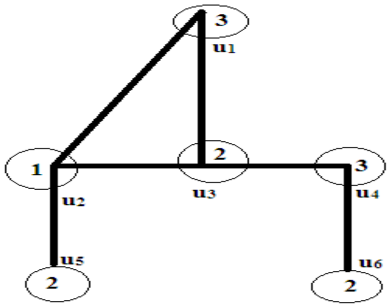

Figure 4. G

Example 3.14. *An example of a graph G ($\neq K_n$) for which $\gamma_{tstd}(G) = \gamma_{std}(G) \neq \chi(G)$ and $\gamma(G) \neq \gamma_t(G)$. See graph G in Fig.5, $\gamma_{tstd}(G) = \gamma_{std}(G) = 3 \neq \chi(G) = 2$ and $\gamma(G) = 2 \neq \gamma_t(G) = 3$.*

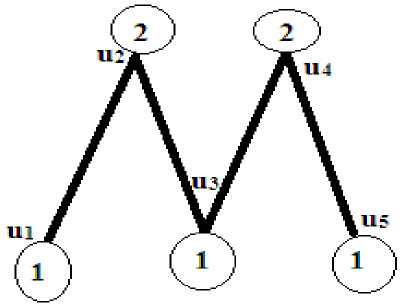

Figure 5. G

Example 3.15. *An example of a graph G for which $\gamma_{tstd}(G) = \gamma_{std}(G) \neq \chi(G)$, $\gamma(G) = \gamma_t(G)$. See Graph G in Fig.6, where $\gamma_{tstd}(G) = \gamma_{std}(G) = 5$ and $\chi(G) = 3$, $\gamma(G) = \gamma_t(G) = 4$.*

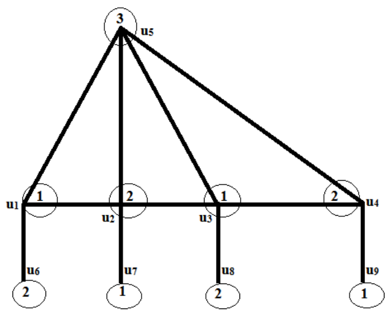

Figure 6. G

Theorem 3.16. *Let G be a graph. If $\chi(G) = n =$ order of the graph G then $\gamma_{tstd}(G) = \gamma_{std}(G) = n$.*

Theorem 3.17. *Let G be a graph. If $\gamma_{tstd}(G) = \chi(G)$ then $\gamma_{std}(G) = \chi(G)$.*

Proof. It is Obvious as $\chi(G) \leq \gamma_{std}(G) \leq \gamma_{tstd}(G) = \chi(G)$. □

Remark 3.18. *Converse of above Theorem 3.17 is not true in general. Following Example 3.19 justifies this.*

Example 3.19.

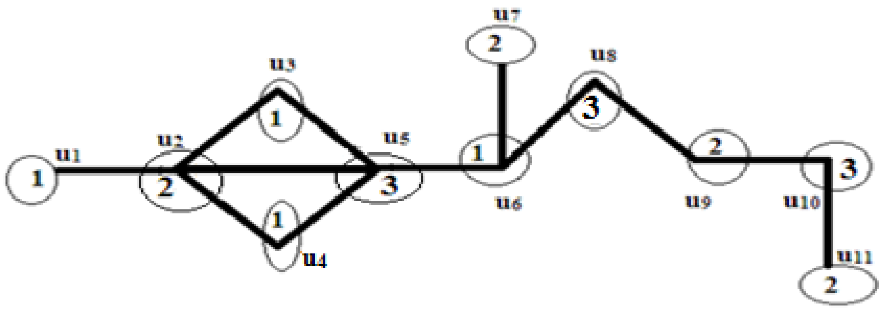

Figure 7. G

γ_{tstd}-Set of G is $\{u_2, u_5, u_6, u_9, u_{10}\}$ and γ_{std}-Set of G is $\{u_2, u_6, u_{10}\}$. So $\gamma_{std}(G) = 3 = \chi(G)$ but $\gamma_{tstd}(G) = 5 \neq 3 = \chi(G)$.

Let us note down one important theorem from [2].

Theorem 3.20 ([2])**.** *Let G be a graph of order n. Then $\gamma_{std}(G) = n$ if and only if $G = K_n$ (K_n denotes a Complete graph with n vertices)*

Remark 3.21. *It is interesting to know that if the graph is connected then the Theorem 3.20 is also true if we replace γ_{std} by γ_{tstd}. But first we note down the following necessary results.*

Result 3.22. *Let G be a graph. Then following are equivalent:*

(1) $\chi(G) = n$, (2) $G = K_n$ and (3) $\omega(G) = n$. ($\omega(G)$ denotes the Clique number of a graph G.)

Example 3.23. *Consider the following example which is about an independent set that is color class of every χ-Partition of G.*

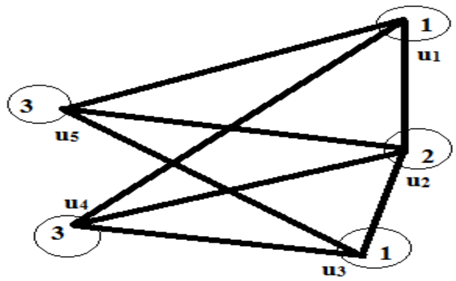

Figure 8. G

Note that $\{u_4, u_5\}$ is a color class of every χ-Partition of G.

Below given theorem determines a condition under which a given independent set is not a color class of some χ-Partition of G. Before that we take a look at the following note which plays vital role in our research paper.

Note 3.24. *If $\{V_1, V_2, \ldots, V_\chi\}$ is a χ-Partition of G then in each color class there exists a vertex which is adjacent to at least one vertex of each remaining color classes.*

Theorem 3.25. *Let G = (V, E) be a graph and $S \subset V$ is an independent set. If S is not maximal independent set then it is a not a color class of some χ-Partition of G.*

Proof. Assume that S is an independent set but not a maximal independent set. If S is not a color class of every χ-Partition of G then we are done. So suppose S is a color class of some χ-Partition $\{V_1, V_2, \ldots, V_\chi\}$ of G. Without loss of generality assume $S = V_1$. As S is not a maximal independent set, there exists vertex $v \in V \setminus S$ such that $S \cup \{v\}$ is an independent set. Let $v \in V_i$, for some $i \in \{2, 3, \ldots, \chi\}$. Note that $\{v\}$ cannot be color class of $\{V_1, V_2, \ldots, V_\chi\}$ for otherwise v has to be adjacent to some vertex of $V_1 = S$. Clearly $\{V_1 \cup \{v\}, V_2, \ldots, V_{i-1}, V_i \setminus \{v\}, V_{i+1}, \ldots, V_\chi\}$ is a χ-Partition of G in which $S = V_1$ is not a color class. □

Theorem 3.26. *Let G = (V, E) be a graph and $S \subset V$ is an independent set. If S is not maximal independent set then $V \setminus S$ is a transversal of some χ-Partition of G.*

Proof. Assume that S is an independent set but not a maximal independent set of G. If $V \setminus S$ is a transversal of every χ- Partition of G then we are done. So assume that $V \setminus S$ is not a transversal of some χ-Partition $\Pi = \{V_1, V_2, \ldots, V_\chi\}$ of G. Then $(V \setminus S) \cap V_i = \phi$, for some $i \in \{1, 2, 3, \ldots, \chi\}$. Without loss of generality assume that $i = 1$. This implies that $V_1 \subset (V \setminus S)^c = S$. ($(V \setminus S)^c$ is a complement of $(V \setminus S)$)

Case 1. $V_1 = S$.

Then $\Pi = \{S, V_2, \ldots, V_{i-1}, V_i, V_{i+1}, \ldots, V_\chi\}$ is a χ-Partition of G. Note that S is not a maximal independent set of G. Then as Theorem 3.24, S is not a color class of χ-Partition $\{V_1 \cup \{v\}, V_2, \ldots, V_{i-1}, V_i \setminus \{v\}, V_{i+1}, \ldots, V_\chi\}$ of G. Clearly $V \setminus S = V \setminus V_1$ is a transversal of this χ-Partition of G.

Case 2. $V_1 \subsetneq S$.

Then there exists $S_1 \subsetneq S$ with $V_1 \cap S_1 = \phi$ such that $V_1 \cup S_1 = S$. Note that S_1 is also an independent set. Also no subset of S_1 can be a color class of Π for otherwise at least one vertex in S_1 have to be adjacent to at least one vertex in V_1. Then one can induce, from χ-Partition Π, a χ-Partition, say Π', of G in which $V_1 \cup S_1 = S$ is a color class. As above, as S is not maximal independent set, one can easily prove that $V \setminus S$ is a transversal of some χ-Partition of G. □

Remark 3.27. *Converse of above Theorem 3.24 and Theorem 3.25 are not true in general. Following Example 3.27 justifies this.*

Example 3.28.

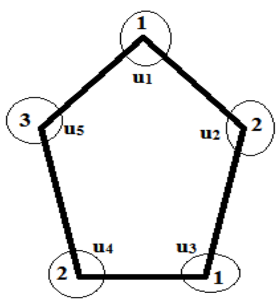

Figure 9. G

In graph G, $S = \{u_1, u_4\}$ is not a color class and $V \setminus S$ is a transversal of the χ-Partition defined above but S is a maximal independent set.

Theorem 3.29. *Let G = (V, E) be a graph and $S \subset V$ be an independent set. If each vertex in S is adjacent to every vertex in $V \setminus S$ then S is a color class of every χ-Partition of G.*

Proof. Assume that each vertex in S is adjacent to every vertex in $V \setminus S$. Let $\Pi = \{V_1, V_2, \ldots, V_{\chi^1}\}$ be a χ^1-Partition of $G \setminus S$. Note that, in G, as each vertex in S is adjacent to every vertex in $V \setminus S$, no vertex of S can be included in any color class of Π and so $\chi^1(G \setminus S) < \chi(G)$. Clearly $\Pi \cup S$ is a χ-Partition of G with $\chi^1(G \setminus S) + 1 = \chi(G)$. Hence S is a color class of every χ-Partition of G. □

Remark 3.30. *Converse of above Theorem 3.28 is not true in general. Following Example 3.30 justifies this.*

Example 3.31.

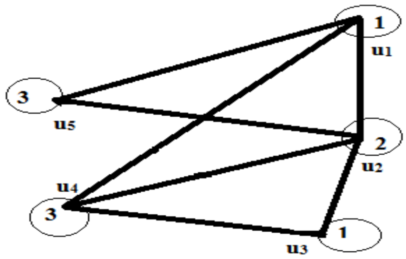

Figure 10. G

Note that $S = \{u_4, u_5\}$ is a color class of every χ-Partition of G but each vertex in S is not adjacent to every vertex in $V \setminus S$.

Now Let us prove the theorem which we were waiting for.

Theorem 3.32. *Let G be a graph with order n. $\gamma_{tstd}(G) = n$ if and only if $G = K_n$.*

Proof. Assume $\gamma_{tstd}(G) = n$.

Suppose $G \neq K_n$. Then there exists two non-adjacent vertices u and v in G and also $|V| \geq 3$ as G is connected. Note that $\{u\}$ is not maximal independent set of G. So by Theorem 3.25, $V \setminus \{u\}$ is a transversal of some χ-Partition of G.

Case 1. u is not a support vertex.

Then trivially $V \setminus \{u\}$ is a total dominating set of G. So $V \setminus \{u\}$ is a Total Dominating Color Transversal Set of G. Hence $\gamma_{tstd}(G) \leq n - 1$ which is contradiction.

Case 2. u is a support vertex.

In such case u is adjacent to a pendant vertex say u^1 of G. Note that u^1 cannot be support vertex for otherwise G becomes disconnected. So $V \setminus \{u^1\}$ is a total dominating set of G. Note that $\{u^1\}$ is not maximal independent set of G as $|V| \geq 3$. So by Theorem 3.25, $V \setminus \{u^1\}$ is a transversal of some χ-Partition of G. Hence $V \setminus \{u^1\}$ is a Total Dominating Color Transversal Set of G. Therefore $\gamma_{tstd}(G) \leq n - 1$ which is contradiction. Hence our assumption that $G \neq K_n$ is wrong. So $G = K_n$.
Converse is obvious. □

Corollary 3.33. *Let G be a graph with order n. Then followings are equivalent:*
(1) $\gamma_{tstd}(G) = n$ *(2)* $\gamma_{std}(G) = n$ *(3)* $G = K_n$ *(4)* $\omega(G) = n$.

4. Concluding Remarks

We have obtained some results that reflects relation between Dominating and Total Dominating Color Transversal number of a graph. Intuitively it seems like there are many amazing relations between these two numbers and many more exciting results are still to come.

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