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Common Fixed Point Theorems For A Pair Of Weakly Increasing/Decreasing Self Maps Under ψ-Weak Generalized Geraghty Contractions in Partially Ordered Partial b-Metric Spaces

Research Article

Vedula Perraju[1*]

1 Principal, Mrs.A.V.N. College, Visakhapatnam, India.

Abstract: In this paper we consider the concept of $\psi-$ weak generalized Geraghty contractive pair of weakly increasing/decreasing self mappings in a complete partially ordered partial b-metric space. We study the existence of fixed points for such a pair of weakly increasing/decreasing self mappings in complete partially ordered partial b-metric spaces controlled by $\psi-$ weak generalized Geraghty contractive type condition and obtain some fixed point results of G.V.R.Babu et.al [3] in complete partially ordered metric spaces as corollaries. Supporting example is also provided. An open problem is given at the end of the paper.

MSC: 54H25, 47H10.

Keywords: Fixed point theorems, weakly increasing/decreasing self mappings, $\psi-$ contractive mappings, partial b-metric, ordered partial b-metric space, partially ordered partial b-metric space, Geraghty contraction.

1. Introduction and preliminaries

Most of the generalizations of fixed point theorems usually start from Banach [5] contraction principle. But all the generalizations may not be from this principle. In 1973, Geraghty [9] introduced an extension of the contraction in which the contraction constant was replaced by a function having some specified properties. In 1989, Bakktin [4] introduced the concept of a b-metric space as a generalization of a metric spaces. In 1993, Czerwik [8] extended many results related to the b-metric spaces. In 1994, Matthews [17] introduced the concept of partial metric space in which the self distance of any point of space may not be zero. In 1996, S.J.O'Neill [22] generalized the concept of partial metric space by admitting negative distances. In 2013, Shukla [28] generalized both the concepts of b-metric and partial metric space by introducing the partial b-metric spaces. Many authors recently studied the existence of fixed points of self maps in different types of metric spaces [1, 2, 14, 21, 26, 29]. Xian Zhang [31] proved a common fixed point theorem for two self maps on a metric space satisfying generalized contractive type conditions. Some authors studied some fixed point theorems in b-metric spaces [18, 24, 25, 32]. After that some authors proved $\alpha-\psi$ versions of certain fixed point theorems in different type metric spaces [13, 19, 24]. Mustafa [20] gave a generalization of Banach contraction principle in complete ordered partial b - metric

* *E-mail: perrajuvedula2004@gmail.com*

space by introducing a generalized $\alpha - \psi$ weakly contractive mapping.

In this paper we prove fixed point theorems for ψ-weak generalized Geraghty contractive pair of weakly increasing/decreasing self mappings in complete partially ordered partial b - metric spaces satisfying a contractive type condition by considering partial b - metric p as in Definition 1.1 (Shukla [28]) which is more general than that of any partial b-metric and obtained some fixed point results of G.V.R.Babu et.al [3] in complete partially ordered metric space as corollaries A supporting example is given and an open problem is also given at the end of the paper. Shukla [28] introduced the notation of a partial b-metric space as follows.

Definition 1.1 ([28]). *Let X be a non empty set and let $s \geq 1$ be a given real number. A function $p : X \times X \to [0, \infty)$ is called a partial b-metric if for all $x, y, z \in X$ the following conditions are satisfied.*

(1). $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$

(2). $p(x, x) \leq p(x, y)$

(3). $p(x, y) = p(y, x)$

(4). $p(x, y) \leq s\{p(x, z) + p(z, y)\} - p(z, z)$. The pair (X, p) is called a partial b-metric space. The number $s \geq 1$ is called a coefficient of (X, p).

Definition 1.2 ([13]). *Let $(X, \leq)$ be a partially ordered set and $T : X \to X$ be a mapping. We say that T is non decreasing with respect to $\leq$ if $x, y \in X$, $x \leq y \Rightarrow Tx \leq Ty$.*

Definition 1.3 ([13]). *Let $(X, \leq)$ be a partially ordered set. A sequence $\{x_n\} \in X$ is said to be non decreasing with respect to $\leq$ if $x_n \leq x_{n+1} \forall\ n \in \mathbb{N}$.*

Definition 1.4 ([20]). *A triple $(X, \leq, p)$ is called an ordered partial b-metric space if $(X, \leq)$ is a partially ordered set and p is a partial b-metric on X.*

Definition 1.5 ([19]). *Define $\Psi = \{\psi / \psi : [0, \infty) \to [0, \infty)$ is non-decreasing and satisfies (1) $\}$ ψ is continuous and*

$$\psi(t) = 0 \Leftrightarrow t = 0 \tag{1}$$

Definition 1.6 ([9]). *A self map $f : X \to X$ is said to be a Geraghty contraction if there exists $\beta \in \Omega$ such that $d(f(x), f(y)) \leq \beta(d(x, y))d(x, y)$ where $\Omega = \{\beta : [0, \infty) \to [0, 1) / \beta(t_n) \to 1 \Rightarrow t_n \to 0 \text{ as } n \to \infty\}$.*

Definition 1.7 ([7]). *Suppose $(X, \leq)$ is a partially ordered set and $f, g : X \to X$ are self maps. f is said to be $g-$ non-decreasing if for $x, y \in X$, $gx \leq gy \Rightarrow fx \leq fy$.*

Definition 1.8 ([3]). *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d such that (X, d) is a metric space. Let f and g be two self mappings on X. Suppose there exists $\psi \in \Psi, \beta \in \Omega$ and $L > 0$ such that*

$$\psi(d(f(x), f(y)) \leq \beta(M(x, y))M(x, y) + L.N(x, y) \tag{2}$$

for all $x, y \in X$ with $gx \geq gy$, where $M(x, y) = \max\{d(gx, gy), d(gx, fx), d(gy, fy), \frac{1}{2}[d(gx, fy) + d(fx, gy)]\}$ and $N(x, y) = \min\{d(gx, gy), d(gx, fy), d(fx, gy)\}$. Then we say that (f, g) is a pair of ψ weak generalized Geraghty contraction maps.

Definition 1.9 ([10]). *Two self maps f and g of a metric space (X, d) are said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$.*

Definition 1.10 ([11]). *Two self maps f and g of a metric space (X, d) are said to be weakly compatible if they commute at their coincidence points, that is if $fu = gu$ for some $u \in X$, then $fgu = gfu$.*

Definition 1.11 ([23]). *Two self maps f and g of a metric space (X, d) are said to be reciprocally continuous if $\lim_{n\to\infty} fgx_n = fz$ and $\lim_{n\to\infty} gfx_n = gz$ whenever $\{x_n\}$ is a sequence in X with $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$. G.V.R.Babu et.al [3] proved the following theorems:*

Theorem 1.12 ([3]). *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d such that (X, d) is a complete metric space. Let f and g be two self maps on X such that f is g-non-decreasing. Suppose that (f, g) is a pair of generalized Geraghty contraction maps satisfying (2). Assume that*

(1). $fX \subseteq gX$

(2). there exists $x_0 \in X$ such that $gx_0 \leq fx_0$

(3). $g(X)$ is is a closed subset of X.

(4). if any non-decreasing $\{x_n\}$ in X, converges to u, then $x_n \leq u \ \forall\ n \in \mathbb{N}$. Then f and g have a coincidence point in X

Theorem 1.13 ([3]). *In addition to the hypothesis of Theorem 1.12, if $gu < ggu$ where u is as in (iv) and f and g are weakly compatible then f and g have a common fixed point in X.*

Theorem 1.14 ([3]). *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d such that (X, d) is a complete metric space. Let f and g be two self maps on X, f is g-non-decreasing. Suppose that (f, g) is a pair of ψ-weak generalized Geraghty contraction maps. Assume that*

(1). $fX \subseteq gX$

(2). f and g are compatible.

(3). there exists $x_0 \in X$ such that $gx_0 \leq fx_0$

(4). f and g are reciprocally continuous. Then f and g have a coincidence point in X.

2. Main Result

In this section we prove coincident point and common fixed point theorems for a pair of weakly increasing/decreasing self maps on partially ordered partial b - metric spaces by using by partial b - metric p of definition 1.1 and obtain Theorems 1.12, 1.13 and 1.14 as corollaries. A supporting example is also given. An open problem is also given at the end. We begin this section with the following definition

Definition 2.1 ([20]). *Suppose $(X, \leq)$ is a partially ordered set and p is a partial b-metric with $s \geq 1$ as the coefficient of (X, p). Then we say that the triplet $(X, \leq, p)$ is a partially ordered partial b-metric space. We observe that every ordered partial b-metric space is a partially ordered partial b-metric space.*

Definition 2.2 ([20]). *A sequence $\{x_n\}$ in a partial b - metric space (X, p) is said to be:*

(1). convergent to a point $x \in X$ if $\lim_{n\to\infty} p(x_n, x) = p(x, x)$

(2). a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ *exists and is finite*

(3). a partial metric space (X, p) *is said to be complete if every Cauchy sequence* $\{x_n\}$ *in* X *converges to a point* $x \in X$ *such that* $\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x) = p(x, x)$.

Now we introduce the notions of a pair of weakly increasing/decreasing self maps, compatibility, weak compatibility and reciprocal continuity of two self maps on a partially ordered partial b-metric space.

Definition 2.3 ([6]). *Let* $(X, \le)$ *be a partially ordered set and* $S, T : X \to X$ *be such that* $Sx \le TSx$ *and* $Tx \le STx$ *(*$Sx \ge TSx$ *and* $Tx \ge STx$*)* $\forall x \in X$*. Then* S *and* T *are said to be weakly increasing/decreasing mappings.*

Definition 2.4. *A pair of weakly increasing/decreasing self maps* S, T *and a self map* g *on a partially ordered partial b - metric space* $(X, \le, p)$ *are said to be compatible if* $\lim_{n\to\infty} p(Sgx_n, gSx_n) = 0 = \lim_{n\to\infty} p(Tgx_n, gTx_n)$ *whenever sequence* $\{x_n\}$ *in* X *such that* $\lim_{m,n\to\infty} p(Sx_m, Tx_n) = \lim_{n\to\infty} p(Tx_n, u) = \lim_{m\to\infty} p(Sx_m, u) = p(u, u) = 0$ *and* $\lim_{m,n\to\infty} p(gx_m, gx_n) = \lim_{n\to\infty} p(gx_n, u) = p(u, u) = 0$ *for some* $u \in X$

Definition 2.5. *A pair of weakly increasing/decreasing self maps* S, T *and a self map* g *on a partially ordered partial b-metric space* $(X, \le, p)$ *are said to be weakly compatible if they commute at their coincidence points, that is* $Su = Tu = gu$ *for some* $u \in X$*, then* $Sgu = gSu = Tgu = gTu$.

Definition 2.6. *A pair of weakly increasing/decreasing self maps* S, T *and a self map* g *on a partially ordered partial b - metric space* $(X, \le, p)$ *are said to be reciprocally continuous if* $\lim_{m,n\to\infty} p(Sgx_m, Tgx_n) = \lim_{m\to\infty} p(Sgx_m, Sz) = \lim_{n\to\infty} p(Tgx_n, Tz) = p(Sz, Tz) = 0$ *and* $\lim_{m,n\to\infty} p(gSx_m, gTx_n) = \lim_{m\to\infty} p(gSx_m, Sz) = \lim_{n\to\infty} p(gTx_n, Tz) = p(gz, gz) = 0$ *whenever* $\{x_n\}$ *is a sequence in* X *with* $\lim_{m,n\to\infty} p(Sx_m, Tx_n) = \lim_{m\to\infty} p(Sx_m, z) = \lim_{n\to\infty} p(Tx_n, z) = p(z, z) = 0$ *and* $\lim_{m,n\to\infty} p(gx_m, gx_n) = \lim_{n\to\infty} p(gx_n, z) = p(z, z) = 0$ *for some* $z \in X$

In the following definition we extend the notion of $\psi -$ weak generalized Geraghty contraction for a pair of weakly increasing/decreasing self maps S, T and a self map g on a partially ordered partial b-metric space $(X, \le, p)$.

Definition 2.7. *Let* $(X, \le)$ *be a partially ordered set and suppose that there exists a partial b-metric* p *such that* (X, p) *is a partial b-metric space. Let* S, T *be a pair of weakly increasing/decreasing self maps and* g *be a self mapping on* X*. Suppose there exists*

$$\psi \in \Psi, \beta \in \Omega \ \text{ such that } \ \psi(sp(Sx, Ty) \le \beta(\psi(M(x, y)))\psi(M(x, y)) \tag{3}$$

for all $x, y \in X$ *whenever* gx *and* gy *are comparable, where* $M(x, y) = \max\{p(gx, gy), p(gx, Sx), p(gy, Ty), \frac{1}{2s}[p(gx, Ty) + p(Sx, gy)]\}$*. Then we say that* g *is a pair of* ψ *weak generalized Geraghty contraction maps* S, T*. We also say that* g *is a pair of weak generalized Geraghty contraction maps* S, T *if* $\psi(t) = t \ \forall \ t \in [0, \infty)$.

Definition 2.8 ([7]). *Suppose* $(X, \le)$ *is a partially ordered set and* $S, T, g : X \to X$ *are self maps on* X*.* S, T *are said to be g-non-decreasing if for* $x, y \in X$*,* $gx \le gy \Rightarrow Sx \le Sy$ *and* $Tx \le Ty$.

Now we state the following useful lemmas, whose proofs can be found in Sastry. et. al [26].

Lemma 2.9. *Let* $(X, \le, p)$ *be a complete partially ordered partial b - metric space with coefficient* $s \ge 1$*. Let* $\{x_n\}$ *be a sequence in* X *such that*

(1). $\lim_{n\to\infty} x_n = x$*,* $\lim_{n\to\infty} y_n = y$ *and* $\lim_{n\to\infty} p(x_n, y_n) = 0 \Rightarrow x = y$

(2). $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$ *and* $\lim_{n\to\infty} x_n = x$ *,* $\lim_{n\to\infty} x_n = y$
Then $\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, y) = p(x, y)$ *and hence* $x = y$

Lemma 2.10.

(1). $p(x, y) = 0 \Rightarrow x = y$

(2). $\lim_{n\to\infty} p(x_n, x) = 0 \Rightarrow p(x, x) = 0$ *and hence* $x_n \to x$ *as* $n \to \infty$.

Lemma 2.11. *Let* $(X, \leq, p)$ *be a partially ordered partial b - metric space with coefficient* $s \geq 1$. *Let* $\{x_n\}$ *be a sequence in* X *such that* $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$. *Then*

(1). $\{x_n\}$ *is a Cauchy sequence* $\Rightarrow \lim_{m,n\to\infty} p(x_m, x_n) = 0$.

(2). $\{x_n\}$ *is not a Cauchy sequence* $\Rightarrow \exists\ \epsilon > 0$ *and sequences* $\{m_i\}, \{n_i\} \ni m_k > n_k > k \in \mathbb{N}$; $p(x_{m_k}, x_{n_k}) > \epsilon$ *and* $p(x_{n_k}, x_{m_k-1}) \leq \epsilon$.

Proof.

(1). Suppose $\{x_n\}$ is a Cauchy sequence then $\lim_{m,n\to\infty} p(x_m, x_n)$ exists and finite. Therefore $0 = \lim_{n\to\infty} p(x_n, x_{n+1}) = \lim_{m,n\to\infty} p(x_m, x_n)$. Therefore $\lim_{m,n\to\infty} p(x_m, x_n) = 0$.

(2). $\{x_n\}$ is not a Cauchy sequence $\Rightarrow \lim_{m,n\to\infty} p(x_m, x_n) \neq 0$ if it exists $\Rightarrow \exists\ \epsilon > 0$ and for every N and $m, n > N \ni p(x_m, x_n) > \epsilon \because \lim_{n\to\infty} p(x_n, x_{n+1}) = 0 \Rightarrow \exists\ M \ni p(x_n, x_{n+1}) < \epsilon\ \forall\ n > M$. Let $N_1 > M$ and n_1 be the smallest such that $m > n_1$ and $p(x_{n_1}, x_m) > \epsilon$ for at least one m. Let m_1 be the smallest such that $m_1 > n_1 > N_1 > 1$ and $p(x_{n_1}, x_{m_1}) > \epsilon$ so that $p(x_{n_1}, x_{m_1-1}) \leq \epsilon$. Let $N_2 > N_1$ and choose $m_2 > n_2 > N_2 > 2 \ni p(x_{n_2}, x_{m_2}) > \epsilon$ and $p(x_{n_2}, x_{m_2-1}) \leq \epsilon$. Continuing this process we can get sequences of positive integers $\{m_k\}$ and $\{n_k\}$ such that $m_k > n_k > k$ and $p(x_{n_k}, x_{m_k}) > \epsilon$; $p(x_{n_k}, x_{m_k-1}) \leq \epsilon$.

□

Now we state our first main result for a pair of weakly increasing self maps:

Theorem 2.12. *Let* $(X, \leq, p)$ *be a complete partially ordered partial b - metric space with coefficient* $s \geq 1$. *Let* S, T *be a pair of weakly increasing self maps and* g *be a self mapping on* X. S, T *are* g*- non - decreasing. Suppose that* g *is a pair of weak generalized Geraghty contraction maps* S, T, *that is there exist* $\psi \in \Psi$ *and* $\beta \in \Omega$ *such that* $\psi(sp(Sx, Ty) \leq \beta(\psi(M(x, y)))\psi(M(x, y))$ *for all* $x, y \in X$ *whenever* gx *and* gy *are comparable, where*

$$M(x, y) = max\{p(gx, gy), p(gx, Sx), p(gy, Ty), \frac{1}{2s}[p(gx, Ty) + p(Sx, gy)]\} \tag{4}$$

Assume that

(1). $S(X), T(X) \subseteq g(X)$

(2). there exists $x_0 \in X$ *such that* $gx_0 \leq Sx_0$

(3). $g(X)$ *is is a closed subset of* X.

(4). if any non - decreasing sequence $\{x_n\}$ *in* X, *converges to* u, *then* $x_n \leq u\ \ \forall\ n \geq 0$

Then S, T *and* g *have a coincidence point in* X.

Proof. let $x_0 \in X$ be as in (ii). If $gx_0 = Sx_0$ then x_0 is a coincident point and there is nothing to prove. Now suppose $gx_0 < Sx_0$. By (i) $\exists\, x_1 \in X$ such that $gx_1 = Sx_0$. Since $gx_0 < Sx_0 = gx_1$ and S is $g-$ non decreasing, we have $Sx_0 \leq TSx_0 \Rightarrow Sx_0 \leq Tx_1$. Since $S(X), T(X) \subseteq g(X)$ and $Tx_1 \in T(X) \subseteq g(X)$, there exists $x_2 \in X$ such that $gx_2 = Tx_1$ and $gx_1 \leq gx_2$. Continuing this process, we can find sequence $\{x_n\}$ with $Sx_{2n} = gx_{2n+1}$ and $Tx_{2n+1} = gx_{2n+2}$ for $n = 0, 1, 2, 3, \ldots$. Further, since $gx_1 \leq gx_2$ and S, T are weakly increasing $g-$ non decreasing, we have $Tx_1 \leq Sx_2$ so that $gx_2 \leq gx_3$.

$\therefore$ By induction, we get $gx_n \leq gx_{n+1}\ \forall\ n = 0, 1, 2, 3, \ldots$ Suppose n is odd and $gx_n = gx_{n+1} \Rightarrow gx_{n+1} = Tx_n = gx_n \Rightarrow x_n$ is a coincident point of T and g in X. Suppose n is even and $gx_n = gx_{n+1} \Rightarrow gx_{n+1} = Sx_n = gx_n \Rightarrow x_n$ is a coincident point of S and g in X. Suppose n is odd and x_n is a coincident point of T and g in X. Then $gx_n = gx_{n+1} \Rightarrow gx_{n+1} = Tx_n = gx_n$ and assume that $gx_{n+1} \neq gx_{n+2}$ we have

$$\begin{aligned}
\psi(sp(gx_{n+2}, gx_{n+1})) &= \psi(sp(Sx_{n+1}, Tx_n)) \\
&\leq \beta(\psi(M(x_{n+1}, x_n)))\psi(M(x_{n+1}, x_n)), \text{ where } M(x_{n+1}, x_n) \\
&= \max\{p(gx_{n+1}, gx_n), p(gx_{n+1}, Sx_{n+1}), p(gx_n, Tx_n), \frac{1}{2s}[p(gx_{n+1}, Tx_n) + p(Sx_{n+1}, gx_n)]\} \\
&= \max\{p(gx_{n+1}, gx_n), p(gx_{n+1}, gx_{n+2}), p(gx_n, gx_{n+1}), \frac{1}{2s}[p(gx_{n+1}, gx_{n+1}) + p(gx_{n+2}, gx_n)]\} \\
&\leq \max[p(gx_{n+1}, gx_n), p(gx_{n+1}, gx_{n+2}), \frac{1}{2s}\{p(gx_{n+1}, gx_{n+1}) + s(p(gx_{n+2}, gx_{n+1}) + p(gx_{n+1}, gx_n)) \\
&\quad - p(gx_{n+1}, gx_{n+1})\}] \\
&= p(gx_{n+1}, x_{n+2}) \\
\therefore\ \psi(sp(gx_{n+2}, gx_{n+1})) &\leq \beta(\psi(p(gx_{n+2}, gx_{n+1}))\psi(p(gx_{n+2}, gx_{n+1}) \\
&< \psi(p(gx_{n+2}, gx_{n+1})) \\
\Rightarrow sp(gx_{n+2}, gx_{n+1}) &< p(gx_{n+2}, gx_{n+1}), \quad \text{a contradiction.} \\
\therefore gx_{n+1} &= gx_{n+2} \\
\therefore gx_{n+2} &= Sx_{n+1} = gx_{n+1}
\end{aligned}$$

$\therefore x_{n+1} = x_n$ is a coincident point of S and g in X. $\therefore x_n$ is a coincident point of T and g in X then x_n is a coincident point of S and g in X. Similarly by considering n to be even x_n is a coincident point of S and g in X, then x_n is a coincident point of T and g in X. Let n be odd and we may assume that $gx_{n+1} \neq gx_{n+2}\ \forall\ n \in \mathbb{N}$. Then we have $p(gx_{n+2}, gx_{n+1}) > 0$, therefore by (4),

$$\begin{aligned}
\psi(sp(gx_{n+2}, gx_{n+1})) &= \psi(sp(Sx_{n+1}, Tx_n)) \\
&\leq \beta(\psi(M(x_{n+1}, x_n)))\psi(M(x_{n+1}, x_n)), \text{ where } M(x_{n+1}, x_n) \\
&= \max\{p(gx_{n+1}, gx_n), p(gx_{n+1}, Sx_{n+1}), p(gx_n, Tx_n), \frac{1}{2s}[p(gx_{n+1}, Tx_n) + p(Sx_{n+1}, gx_n)]\} \\
&= \max\{p(gx_{n+1}, gx_n), p(gx_{n+1}, gx_{n+2}), p(gx_n, gx_{n+1}), \frac{1}{2s}[p(gx_{n+1}, gx_{n+1}) + p(gx_{n+2}, gx_n)]\} \\
&\leq \max[p(gx_{n+1}, gx_n), p(gx_{n+1}, gx_{n+2}), \frac{1}{2s}\{p(gx_{n+1}, gx_{n+1}) + s(p(gx_{n+2}, gx_{n+1}) + p(gx_{n+1}, gx_n)) \\
&\quad - p(gx_{n+1}, gx_{n+1})\}] \\
&= \max[p(gx_{n+1}, gx_n), p(gx_{n+1}, x_{n+2})]
\end{aligned}$$

Suppose

$$p(gx_{n+1}, gx_n) \leq p(gx_{n+1}, gx_{n+2}) \tag{5}$$

Then $M(x_{n+1}, x_n) = p(gx_{n+1}, gx_{n+2})$

$$\therefore \psi(sp(gx_{n+2}, gx_{n+1})) \leq \beta(\psi(p(gx_{n+2}, gx_{n+1}))\psi(p(gx_{n+2}, gx_{n+1}) < \psi(p(gx_{n+2}, gx_{n+1}))$$

$\Rightarrow sp(gx_{n+1}, gx_n) < p(gx_{n+2}, gx_{n+1})$, a contradiction.

$$\therefore M(x_{n+1}, x_n) = p(gx_{n+1}, gx_n) \tag{6}$$

$$\begin{aligned}\therefore \psi(p(gx_{n+2}, gx_{n+1})) &\leq \psi(sp(gx_{n+2}, gx_{n+1}))\\ &\leq \beta(\psi(p(gx_{n+1}, gx_n))\psi(p(gx_{n+1}, gx_n)\\ &< \psi(p(gx_{n+1}, gx_n)))\\ \Rightarrow p(gx_{n+2}, gx_{n+1}) &\leq sp(gx_{n+2}, gx_{n+1}) < p(gx_{n+1}, gx_n) \end{aligned}\tag{7}$$

$\therefore$ sequence $\{\psi(p(gx_{n+1}, gx_n))\}$ is strictly decreasing and converges to r (say). Also sequence $p(gx_{n+1}, gx_n)$ is strictly decreasing and converges to λ (say).

$$\therefore r = \psi(\lambda) \tag{8}$$

Suppose $r \neq 0$

$$\therefore \frac{\psi(p(gx_{n+2}, gx_{n+1}))}{\psi(p(gx_{n+1}, gx_n))} \leq \beta(\psi(p(gx_{n+1}, gx_n)) < 1 \tag{9}$$

taking limits as $n \to \infty$

$$\therefore \lim_{n\to\infty} \beta(\psi(p(gx_{n+1}, gx_n))) = 1 \Rightarrow \lim_{n\to\infty} \psi(p(gx_{n+1}, gx_n)) = 0$$

$$\therefore r = 0 \Rightarrow \psi(\lambda) = 0 \Rightarrow \lambda = 0 \tag{10}$$

In the similar lines we can discuss the case when n is even and arrive the same conclusions.

$$\therefore r = 0 \Rightarrow \psi(\lambda) = 0 \Rightarrow \lambda = 0 \tag{11}$$

Now we claim sequence $\{gx_n\}$ is a Cauchy sequence. Assume that $\{gx_n\}$ is not a Cauchy sequence. Then by lemma 2.11 $\exists\ \epsilon > 0$ and sequences $\{m_k\}$, $\{n_k\}$; $m_k > n_k > k$ such that $p(gx_{m_k}, gx_{n_k}) \geq \epsilon$ and $p(gx_{m_k-1}, gx_{n_k}) < \epsilon$. Let us observe the following cases:

Case(i): Let m_k is even and n_k is odd

$$\begin{aligned}\therefore \psi(s\epsilon) \leq \psi\{sp(gx_{m_k}, gx_{n_k})\} &= \psi\{sp(Tx_{m_k-1}, Sx_{n_k-1})\}\\ &\leq \beta(\psi(M(x_{m_k-1}, x_{n_k-1}))\psi\{M(x_{m_k-1}, x_{n_k-1})\} \text{ where } M(x_{m_k-1}, x_{n_k-1}) \qquad (12)\\ &= \max[p(gx_{m_k-1}, gx_{n_k-1}), p(gx_{n_k-1}, Sx_{n_k-1}), p(gx_{m_k-1}, Tx_{m_k-1}),\\ &\quad \frac{1}{2s}[\{p(gx_{m_k-1}, Sx_{n_k-1}) + p(Tx_{m_k-1}, gx_{n_k-1})\}]\\ &= \max[p(gx_{m_k-1}, gx_{n_k-1}), p(gx_{n_k-1}, gx_{n_k}), p(gx_{m_k-1}, gx_{m_k}),\\ &\quad \frac{1}{2s}[\{p(gx_{m_k-1}, gx_{n_k}) + p(gx_{m_k}, gx_{n_k-1})\}]\\ &\leq \max[p(gx_{m_k-1}, gx_{n_k-1}), p(gx_{n_k-1}, gx_{n_k}), p(gx_{m_k-1}, gx_{m_k}),\\ &\quad \frac{1}{2s}[\{sp(gx_{m_k-1}, gx_{n_k-1}) + sp(gx_{n_k-1}, gx_{n_k}) - p(gx_{n_k-1}, gx_{n_k-1}) + sp(gx_{m_k-1}, gx_{n_k-1})\\ &\quad + sp(gx_{m_k}, gx_{m_k-1}) - p(gx_{m_k-1}, gx_{m_k-1})\}]\end{aligned}$$

$$\begin{aligned}
&\le \max[p(gx_{m_k-1}, gx_{n_k-1}), p(gx_{n_k-1}, gx_{n_k}), p(gx_{m_k-1}, gx_{m_k}), \frac{1}{2s}[\{2sp(gx_{m_k-1}, gx_{n_k-1})\\
&\quad + sp(gx_{n_k-1}, gx_{n_k}) + sp(gx_{m_k}, gx_{m_k-1})\}]\\
&= p(gx_{m_k-1}, gx_{n_k-1}) + \frac{1}{2}p(gx_{n_k-1}, gx_{n_k}) + \frac{1}{2}p(gx_{m_k}, gx_{m_k-1})\\
&\le sp(gx_{m_k-1}, gx_{n_k}) + sp(gx_{n_k}, gx_{n_k-1}) - p(gx_{n_k}, gx_{n_k}) + \frac{1}{2}p(gx_{n_k-1}, gx_{n_k}) + \frac{1}{2}p(gx_{m_k}, gx_{m_k-1})\\
&\le sp(gx_{m_k-1}, gx_{n_k}) + sp(gx_{n_k}, gx_{n_k-1}) + \frac{1}{2}p(gx_{n_k-1}, gx_{n_k}) + \frac{1}{2}p(gx_{m_k}, gx_{m_k-1})\\
&\le s\epsilon + s\eta + \frac{1}{2}\eta + \frac{1}{2}\eta \text{ where } \eta > 0 \text{ for large } k
\end{aligned}$$

$$\therefore \psi(s\epsilon) \le \beta(\psi(M(x_{m_k-1}, x_{n_k-1}))\psi(s\epsilon + s\eta + \eta) < \psi(s\epsilon + s\eta + \eta) \tag{13}$$

(This being for large k and true for every $\eta > 0$). Since ψ is continuous , then we get for large k, $\psi(s\epsilon) \le \lim_{k\to\infty} \beta(\psi(M(x_{m_k-1}, x_{n_k-1})))\ \psi(s\epsilon) \le \psi(s\epsilon)$. Therefore $\lim_{k\to\infty} \beta(\psi(M(x_{m_k-1}, x_{n_k-1}))) = 1$. Therefore $\lim_{k\to\infty} M(x_{m_k-1}, x_{n_k-1}) = 0 \Rightarrow \psi(s\epsilon) \le 0 \Rightarrow \psi(s\epsilon) = 0 \Rightarrow s\epsilon = 0$, a contradiction.

Case(ii): Let m_k is odd and n_k is odd

$$\begin{aligned}
\therefore \psi(sp(gx_{m_k}, gx_{n_k+1})) &= \psi(sp(Sx_{m_k-1}, Tx_{n_k})))\\
&\le \beta(\psi(M(x_{m_k-1}, x_{n_k}))\psi(M(x_{m_k-1}, x_{n_k}))\\
&< M(x_{m_k-1}, x_{n_k}) \text{ where } M(x_{m_k-1}, x_{n_k}) \qquad (14)\\
&= \max[p(gx_{m_k-1}, gx_{n_k}), p(gx_{m_k-1}, Sx_{m_k-1}), p(gx_{n_k}, Tx_{n_k}),\\
&\quad \frac{1}{2s}[\{p(Sx_{m_k-1}, gx_{n_k}) + p(gx_{m_k-1}, Tx_{n_k})\}]\\
&= \max[p(gx_{m_k-1}, gx_{n_k}), p(gx_{m_k-1}, gx_{m_k}), p(gx_{n_k}, gx_{n_k+1}),\\
&\quad \frac{1}{2s}[\{p(gx_{m_k}, gx_{n_k}) + p(gx_{m_k-1}, gx_{n_k+1})\}]\\
&= p(gx_{m_k-1}, gx_{n_k}) \quad or \quad \frac{1}{2s}[\{p(gx_{m_k}, gx_{n_k}) + p(gx_{m_k-1}, gx_{n_k+1})\}]
\end{aligned}$$

Suppose $M(x_{m_k-1}, x_{n_k}) = p(gx_{m_k-1}, gx_{n_k}) < \epsilon$. But

$$\begin{aligned}
\epsilon \le p(gx_{m_k}, gx_{n_k}) &\le sp(gx_{m_k}, gx_{n_k+1}) + sp(gx_{n_k+1}, gx_{n_k}) - p(gx_{n_k+1}, gx_{n_k+1})\\
&\le sp(gx_{m_k}, gx_{n_k+1}) + s\eta \text{ where } \eta > 0 \ni p(gx_{n_k+1}, gx_{n_k}) < \eta \qquad (15)\\
\Rightarrow \epsilon - s\eta &\le sp(gx_{m_k}, gx_{n_k+1}) \qquad (16)
\end{aligned}$$

$$\begin{aligned}
\therefore \psi(\epsilon - s\eta) \le \psi(sp(gx_{m_k}, gx_{n_k+1})) &\le \psi(sp(Sx_{m_k-1}, Tx_{n_k}))\\
&\le \beta(\psi(p(gx_{m_k-1}, gx_{n_k})))\psi(p(gx_{m_k-1}, gx_{n_k}))\\
&< \psi(p(gx_{m_{k-1}}, gx_{n_k})) < \psi(\epsilon) \qquad (17)
\end{aligned}$$

Allowing $k \to \infty$, then $\eta \to 0$ and ψ is continuous. $\therefore\ \psi(\epsilon) \le \lim_{k\to\infty} \beta(\psi(p(gx_{m_k-1}, gx_{n_k})))\psi(\epsilon) \le \psi(\epsilon)$ and $\lim_{k\to\infty} p(gx_{m_k-1}, gx_{n_k}) = \epsilon$. $\therefore \lim_{k\to\infty} \beta(\psi(p(gx_{m_k-1}, gx_{n_k}))) = 1 \Rightarrow \lim_{k\to\infty} p(gx_{m_k-1}, gx_{n_k}) = 0 \Rightarrow \epsilon = 0$, a contradiction. Suppose $M(x_{m_k-1}, x_{n_k}) = \frac{1}{2s}\ [\{p(gx_{m_k}, gx_{n_k}) + p(gx_{m_k-1}, gx_{n_k+1})\}]$. On the other hand

$$\begin{aligned}
p(gx_{m_k}, gx_{n_k}) + p(gx_{m_k-1}, gx_{n_k+1}) &\le sp(gx_{n_k}, gx_{m_k-1}) + sp(gx_{m_k-1}, gx_{m_k}) - p(gx_{m_k-1}, gx_{m_k-1}) + sp(gx_{n_k+1}, gx_{n_k})\\
&\quad + sp(gx_{n_k}, gx_{m_k-1}) - p(gx_{n_k}, gx_{n_k})\\
&\le sp(gx_{n_k}, gx_{m_k-1}) + sp(gx_{m_k-1}, gx_{m_k}) + sp(gx_{n_k}, gx_{m_k-1}) + sp(gx_{n_k+1}, gx_{n_k})\\
&\le 2sp(gx_{n_k}, gx_{m_k-1}) + 2s\eta \le 2s\epsilon + 2s\eta.
\end{aligned}$$

where $p(gx_{n_k+1}, gx_{n_k}) \leq \eta$ and $p(gx_{m_k}, gx_{m_k-1}) \leq \eta$ for some $\eta > 0$ for large k

$$\therefore \frac{1}{2s}[\{p(gx_{m_k}, gx_{n_k}) + p(gx_{m_k-1}, gx_{n_k+1})\}] \leq \epsilon + \eta. \tag{18}$$

Therefore,

$$M(x_{m_k-1}, x_{n_k}) = \frac{1}{2s}[\{p(gx_{m_k}, gx_{n_k}) + p(gx_{m_k-1}, gx_{n_k+1})\}] \leq \epsilon + \eta$$

Therefore from (16), (17) and (18)

$$\begin{aligned}
\epsilon - s\eta &\leq sp(gx_{m_k}, gx_{n_k+1}) \\
\therefore \psi(\epsilon - s\eta) &\leq \psi(sp(gx_{m_k}, gx_{n_k+1})) \\
&\leq \beta(\psi(M(x_{m_k-1}, x_{n_k})))\psi(M(x_{m_k-1}, x_{n_k})) \\
&\leq \psi(M(x_{m_k-1}, x_{n_k})) \\
&\leq \psi(\epsilon + \eta)
\end{aligned}$$

Allowing $k \to \infty$, then $\eta \to 0$.

$$\therefore \psi(\epsilon) \leq \lim_{k\to\infty} \beta(\psi(M(x_{m_k-1}, x_{n_k}))) \lim_{k\to\infty} \psi(M(x_{m_k-1}, x_{n_k})) \leq \psi(\epsilon)$$

and $\lim_{k\to\infty} M(x_{m_k-1}, x_{n_k}) = \epsilon$. $\therefore \lim_{k\to\infty} \beta(\psi(M(x_{m_k-1}, x_{n_k}))) = 1 \Rightarrow \lim_{k\to\infty} M(x_{m_k-1}, x_{n_k}) = 0 \Rightarrow \epsilon = 0$, a contradiction. Similarly the other two cases can be discussed. Therefore $\{gx_n\}$ is a Cauchy sequence. Therefore $\{gx_n\} \to gy$ for some $y \in X$ by (iii). Also

$$0 = \lim_{n,m\to\infty} p(gx_n, gx_m) = \lim_{n\to\infty} p(gx_n, gy) = p(gy, gy) \tag{19}$$

Now by (iv) of the hypothesis $gx_n \leq gy \ \ \forall\, n \in \mathbb{N}$. Therefore $gx_{n+1} \leq gy \Rightarrow Tx_n \leq Ty$ and $Sx_n \leq Sy \ \ \forall\, n \in \mathbb{N}$(since S, T are $g-$ non-decreasing). Let n be even $\psi\{sp(Sx_n, Ty)\} \leq \beta(\psi(M(x_n, y))\psi(M(x_n, y))$, where

$$\begin{aligned}
M(x_n, y) &= \max\{p(gx_n, gy), p(gy, Ty), p(gx_n, Sx_n), \frac{1}{2s}[p(gx_n, Ty) + p(Sx_n, gy)]\} \\
&= \max\{p(gx_n, gy), p(gy, Ty), p(gx_n, gx_{n+1}), \frac{1}{2s}[p(gx_n, Ty) + p(gx_{n+1}, gy)]\} \\
&= p(gy, Ty) \quad \text{for large } \ n.
\end{aligned} \tag{20}$$

Now, $\lim_{n\to\infty} \beta(\psi(M(x_n, y))) = 1 \Rightarrow \lim_{n\to\infty} \psi(M(x_n, y)) = 0$

$$\Rightarrow \psi(p(gy, Ty)) = 0 \ \ by \ \ (20) \tag{21}$$

$\Rightarrow p(gy, Ty) = 0 \Rightarrow gy = Ty$ (by Lemma 2.10 (i)). Therefore y is a coincident point of T and g. Suppose $\exists\ \lambda$ such that

$$\beta(\psi(M(x_n, y))) = \lambda, \quad \text{for infinitely many n} \tag{22}$$

$\therefore 0 \leq \lambda < 1$, $\psi(sp(Sx_n, Ty)) \leq \lambda\psi(M(x_n, y)) \leq \lambda\psi(p(gy, Ty)) < \psi(p(gy, Ty))$

$$\Rightarrow sp(Sx_n, Ty) < p(gy, Ty) \Rightarrow \lim_{n\to\infty} \sup sp(Sx_n, Ty) \leq p(gy, Ty) \tag{23}$$

Now

$$\begin{aligned} p(gy,Ty) &\leq sp(gy,gx_{n+1}) + sp(gx_{n+1},Ty) - p(gx_{n+1},gx_{n+1}) \\ &\leq sp(gy,gx_{n+1}) + sp(gx_{n+1},Ty) \\ \Rightarrow p(gy,Ty) - sp(gy,gx_{n+1}) &\leq sp(Sx_n,Ty) \\ \Rightarrow p(gy,Ty) &\leq \lim_{n\to\infty}\inf sp(Sx_n,Ty) \end{aligned} \tag{24}$$

Therefore $\lim_{n\to\infty}\sup sp(Sx_n,Ty) \leq p(gy,Ty) \leq \lim_{n\to\infty}\inf sp(Sx_n,Ty)$. Therefore $\lim_{n\to\infty} sp(Sx_n,Ty) = p(gy,ty)$.

$$\begin{aligned} \therefore \psi(p(gy,Ty)) &= \psi(\lim_{n\to\infty} sp(Sx_n,Ty)) \\ &= \lim_{n\to\infty} \psi(sp(Sx_n,Ty)) \ \ (\textit{since } \psi \textit{ is continuous}) \\ &\leq \lambda\psi(p(gy,Ty)) \end{aligned}$$

$$\Rightarrow \psi(p(gy,Ty)) = 0 \Rightarrow p(gy,Ty) = 0 \Rightarrow gy = Ty \tag{25}$$

Therefore y is a coincident point of T and g. Let n be odd. Interchanging the roles of S and T in the above discussion we can conclude y is a coincident point of S and g. Hence y is a coincident point of a pair of weakly increasing self maps S,T and g. □

Now we state and prove our second main result.

Theorem 2.13. *Let $(X,\leq,p)$ be a complete partially ordered partial b-metric space with coefficient $s \geq 1$. Let S,T be a pair of weakly increasing self maps and g be a self mapping on X. S,T are g-non-decreasing. Suppose that g is a pair of weak generalized Geraghty contraction maps S,T, that is there exist $\psi \in \Psi$ and $\beta \in \Omega$ such that $\psi(sp(Sx,Ty) \leq \beta(\psi(M(x,y)))\psi(M(x,y))$ for all $x,y \in X$ whenever gx and gy are comparable, where*

$$M(x,y) = \max\{p(gx,gy), p(gx,Sx), p(gy,Ty), \frac{1}{2s}[p(gx,Ty) + p(Sx,gy)]\} \tag{26}$$

Assume that

(1). $S(X),T(X) \subseteq g(X)$.

(2). there exists $x_0 \in X$ such that $gx_0 \leq Sx_0$.

(3). $g(X)$ is a closed subset of X.

(4). if any non - decreasing $\{x_n\}$ in X, converges to y, then that $x_n \leq y \ \ \forall\ n \geq 0$.

Further if S,T and g are weakly compatible and if $gy \leq ggy\ \forall\ y \in X$, then S,T and g have a common fixed point in X.

Proof. We have by Theorem 2.12, $\{gx_n\}$ is a Cauchy sequence, which is non-decreasing that converges to gy and $gy = Sy = Ty$. Therefore $\lim_{n,m\to\infty} p(gx_n,gx_m)$ exists and is equal to 0. As sequence $\{gx_n\} \to gy$ implies $0 = \lim_{n,m\to\infty} p(gx_n,gx_m) = \lim_{n\to\infty} p(gx_n,gy) = p(gy,gy)$. Since S,T and g are weakly compatible, we have $Sgy = gSy = Tgy = gTy$. Let

$$gy = Sy = Ty = u \ \ \text{(say)} \tag{27}$$

$$\therefore Tu = Tgy = gTy = gu \tag{28}$$

If $y = u$, then $u = Su = Tu = gu \Rightarrow u$ is a common fixed point of T and g in X. Let $y \neq u \Rightarrow gy \neq gu \Rightarrow p(gy, gu) \neq 0$ (by Lemma 2.10 (i)). We have from (26),

$$\begin{aligned}
&\psi(sp(gy, gu)) = \psi(sp(Sy, Tu)) \leq \beta(\psi(M(y, u)))\psi(M(y, u)) \\
&\text{where } M(y, u) = \max\{p(gy, gu), p(gy, Sy), p(gu, Tu), \frac{1}{2s}[p(gy, Tu) + p(Sy, gu)]\} \\
&\qquad = p(gy, gu) \text{ (by (27) and Lemma 2.10 (i))} \\
&\therefore \psi(sp(gy, gu)) \leq \beta(\psi(M(y, u)))\psi(M(y, u)) \\
&\Rightarrow \psi(sp(gy, gu)) \leq \beta(\psi(p(gy, gu)))\psi(p(gy, gu)) \\
&\Rightarrow \psi(p(gy, gu)) \leq \psi(sp(gy, gu)) < \psi(p(gy, gu)) \text{ if } \psi(p(gy, gu)) > 0, \quad \text{a contradiction.}
\end{aligned}$$

Therefore $\psi(p(gy, gu)) = 0 \Rightarrow p(gy, gu) = 0$. Therefore $gy = gu$. Therefore By (27) and (28) $u = Su = Tu = gu$. Therefore u is a common fixed point of S, T and g in X. □

Now we state and prove our third main result.

Theorem 2.14. *Let $(X, \leq, p)$ be a complete partially ordered partial b - metric space with coefficient $s \geq 1$. Let S, T be a pair of weakly increasing self maps and g be a self mapping on X. S, T are g- non - decreasing. Suppose that g is a pair of weak generalized Geraghty contraction maps S, T, that is there exist $\psi \in \Psi$ and $\beta \in \Omega$ such that $\psi(sp(Sx, Ty) \leq \beta(\psi(M(x, y)))\psi(M(x, y))$ for all $x, y \in X$ whenever gx and gy are comparable, where*

$$M(x, y) = \max\{p(gx, gy), p(gx, Sx), p(gy, Ty), \frac{1}{2s}[p(gx, Ty) + p(Sx, gy)] \tag{29}$$

Assume that

(1). $S(X), T(X) \subseteq g(X)$

(2). S, T and g are compatible

(3). there exists $x_0 \in X$ such that $gx_0 \leq Sx_0$

(4). S, T and g are reciprocally continuous.

Then S, T and g have a coincidence point in X.

Proof. We have by Theorem 2.12, $\{gx_n\}$ is a Cauchy sequence, which is non-decreasing that converges to z (say). Therefore $\lim_{n,m\to\infty} p(gx_n, gx_m)$ exists and is equal to 0. As sequence $\{gx_n\} \to z$ implies $\lim_{n,m\to\infty} p(gx_n, gx_m) = \lim_{n\to\infty} p(gx_n, z) = p(z, z) = 0$.

For n is even

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} gx_{n+1} = z$$
$$\therefore \lim_{n\to\infty} gx_n = \lim_{n\to\infty} Sx_n = z$$

For n is odd

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} gx_{n+1} = z$$
$$\therefore \lim_{n\to\infty} gx_n = \lim_{n\to\infty} Tx_n = z$$

$$\therefore \lim_{n\to\infty} gx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = z$$

Since S, T and g are reciprocally continuous,

$$\lim_{n\to\infty} Sgx_n = Sz \quad \text{and} \quad \lim_{n\to\infty} gSx_n = gz$$

$$\lim_{n\to\infty} Tgx_n = Tz \quad \text{and} \quad \lim_{n\to\infty} gTx_n = gz$$

Also since S, T and g are compatible, $\therefore \lim_{n\to\infty} p(Sgx_n, gSx_n) = 0 = p(Tgx_n, gTx_n)$. Then by Lemma 2.10 (i), we get $Sz = gz$ and $Tz = gz$. Hence z is a coincidence point of S, T and g in X. □

The following corollaries can be established for the Theorems 2.12, 2.13 and 2.14

Corollary 2.15. *Let $(X, \leq, p)$ be a complete partially ordered partial b - metric space with coefficient $s \geq 1$. Let $S, T : X \to X$ be a pair of weakly increasing self maps under ψ weak generalized Geraghty contraction and there exists $x_0 \in X$ such that $x_0 \leq Sx_0$. If any non decreasing sequence $\{x_n\}$ in X converges to u, then we assume that $x_n \leq u\ \forall\ n \geq 0$. Then S, T have a fixed point in X.*

Proof. Follows from the theorem 2.12 by choosing $g = I_x$ □

Corollary 2.16. *Let $(X, \leq, p)$ be a complete partially ordered partial b - metric space with coefficient $s \geq 1$. Let $S, T : X \to X$ be a pair of weakly increasing self maps under weak generalized Geraghty contraction and there exists $x_0 \in X$ such that $x_0 \leq Sx_0$. if any non decreasing sequence $\{x_n\}$ in X converges to u, then we assume that $x_n \leq u\ \forall\ n \geq 0$. Then S, T has a fixed point.*

Proof. Follows from the theorem 2.12 by choosing $g = I_x$ and $\psi(t) = t$. □

Corollary 2.17. *Let $(X, \leq, p)$ be a complete partially ordered partial b - metric space with coefficient $s \geq 1$. Let $S, T : X \to X$ be a pair of weakly increasing self maps under ψ weak generalized Geraghty contraction and there exists $x_0 \in X$ such that $x_0 \leq Sx_0$ and S, T are continuous. Then S, T has a fixed point.*

Proof. Follows from the theorem 2.12 by choosing $g = I_x$. □

Corollary 2.18. *Let $(X, \leq, p)$ be a complete partially ordered partial b - metric space with coefficient $s \geq 1$. Let $S, T : X \to X$ be a pair of weakly increasing self maps under weak generalized Geraghty contraction and there exists $x_0 \in X$ such that $x_0 \leq Sx_0$. and S, T is non decreasing and continuous. Then S, T has a fixed point.*

Proof. Follows from the theorem 2.12 by choosing $g = I_x$ and $\psi(t) = t$. □

Now we give an example in support of Theorem 2.12

Example 2.19. *Let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{10}\}$ with usual ordering. Define*

$$p(x,y) = \begin{cases} 0 \ if \ x = y \\ 1 \ if \ x \neq y \in \{0,1\} \\ |x-y| \ if \ x, y \in \{0, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}\} \\ 4 \ otherwise \end{cases}$$

Clearly, $(X, \leq, p)$ *is a partially ordered partial b - metric space with coefficient* $s = \frac{8}{3}$ *(P.Kumam et.al [16]). Define* $T : X \to X$ *by* $T1 = T\frac{1}{2} = T\frac{1}{3} = T\frac{1}{4} = T\frac{1}{5} = \frac{1}{2}$; $T0 = T\frac{1}{6} = T\frac{1}{7} = T\frac{1}{8} = T\frac{1}{9} = T\frac{1}{10} = \frac{1}{4} \Rightarrow T(X) = \{\frac{1}{2}, \frac{1}{4}\}$. *Define* $S : X \to X$ *by* $S1 = S\frac{1}{2} = S\frac{1}{3} = S\frac{1}{4} = S\frac{1}{5} = S0 = S\frac{1}{6} = S\frac{1}{7} = S\frac{1}{8} = S\frac{1}{9} = S\frac{1}{10} = \frac{1}{2} \Rightarrow S(X) = \{\frac{1}{2}\}$

$$g(x) = \begin{cases} \frac{1}{2n-2} \ if \ 2 \leq n \leq 5 \\ \frac{1}{9} \ if \ 6 \leq n \leq 10 \\ g0 = \frac{1}{9}, g1 = 1 \end{cases}$$

$\Rightarrow g(X) = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{9}\}$. *Therefore* $T(X), S(X) \subset g(X) \subset X$ *and* $g(x) \leq g(y) \Rightarrow T(x) \leq T(y)$ *and* $S(x) \leq S(y)$. *Therefore S, T are g-non decreasing. Define* $\psi : [0, \infty) \to [0, \infty)$ *by* $\psi(t) = \frac{t}{2}$ *and*

$$\beta(t) = \begin{cases} \frac{1}{1+t} \quad if \ \ t \in (0, \infty) \\ 0 \quad if \, t = 0 \end{cases}$$

For $x, y \in X \Rightarrow 0 \leq x = \frac{1}{m} \leq \frac{1}{10}$ *and* $0 \leq y = \frac{1}{n} \leq \frac{1}{10}$, *the following cases can be observed*

(1). For $x = 0, y \in X \Rightarrow \psi(sp(Sx, Ty)) = 0$ *or* $\frac{1}{3} \leq \frac{2}{3} = \beta(\psi(M(x, y)))\psi(M(x, y))$ *where* $M(x, y) = max\{p(gx, gy), p(gx, Sx), p(gy, Ty), \frac{1}{2s}[p(gx, Ty) + p(Sx, gy)]\} = 4$.

(2). For $1 \leq m \leq 5$ *and* $6 \leq n \leq 10$, $\Rightarrow \psi(sp(Sx, Ty)) \leq \frac{1}{3} \leq \frac{2}{3} = \beta(\psi(M(x, y)))\psi(M(x, y))$ *where* $M(x, y) = 4$.

(3). For $6 \leq m \leq 10$ *and* $1 \leq n \leq 5 \Rightarrow \psi(sp(Sx, Ty)) = 0 \leq \frac{2}{3} = \beta(\psi(M(x, y)))\psi(M(x, y))$ *where* $M(x, y) = 4$.

(4). For $6 \leq m \leq 10$ *and* $6 \leq n \leq 10 \Rightarrow \psi(sp(Sx, Ty)) \leq \frac{1}{3} \leq \frac{2}{3} = \beta(\psi(M(x, y)))\psi(M(x, y))$ *where* $M(x, y) = 4$.

$T\frac{1}{2} = \frac{1}{2} = g\frac{1}{2} \Rightarrow Tg\frac{1}{2} = \frac{1}{2} = gT\frac{1}{2} \Rightarrow T$ *and g are weakly compatible at* $\frac{1}{2} \in X$. *Also* $S\frac{1}{2} = \frac{1}{2} = g\frac{1}{2} \Rightarrow Sg\frac{1}{2} = \frac{1}{2} = gS\frac{1}{2} \Rightarrow S$ *and g are weakly compatible at* $\frac{1}{2} \in X$. *Clearly* $g\frac{1}{10} = \frac{1}{9} < \frac{1}{4} = f\frac{1}{10}$. *Let* $x_0 = \frac{1}{10} \Rightarrow gx_0 < Tx_0 = \frac{1}{4} = g\frac{1}{3} = gx_1 \Rightarrow Sx_1 = S\frac{1}{3} = \frac{1}{2} = g\frac{1}{2} = gx_2 \Rightarrow Tx_2 = T\frac{1}{2} = \frac{1}{2} = g\frac{1}{2} = gx_2$. *Therefore* $\frac{1}{2} \in X$ *is a fixed point of T. Also since* $S\frac{1}{2} = \frac{1}{2}$. *Therefore* $\frac{1}{2} \in X$ *is a fixed point of S. Therefore* $\frac{1}{2} \in X$ *is a unique common fixed point of S, T. The hypothesis and conclusions of of Theorem 2.12 satisfied.*

We observe that Theorems 1.12, 1.13 and 1.14 are corollaries of our main results.

Open Problem: Are the Theorems 2.12, 2.13, 2.14 and their corollaries true if continuity of ψ is dropped?.

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References

[1] T.Abdeljawad, *Meir-Keeler α - Contractive fixed and common fixed point theorems*, Fixed Point Theory and Applications, 2013(2013).

[2] H.Aydi, M.Bota, E.Karapinar and S.Mitrovic, *A fixed point theorem for set valued quasi-contractions in b - metric spaces*, Fixed Point Theory and its Applications, 5(2012).

[3] G.V.R.Babu, K.K.M.Sarma and V.A.Kumari, *Common fixed points of $\psi-$ weak generalized geraghty contractions in partially ordered metric spaces*, International Journal of Mathematics and Scientific Computing, 4(2)(2014), 88-93.

[4] I.A.Bakhtin, *The Contraction Principle in quasimetric spaces*, It. Funct. Anal., 30(1989), 26-37.

[5] S.Banach, *Sur les operations dans les ensembles abstraits et leur application oux equations integrals*, Fundamenta Mathematicae, 3(1922), 133-181.

[6] I.Beg and A.R.Butt, *Fixed point for set-valued mappings an implicit relation in partially ordered matrix spaces*, Nonlinear Appl., 71(9)(2009), 3699-3704.

[7] L.Ciric, N.Cakic, M.Rajovic and J.S.Ume, *Monotone generalized nonlinear contractions in partially ordered metric spaces*, Fixed Point Theory and Applications, 2008(2008), Article ID 131294.

[8] S.Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Univ. Ostrav., 1(1993), 511.

[9] M.A.Geraghty, *On contractive maps*, Proc. of Amer. Math. Soc., 40(1973), 604-608.

[10] G.Jungck, *Compatible mappings and common fixed points*, Int. J. Math.and Math. Sci., 9(1986), 771-779.

[11] G.Jungck and B.E.Rhoades, *Fixed point for set valued function without continuity*, Indian J. Pure and Appl. Math., 29(3)(1998), 227-238.

[12] R.Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc., 60(1968), 71-76.

[13] E.Karapinar and R.P.Agarwal, *A note on coupled fixed point theorems for $\alpha-\psi$ Contractive-type mappings in partially ordered metric spaces*, Fixed Point Theory and Applications, 2013(2013).

[14] E.Karapinar and B.Samet, *Generalized $\alpha-\psi$ contractive type mappings and related fixed point theorems with applications*, Abstract and Applied Analysis, (2012), Art. ID 793486.

[15] M.Khan, M.Swaleh and S.Sessa, *Fixed point theorems by altering distances between the points*, Bull. Aust. Math. Soc., 30(1984), 1-9.

[16] P.Kumam, N.V.Dung and V.T.L.Hang, *Some equivalences between cone b-metric spaces and b-metric spaces*, Abstr. Appl. Anal., 2013(2013), 18.

[17] S.G.Matthews, *Partal metric topology Proc. 8th Summer Conference on General Topology and Applications*, Ann. Acad. N.Y., Sci., 728(1994), 183-197.

[18] K.Mehmet and H.Kiziltunc, *On Some Well Known Fixed Point Theorems in b-metric Spaces*, Turkish Journal of Analysis and Number Theory, 1(2013), 13-16.

[19] Mohammad Murasaleen, Syed abdul mohiuddine and Ravi P Agarwal, *Coupled Fixed point theorems for α - ψ contractive type mappings in partially ordered metric spaces*, Fixed Point Theory and Applications 2012(2012).

[20] Z.Mustafa, J.R.Roshan, V.Parveneh and Z.Kadelburg, *Some common fixed point result in ordered partal b-metric spaces*, Journal of Inequalities and Applications, 2013(2013).

[21] H.Nashinea, M.Imdadb and M.Hesanc, *Common fixed point theorems under rational contractions in complex valued metric spaces*, Journal of Nonlinear Sciences and Applications, 7(2014), 42-50.

[22] S.J.ÓNeill, *Partial metrics, valuations and domain theory*, Proc. 11th Summer Conference on General Topology and Applications. Ann. New York Acad. Sci., 806(1996), 304-315.

[23] R.P.Pant, *A common fixed point theorem under a new condition*, Indian J. Pure and Appl. Math., 30(2)(1999), 147-152.

[24] B.Samet, C.Vetro and P.Vetro, *Fixed Point theorems for $\alpha-\psi$ Contractive type mappings*, Nonlinear Analysis, 75(2012), 2154-2165.

[25] K.P.R.Sastry, K.K.M.Sarma, Ch.Srinivasarao and Vedula Perraju, *Coupled Fixed point theorems for α - ψ contractive type mappings in partially ordered partial metric spaces*, International J. of Pure and Engg. Mathematics, 3(I)(2015), 245-262.

[26] K.P.R.Sastry, K.K.M.Sarma, Ch.Srinivasarao and Vedula Perraju, $\alpha-\psi-\varphi$ *contractive mappings in complete partially ordered partial b-metric spaces*, International J. of Math. Sci. and Engg. Appls., 9(2015), 129-146.

[27] W.Shatanawiam and H.Nashineb, *A generalization of Banach's contraction principle for nonlinear contraction in a partial metric space*, Journal of Nonlinear Sciences and Applications, 5(2012), 37-43.

[28] S.Shukla, *Partial b-metric spaces and fixed point theorems*, Mediterranean Journal of Mathematics, (2013), doi:10.1007/s00009-013-0327-4.

[29] S.Singh and B.Chamola, *Quasi-contractions and approximate fixed points*, Natur. J., Phys. Sci., 16(2002), 105-107.

[30] C.Vetroa and F.Vetrob, *Common fIxed points of mappings satisfying implicit relations in partial metric spaces*, Journal of Nonlinear Sciences and Applications, 6(2013), 152-161.

[31] Xian Zhang, *Common fixed point theorems for some new generalised contractive type mappings*, J.Math.Aual.Appl., 333(2007), 2287-301.

[32] H.Yingtaweesittikul, *Suzuki type fixed point for generalized multi-valued mappings in b-metric spaces*, Fixed Point Theory and Applications, 2013(2013).