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Characteristic Properties for the Beta Exponentiated Weibull family

Research Article

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Abstract: The beta exponentiated Weibull family includes the beta Weibull, beta exponential, beta generalized exponential, beta Weibull and beta generalized Rayleigh distributions, among others. We provide a comprehensive treatment of new mathematical and statistical properties of the BEW distribution as a head member of this family of distributions. We derived an explicit mathematical formula for the mode. Impact of new parameters (a,b) on the corresponding first four moments are illustrated. Further, we discuss parametric characterizations for the failure-rate and associate mean residual life function. A general expression for the mean residual life function is derived. Simulation results are reported in several tables and graphs. The usefulness of some results is illustrated by applications to real data.

MSC: 60E05, 62E10, 62F10, 62N02, 93A10.

Keywords: Beta Exponentiated Weibull; Mode, Unimodal, Failure rate, Mean Residual Life Function.

1. Introduction

Many generalized Weibull models have been introduced in several literatures based on the essential relationship between the reliability function and its associate hazard rate function. These generalizations aimed to provide a better fitting of the reliability data sets than the conventional Weibull model. The Weibull distribution is worth generalizing because it is primarily used for modeling product failures in reliability engineering as well as it is sometimes used to model the human aging, see Eakin et al. [8].

In general, many aging distributions aimed to overcome the disability of the traditional distributions including Weibull to model the behavior of many practical lifetime data sets that have non-monotone failure rate (FR) functions: Bathtub and unimodal FR functions. Life times can exhibit increasing FR (IFR) or decreasing FR (DFR). FR functions that first decreases (increases) and then increases (decreases) are called bathtub (upside-down bathtub) shaped, BFR (UFR). The BFR function plays leading role in reliability applications in many fields such as human aging process and electrical devices; See Nadarajah [22] and Bibbington et al. [4].

In recent years, new classes of distributions were proposed based on generalizations or modifications of Weibull distribution. Pham and Lai [24] offered a review of many generalizations and modifications of Weibull distribution. Among these, the exponentiated Weibull (EW) distribution introduced by Mudholkar and Srivastava [18], Generalized Weibull distribution by

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Mudholkar et al. [19, 20], the additive Weibull distribution by Xie and Lai [30], the extended Weibull distribution by Xie et al. [31], the modified Weibull distribution by Lai et al. [15], and the extended flexible Weibull distribution by Bebbington et al. [3].

Other generalizations of Weibull distribution based on beta-G distribution construct increased class of generalized distributions (G is the cdf of a random variable). Among them, the beta exponential (BE) distribution by Nadarajah and Kotz [21], the beta Weibull (BW) distribution by Famoye et al. [9], the beta generalized exponential (BGE) distribution by Souza et al. [27], the beta inverse Weibull (BInW) distribution by Khan [13], the beta Weibull-geometric distribution by Bidram et al. [5] and the beta exponentiated Weibull (BEW) distribution introduced by two works of both Singla et al. [26] and Corderio et al. [7].

These new parametric distributions have flexibility to model both monotonic and non-monotonic failure rates even though the baseline failure may be merely monotonic. They thus lead to a good alternative to several existing conventional lifetime distributions in modeling lifetime data in practice. The BEW distribution has been proposed and studied by Singla et al. [26] and Corderio et al. [7]. The motivation of the work is to derive new characteristic properties for the BEW distribution through studying the effects of the new parameters on the behaviors of: probability density function (pdf), FR function, and MRL function of the BEW family. We provide and discuss some statistical measures of its (pdf). An explicit mathematical formula of the mode is derived. Parametric characterizations for the pdf and failure rate functions are discussed. Mathematical formula for the associate mean residual life function is derived and its behaviors are studied.

The paper is organized as follows. In Section 2, we present the definition of the BEW distribution. In Section 3, statistical properties of the pdf will be discussed. In Sections 4 and 5, parametric characterizations of the FR and the mean residual life functions will be investigated.

2. Distribution and Special Cases

The BEW distribution is considered as a member of a generalized class of distributions called the beta-G family. The cumulative distribution function (cdf) of the BEW random variable X is defined as the logit of the beta random variable and it is given by

$$F(x) = \frac{1}{B(a,b)} \int_0^{F_{EW}(x)} t^a \, (1-t)^{b-1} dt \tag{1}$$

where $F_{EW}(x) = \left(1-e^{-(\lambda x)^\beta}\right)^\alpha$, x>0, for all positive values of the parameters λ, β, α, a and b, and $F_{EW}(x)$ denotes the cdf of the well-known EW distribution. We write $X \sim$BEW$(a,\ b,\ \lambda,\ \beta,\ \alpha)$. When a and b are both integers the $F(x)$ form is the same as the cdf of order statistics of X. The pdf of BEW$(a,\ b,\ \lambda,\ \beta,\ \alpha)$ is

$$f(x) = \frac{\lambda\beta\alpha}{B(a,b)} (\lambda x)^{\beta-1} e^{-(\lambda x)^\beta} \left(1-e^{-(\lambda x)^\beta}\right)^{\alpha a-1} \left[1-\left(1-e^{-(\lambda x)^\beta}\right)^\alpha\right]^{b-1}, \tag{2}$$

$x>0$, for all positive values of $a,\ b,\ \lambda,\ \beta$ and α. The reliability function is

$$R(x) = 1 - F(x).$$

It can be written in terms of the reliability function of the EW distribution as

$$R(x) = \frac{1}{B(a,b)} B_{R_{EW}(x)}(b,a) \tag{3}$$

where $B_y(a,b)$ is the well-known incomplete beta function and $R_{EW}(x) = 1 - F_{EW}(x)$. The BEW family contains a wide class of distributions as special cases: The BW distribution (when $\alpha = 1$), BGE distribution (when $\beta = 1$), BE distribution (when $\alpha = \beta = 1$), EW distribution (when $a = b = 1$), EW distribution with parameters λ, β and αa, generalized exponential (GE) distribution with parameters λ and αa (when $b = \beta = 1$), beta generalized Rayleigh (BGR) distribution (when $\beta = 2$) introduced by Cordeiro et al. [6]. Additionally, the BEW distribution has a desirable physical interpretation in the field of reliability of systems engineering as shown in the following:

If $a = 1$, then $F(x) = 1 - (1 - F_{EW}(x))^b$ and $R(x) = R_{EW}^b(x)$. Thus considering a series system of $b = m$ (an integer) components which are independent EW random variables, the reliability of this system can be modeled as the BEW distribution.

If $b = 1$, then $F(x) = F_{EW}^a(x)$ and $R(x) = 1 - (1 - R_{EW}(x))^a$. Thus we have a parallel system of $a = n$ (an integer) components which are independent EW random variables; the reliability of this system can be modeled as the BEW distribution. Therefore, unlike the EW distribution that represents just a parallel system the BEW is distinct in modeling the reliability of series and parallel systems.

3. Statistical Measures and Curve Behavior

In this section, we provide a comprehensive treatment of mathematical properties of $f(x)$ given by Equation (2). Expressions of the median and mode are derived. The asymptotes of $f(x)$ are obtained. The impact of the new parameters a and b on pdf characterizations is discussed.

3.1. Quantile, Median and Mode

A q-th quantile of the BEW random variable X is the smallest number, x_q, such that $\int_0^{x_q} f(x)\,dx = q$, $0< q<1$. Using $f(x)$ given by equation (2), we get

$$\frac{1}{B(a,b)} \int_0^{\zeta^{\alpha}(x_q)} y^{a-1} (1-y)^{b-1} dy = q$$

where $\zeta(x_q) = 1 - e^{-(\lambda x_q)^{\beta}}$. Thus, the q-th quantile of the BEW distribution is the value of x_q such that

$$I_{\zeta^{\alpha}(x_q)}(a,b) = q \tag{4}$$

where $I_z(a,b) = B_z(a,b)/B(a,b)$ is the regularized incomplete beta function. For $q = 0.5$ (we denote $x_q = x_m$) the median of BEW distribution is obtained by solving the nonlinear equation

$$I_{\zeta^{\alpha}(x_m)}(a,b) = 0.5 \tag{5}$$

The unimodality for the distribution is important to know. When the distribution is unimodal, it can fit a broad range of data sets. In the following, we try to detect a mathematical form of the mode of the BEW distribution. The mode of the density function $f(x)$ given by equation (2) is the high point (maximum) that occurs where the function flattens out or where the slope of the tangent of $f(x)$ curve is zero. The derivative of $f(x)$ tells us, where.

Differentiating $f(x)$ with respect to x, we get $\acute{f}(x) = g(x)\,f(x)$,

$$g(x) = (\beta-1)\,x^{-1} - \lambda\beta(\lambda x)^{\beta-1} + \lambda\beta\,(\alpha a - 1)\,(\lambda x)^{\beta-1} e^{-(\lambda x)^{\beta}} u^{-1} - \lambda\beta\alpha(b-1)(\lambda x)^{\beta-1} e^{-(\lambda x)^{\beta}} u^{-1} u^{\alpha}(1-u^{\alpha})^{-1}, \tag{6}$$

where $u = u(x) = 1 - e^{-(\lambda x)^{\beta}}$, $f(x) > 0$, $x > 0$ and $\acute{f}(x) = 0$ when $g(x) = 0$. We trace the mode from $g(x)$ through the following cases.

(1). $a = b = 1$: We have, from equation (6),

$$(\beta-1)\,x^{-1} - \lambda\beta(\lambda x)^{\beta-1} + \lambda\beta\,(\alpha-1)\,(\lambda x)^{\beta-1}\left(e^{-(\lambda x)^{\beta}}-1\right)^{-1} = 0$$

from which we get the mode, x_d, of the pdf of the EW distribution in the form:

$$x_d = \begin{cases} \frac{1}{\lambda}\left[\frac{2(\beta\alpha-1)}{\beta(\alpha+1)}\right]^{1/\beta}, \beta\alpha > 1 \\ 0, \qquad \beta\alpha \le 1 \end{cases} ;\,See[23].$$

(2). $b = 1,\ \beta = 1$: For any positive values of α and a such that $\alpha a > 1$, we get from equation (6) that $x = \frac{1}{\lambda}\ln\ \alpha a$. In this case, it is clear that the pdf $f(x)$ is increasing function with relatively maximum value, x_d, obtained as follows. Expanding $\ln\ \alpha a$, we get

$$x = \frac{2}{\lambda}\left[\frac{\alpha a-1}{\alpha a+1} + \frac{1}{3}\left(\frac{\alpha a-1}{\alpha a+1}\right)^3 + \frac{1}{5}\left(\frac{\alpha a-1}{\alpha a+1}\right)^5 + \ldots\right].$$

As a first approximation for this series by neglecting the terms with powers greater than one, we get the form of the mode as

$$x_d = \begin{cases} \frac{2}{\lambda}\ \frac{\alpha a-1}{\alpha a+1}, & \alpha a > 1 \\ 0, & \alpha a \le 1 \end{cases}. \tag{7}$$

We note that $0 < \alpha a \le 1$ there is no mode. Thus the pdf $f(x)$ is decreasing when$0 < \alpha a \le 1$.

(3). $b = 1,\ \alpha a > 1$: In this case, equation (6) gives us the form

$$\beta\,(\alpha a-1)\,\frac{(\lambda x)^{\beta}}{e^{(\lambda x)^{\beta}}-1} - \beta(\lambda x)^{\beta} + \beta = 1. \tag{8}$$

We have a series representation of $\frac{y}{e^y-1}$ as $\frac{y}{e^y-1} = 1 - \frac{y}{2} + \sum_{j=1}^{\infty}\frac{B_{2j}\,y^{2j}}{2j},\quad y < 2\pi$; (See [12], page 26). Let $y = (\lambda x)^{\beta}$ in this series, apply in Equation (8) and we just consider the first two terms as a first approximation, we get the mode in the form

$$x_d = \begin{cases} \frac{1}{\lambda}\left[\frac{2(\beta\alpha a-1)}{\beta(\alpha a+1)}\right]^{1/\beta}, & \beta\alpha a > 1 \\ 0, & \beta\alpha a \le 1. \end{cases} \tag{9}$$

(4). $b \ne 1,\ a \ne 1,\ and\ \alpha = 1$ such that $\beta a > 1$:

In this case, equation (6) takes the form

$$(\beta-1)\,x^{-1} - \lambda\beta(\lambda x)^{\beta-1} + \lambda\beta\,(a-1)\,(\lambda x)^{\beta-1}e^{-(\lambda x)^{\beta}}u^{-1} - \lambda\beta\alpha\,(b-1)\,(\lambda x)^{\beta-1}e^{-(\lambda x)^{\beta}}(1-u)^{-1} = 0.$$

After some mathematical manipulations, we can get the form of the mode as

$$x_d = \begin{cases} \frac{1}{\lambda}\left[\frac{2(\beta a-1)}{\beta(a+2b-1)}\right]^{1/\beta}, & \beta a > 1 \\ 0, & \beta a \le 1. \end{cases} \tag{10}$$

It is clear that this form is satisfied for all positive values of the parameter b. From the above four cases, we see that the guarantee of existence the mode of $f(x)$ is $\beta\alpha a > 1$. Thus, for positive real values of b, the mode of the pdf is

$$x_d = \begin{cases} \frac{1}{\lambda}\left[\frac{2(\beta\alpha a-1)}{\beta(\alpha a+2b-1)}\right]^{1/\beta}, & \beta\alpha a > 1 \\ 0, & \beta\alpha a \le 1. \end{cases} \tag{11}$$

Using simulated data, we plot Figures 1a and 1b at different values of the parameters combinations. All figures in this article were created by Mathematica-10 Package.

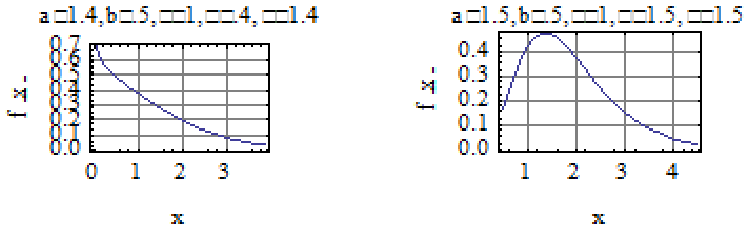

Fig. 1a Plots of pdfs at $\beta\alpha a < 1$ and $\beta\alpha a > 1$, $b = 0.5$

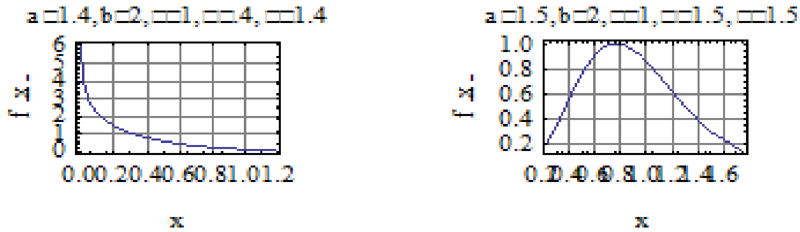

Fig. 1b Plots of pdfs at $\beta\alpha a < 1$ and $\beta\alpha a > 1$, $b = 2$

Therefore, the density function of the BEW distribution is decreasing function when $\beta\alpha a \leq 1$ and unimodal when $\beta\alpha a > 1$. There are three classes of the members of BEW family as follows:

(1). When $\beta\alpha a < 1$ we have monotone decreasing pdf's with unlimited left-tail and right-tail is medium.

(2). When $\beta\alpha a = 1$ we have monotone decreasing pdf's with left-tail ordinate equal to $\lambda\beta\alpha/B(a,b)$ and medium right-tail.

(3). When $\beta\alpha a > 1$ we have unimodal pdf's with short left-tail and medium right-tail. These classes are shown in Figures 2a-2c using simulated data.

From the above discussion, we formulate the following theorem.

Theorem 3.1. *The pdf of BEW*$(a, b, \lambda, \beta, \alpha)$ *is monotonically decreasing function for the region* $\{(\beta, \alpha, a) : \beta\alpha a \leq 1\}$ *and it is unimodal for the region* $\{(\beta, \alpha, a) : \beta\alpha a > 1\}$ *with mode given by Equation* (11).

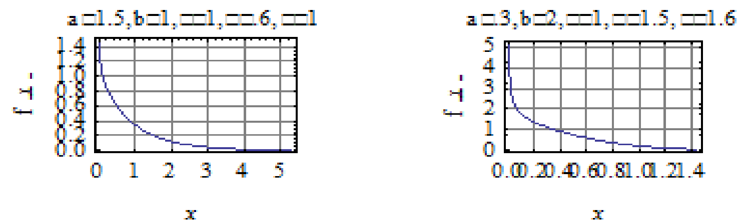

Fig. 2a Plots of pdfs of class (1) of BEW family; $\beta\alpha a < 1$

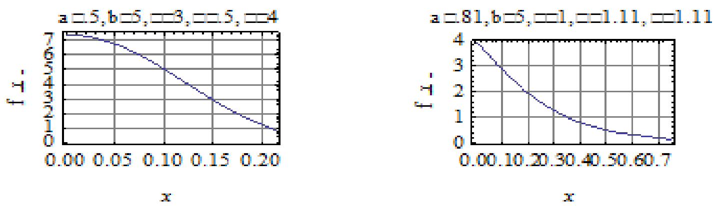

Fig. 2b Plots of pdfs of class (2) of BEW family; $\beta\alpha a = 1$

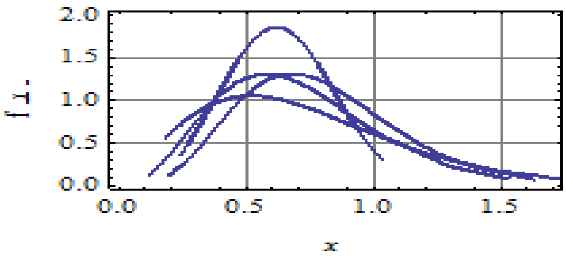

Fig. 2c Plots of pdfs of class (3) of BEW family; $\beta\alpha a > 1$

3.2. Modes of Sub-models

Formula of the mode given by Equation (11) is familiar and easy to use for deriving mode formulas of the new class of Weibull generalizations resulting in beta-G family of distributions. Table 1 offers a brief summary of mathematical formulas of the mode of well-known lifetime distributions, appearing in the reliability engineering literature, which have no known mode formulas. These formulas are special cases of that given by Equation (11).

Distribution	Mode Form	Distribution	Mode Form
BGE	$\frac{2(\alpha a-1)}{\lambda(\alpha a+2b-1)}$, $\alpha a > 1$	GE	$\frac{2(\alpha-1)}{\lambda(\alpha+1)}$, $\alpha > 1$
BW	$\frac{1}{\lambda}\left[\frac{2(\beta a-1)}{\beta(a+2b-1)}\right]^{1/\beta}$, $\beta a > 1$	EW	$\frac{1}{\lambda}\left[\frac{2(\beta\alpha-1)}{\beta(\alpha+1)}\right]^{1/\beta}$, $\beta\alpha > 1$
BE	$\frac{2(a-1)}{\lambda(a+2b-1)}$, $a > 1$	GR	$\frac{1}{\lambda}\sqrt{\frac{2\alpha-1}{\alpha+1}}$, $\alpha > 0.5$
BGR	$\frac{1}{\lambda}\sqrt{\frac{2\alpha a-1}{\alpha a+2b-1}}$, $\alpha a > 1$	W	$\frac{1}{\lambda}(1-\frac{1}{\beta})^{1/\beta}$, $\beta > 1$

Table 1. **Table 1: The mode forms of some sub-models**

Remark 3.2. *The k^{th} moment of the BEW distribution is in terms of improper integration. Instead, we can obtain the k^{th} moment in terms of definite integration as:*

$$E\left(X^k\right) = \frac{\alpha}{\lambda^k B(a,b)} \int_0^1 (lnv^{-1})^{k/\beta}(1-v)^{\alpha a-1}(1-(1-v)^{\alpha})^{b-1}dv \quad \text{(R.1)}$$

$$or\ \ E\left(X^k\right) = \frac{1}{\lambda^k B(a,b)} \int_0^1 [ln\left(1-v^{1/\alpha}\right)^{-1}]^{k/\beta} v^{a-1}(1-v)^{b-1}dv, \quad \text{(R.2)}$$

for $k = 1, 2, 3, \ldots$. For $\beta = \alpha = 1$, it is easy to obtain the k^{th} moment in the case of beta exponential distribution at $a = 1$ as

$$E\left(X^k\right) = \frac{k}{\lambda\, b}\, E\left(X^{k-1}\right) \quad \text{(R.3)}$$

$$or\ \ E\left(X^k\right) = \frac{k!}{(\lambda b)^k}. \quad \text{(R.4)}$$

3.3. Asymptotic Behavior of pdf Curve

The equivalence of $f(x)$ or the asymptote of the pdf in Equation (2) as $x \to 0$ is given by $f(x) \sim \beta\alpha\lambda^{\beta\alpha a}\ x^{\beta\alpha a-1}/B(a,b)$ and the equivalence as $x \to \infty$ is given by $f(x) \sim \beta\alpha^b\lambda^\beta\ \ x^{\beta-1}e^{-b(\lambda x)^\beta}/B(a,b)$. Therefore, in terms of the left and right tails of the pdf $f(x)$ we have the following lemma.

Lemma 3.3. *The limit of BEW density function as $x \to \infty$ is 0 and its limit as $x \to 0$ is given by*

$$\lim_{x\to 0} f(x) = \begin{cases} 0, & for\ \beta\alpha a > 1 \\ \frac{\lambda\beta\alpha}{B(a,b)}, & for\ \beta\alpha a = 1 \\ \infty, & for\ \beta\alpha a < 1. \end{cases} \tag{12}$$

Proof. By using Taylor's expansion for the terms of $f(x)$ given by Equation (2) and taking the limit as $x \to 0$ and as $x \to \infty$, we get the results in Equation (12). The effect of $\beta\alpha a$ on the shape of the pdf $f(x)$ for BEW distribution is similar to the effect of $\beta\alpha$ for the EW distribution case. Further, we have the following shape properties of $f(x)$ given by equation (2).

1. *Case of* $\beta\alpha a \leq 1$: The larger values of b the larger the length of the upper tail while decay is trait for the lower tail and conversely for larger values of a. Figures 3a-3c support these results based on simulated data.

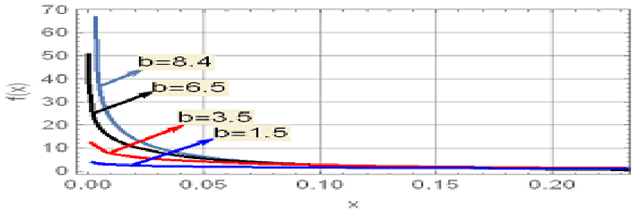

Fig. 3a Effects of increasing values of b on pdf curve; $\beta\alpha a < 1$

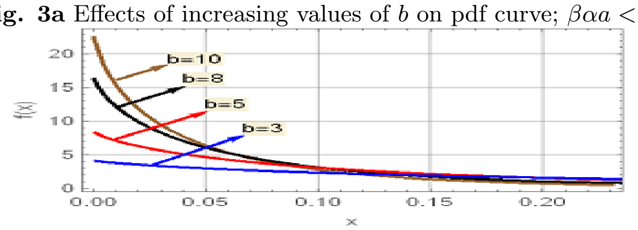

Fig. 3b Effects of increasing values of b on pdf curve; $\beta\alpha a = 1$

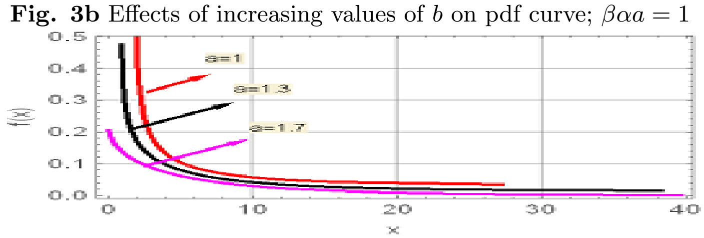

Fig. 3c Effects of increasing values of a on pdf curve; $\beta\alpha a < 1$

2. *Case of* $\beta\alpha a > 1$: When $\beta\alpha a > 1$ the BEW distribution is unimodal; the case in which the distribution can fit a wide range of positive data sets. We provide Table 2 that displays numerical illustrations for various shapes of BEW distribution through computed values for some statistical measures. For BEW$(a, b, \lambda, \beta, \alpha)$ at $\lambda = 1$, $\beta = 6.326$, $\alpha = 2$ and for various values of a and b, we generate various data sets and compute the mean, μ, and standard deviation, σ. The mode, x_d, and median, x_m, are computed using Equations (11) and (4), respectively. On the basis of quantile functions we compute the skewness, S, as defined by Galton (1983) and kurtosis, K, as defined by Moors (1988); that is

$$S = \frac{Q(6/8) - 2Q(4/8) + Q(2/8)}{Q(6/8) - Q(2/8)}, \quad K = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)}$$

where $Q(q) = x_q$ is obtained from equation (4). The values of Galton's skewness and Moors' kurtosis are presented in Table 3.

In view of Table 2, we have the following observations:

The larger the values of b the higher the peak of the distribution. The curve moves to the left; Viz, the mean, median, and mode are decreasing as increasing b values. Figure 4 show this effect of b. The larger the values of a the higher the peak of the distribution. The curve moves to the right; Viz, the mean, median, and mode are increasing as increasing a values. Figure 5 show this effect of a. The larger the values of both a and b the higher the peak of the distribution. The curve remains at the same location when a and b values are the same and the BEW distribution curve tends to resemble the normal distribution curve. The mean, median, and mode are approximately equal when $a = b$. See Figure 6. The values of the mean increases as a increases while it decreases as b increases. The standard deviation decreases as either a or b increases.

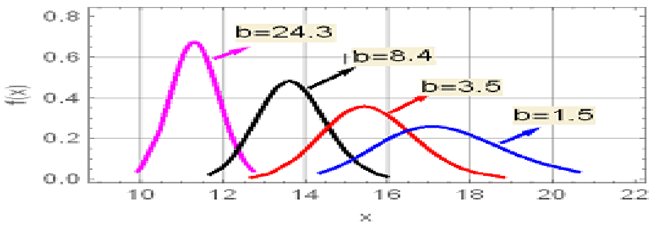

Fig. 4 Effects of the larger values of b on pdf curve; $\beta\alpha a > 1$

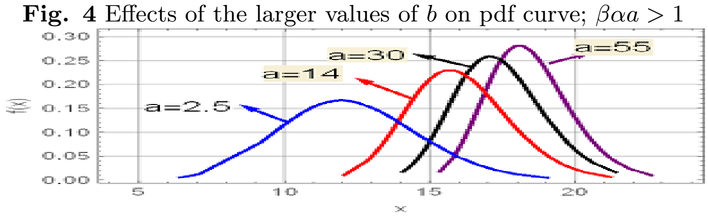

Fig. 5 Effects of the larger values of a on pdf curve; $\beta\alpha a > 1$

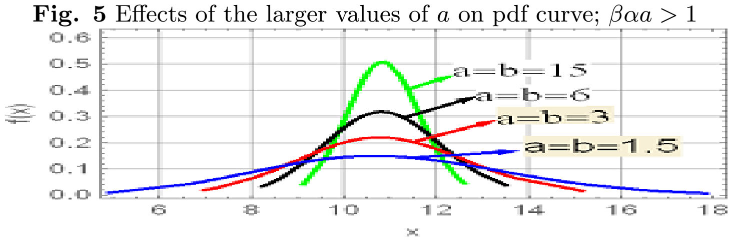

Fig. 6 Effects of the larger values of both a and b $(a = b)$ on pdf curve; $\beta\alpha a > 1$

b	a	x_m	x_d	μ	σ	b	a	x_m	x_d	μ	σ
0.5	0.5	1.0330	1.0859	1.0196	0.1970	7	0.5	0.7759	0.7155	0.7607	0.1274
	0.8	1.0932	1.0976	1.0858	0.1632		0.8	0.8299	0.7738	0.8202	0.1015
	1	1.1167	1.1014	1.1117	0.1502		1	0.8534	0.8009	0.8456	0.0916
	7	1.2495	1.1138	1.2550	0.0893		7	1.0330	1.0040	1.0322	0.0438
	10	1.2665	1.1144	1.2726	0.0836		10	1.0624	1.0296	1.0620	0.0391
	25	1.3049	1.1152	1.3119	0.0722		25	1.1313	1.0752	1.1314	0.0299
0.8	0.5	0.9699	1.0081	0.9568	0.1792	10	0.5	0.7519	0.6763	0.7367	0.1221
	0.8	1.0330	1.0437	1.0252	0.1476		0.8	0.8039	0.7329	0.7940	0.0968
	1	1.0585	1.0566	1.0528	0.1353		1	0.8265	0.7595	0.8186	0.0871
	7	1.2096	1.1064	1.2128	0.0756		7	1.0029	0.9726	1.0021	0.0409
	10	1.2292	1.1092	1.2329	0.0700		10	1.0330	1.0028	1.0325	0.0364
	25	1.2730	1.1131	1.2774	0.0590		25	1.1051	1.0600	1.1050	0.0275
1	0.5	0.9437	0.9732	0.9304	0.1717	25	0.5	0.6955	0.5851	0.6805	0.1105
	0.8	1.0070	1.0165	0.9990	0.1410		0.8	0.7426	0.6358	0.7329	0.0867
	1	1.0330	1.0330	1.0270	0.1290		1	0.7631	0.6601	0.7553	0.0776
	7	1.1915	1.1017	1.1938	0.0704		7	0.9269	0.8781	0.9260	0.0348
	10	1.2123	1.1058	1.2150	0.0648		10	0.9568	0.9163	0.9562	0.0308
	25	1.2586	1.1118	1.2621	0.0539		25	1.0330	1.0011	1.0328	0.0228

Table 2. **Statistical measures for different values of a and b.**

In view of Table 3, we record the following observations:

When $b < a$ the distribution is positive skewed. When $b > a$ the distribution is negative skewed. For fixed b, the skewness increases as a increases. For fixed a, the skewness decreases as b increases. For fixed $b \leq 1$, the kurtosis is an increasing function of a. For fixed $b > 1$, the kurtosis fluctuates; viz, it is an increasing function of $a \leq 1$ while it is a decreasing function of $a > 1$. For fixed $a \leq 1$, the kurtosis is an increasing function of b. For fixed $a > 1$, the kurtosis fluctuates; viz, it is an increasing function of $b \leq 1$ while it is a decreasing function of $b > 1$.

b	a	S	K	b	a	S	K
0.5	0.5	-0.0501	1.2182	7	0.5	-0.0904	1.2471
	0.8	-0.0319	1.2281		0.8	-0.0690	1.2479
	1	-0.0219	1.2302		1	-0.0596	1.2461
	7	0.0471	1.2351		7	-0.0111	1.2348
	10	0.0555	1.2364		10	-0.0064	1.2345
	25	0.0730	1.2403		25	0.0033	1.2342
0.8	0.5	-0.0534	1.2293	10	0.5	-0.0946	1.2481
	0.8	-0.0371	1.2343		0.8	-0.0730	1.2491
	1	-0.0288	1.2352		1	-0.0634	1.2471
	7	0.0306	1.2383		7	-0.0138	1.2349
	10	0.0382	1.2394		10	-0.0092	1.2344
	25	0.0541	1.2424		25	-0.0001	1.2339
1	0.5	-0.0569	1.2334	25	0.5	-0.1030	1.2498
	0.8	-0.0403	1.2366		0.8	-0.0812	1.2512
	1	-0.0323	1.2369		1	-0.0713	1.2492
	7	0.0233	1.2382		7	-0.0190	1.2352
	10	0.0306	1.2390		10	-0.0144	1.2345
	25	0.0457	1.2415		25	-0.0058	1.2336

Table 3. **Skewness and kurtosis for different values of a and b.**

Other values of a and b were also considered and the results are not reported here since they have similar patterns to the cases listed in Tables 2 and 3.

For any fixed values of both β and α with any values of both a and b such that $a = b$, skewness decreases as a and b increase while kurtosis slightly fluctuates. For example for values of $\beta = 3.5$, $\alpha = 2.5$ and $\lambda = 1$, see Table 4.

$(b,\ a)$	(0.3,0.3)	(0.8,0.8)	(1,1)	(7,7)	(10,10)	(30,30)	(60,60)	(120,120)
S	0.0138	0.0068	0.0058	0.0019	0.0016	0.0009	0.0006	0.0004
K	1.1484	1.2265	1.2310	1.2342	1.2339	1.2334	1.2332	1.2331

Table 4. **Skewness and kurtosis for different values of a and b $(a = b)$**

□

4. Failure Rate Function

The failure (hazard) rate function is a measure of sudden or immediate failure. The failure rate FR is important to distinguish between features of different lifetime distributions. By definition, the FR of BEW distribution is given by

$$h\,(x) = f\,(x)/R(x) \tag{13}$$

where $f\,(x)$ and $R(x)$ are given by Equations (2) and (3). Therefore, we have

$$h\,(x) = \frac{\beta\alpha\lambda\ (\lambda x)^{\beta-1} e^{-(\lambda x)^{\beta}}\ u^{\alpha a-1}(1-u^{\alpha})^{b-1}}{B_{R_{EW}(x)}(b,a)}. \tag{14}$$

where $u = u(x)$ is given in Equation (6). Equation (13) can be expressed in the form

$$h\left(x\right)=\varphi\left(x\right)h_{EW}(x) \tag{15}$$

where $h_{EW}\left(x\right)=f_{EW}\left(x\right)/R_{EW}\left(x\right)$, is the failure rate function of the EW distribution and $\varphi\left(x\right)$ is given by

$$\varphi\left(x\right)=\frac{F_{EW}^{a-1}\left(x\right)\ .\ R_{EW}^{b}(x)}{B_{R_{EW}(x)}(b,a)}. \tag{16}$$

We note that when $a=b=1$ we have $B_{R_{EW}(x)}(b,a)\rightarrow R_{EW}\left(x\right)$, $\varphi\left(x\right)=1$ and then $h\left(x\right)=h_{EW}(x)$.

4.1. Asymptotic Characterizations of FR Function

Approximations of FR function in view of Equation (15) are as follows:

(1). x is small: We have $F_{EW}\left(x\right)=F_{W}^{\alpha}(x)$, $F_{W}\left(x\right)=1-e^{-(\lambda x)^{\beta}}$. Using Taylor's series expansion that is useful locally (for x near zero) we see that $F_{W}\left(x\right)$ approaches to $(\lambda x)^{\beta}$. Thus, $F_{W}\left(x\right)\approx(\lambda x)^{\beta}$, for small x and

$$F_{EW}^{a-1}\left(x\right)\approx(\lambda x)^{\beta\alpha(a-1)}. \tag{17}$$

Also, we have the reliability function $R_{EW}\left(x\right)$ that approaches to one at the beginning of the start of operation of a unit (system); That is,

$$R_{EW}\left(x\right)\rightarrow1,\ for\ small\ x.$$

Thus, $B_{R_{EW}(x)}(b,a)\approx B(b,a)$ and

$$\varphi\left(x\right)=\frac{(\lambda x)^{\beta\alpha(a-1)}}{B(a,b)}. \tag{18}$$

For small x, we have also $h_{EW}\left(x\right)\approx\beta\alpha\ \lambda^{\beta\alpha}\ x^{\beta\alpha-1}$, see Jiang and Murthy [14]. Using this and equation (18), $h\left(x\right)$ can be approximated as

$$h\left(x\right)\approx\frac{\beta\alpha\lambda^{\beta\alpha a}}{B(b,a)}\ \ x^{\beta\alpha a-1}. \tag{19}$$

We note that, for small x, we have two cases considered as special cases of the result obtained in Equation (19): (i) For $a=b=1$, we get the approximation of the FR function of EW distribution as special case. (ii) For $\beta=\alpha=1$, Equation (19) is reduced to the result obtained by Nadarajah and Kotz [21] for the case of BE distribution.

(2). x is large: Considering the asymptotic case, $x\rightarrow\infty$, we have $F_{EW}\left(x\right)$ approaches to 1 and using L'Hospital rule with some mathematical manipulations, we find that $\varphi\left(x\right)$ approaches to b; See the appendix. Using this in Equation (15), we get for large x,

$$h\left(x\right)\approx b\ h_{EW}(x). \tag{20}$$

4.2. Parametric Characterizations of FR Function

From the results of sub-Section 4.1, we formulate the following theorem.

Theorem 4.1.

(1). The limit of BEW failure-rate function as $x\rightarrow0$ is

$$\lim_{x\rightarrow0}h\left(x\right)=\begin{cases}\infty, & when\ \beta\alpha a<1;\\ \frac{\beta\alpha\lambda\ \Gamma(a+b)}{\Gamma(a)\Gamma(b)}, & when\ \beta\alpha a=1;\\ 0, & when\ \beta\alpha a>1.\end{cases} \tag{21}$$

(2). The limit of BEW failure-rate function as $x \to \infty$ is

$$\lim_{x\to\infty} h(x) = \begin{cases} \infty, & when\ \beta > 1; \\ \lambda b, & when\ \beta = 1; \\ 0, & when\ \beta < 1. \end{cases} \quad . \tag{22}$$

Proof.

(1). The limit of $h(x)$ when $x \to 0$ is the same as the limit of $f(x)$ given by equation (12).

(2). We have from equation (20) that, as $x \to \infty$, $h_{EW}(x) \approx h_W(x)$, see Jiang and Murthy [14], $h_W(x)$ is the FR of traditional Weibull distribution. Thus, $h(x) = \lambda b\beta(\lambda x)^{\beta-1}$ which gives the results shown in equation (22) as $x \to \infty$. Further, it is not difficult to get the results in (ii) when we discuss the limit:

$$\lim_{x\to\infty} [\beta\lambda(\lambda x)^{\beta-1} - (\beta-1)x^{-1} - \beta\lambda(\alpha a-1)(\lambda x)^{\beta-1}.e^{-(\lambda x)^{\beta}}.u^{-1}$$

$$+\beta\alpha\lambda(b-1)(\lambda x)^{\beta-1}.e^{-(\lambda x)^{\beta}}.u^{-1}u^{\alpha}(1-u^{\alpha})^{-1}], u = u(x) = 1 - e^{-(\lambda x)^{\beta}}.$$

□

Theorem 4.2. *The FR of the BEW distribution is*

(1). constant (=λb) when $\beta = \alpha = a = 1$,

(2). increasing when $\beta \geq 1$ and $\beta\alpha a \geq 1$,

(3). decreasing when $\beta \leq 1$ and $\beta\alpha a \leq 1$,

(4). bathtub shaped when $\beta > 1$ and $\beta\alpha a < 1$, and

(5). upside-down bathtub shaped (unimodal) when $\beta < 1$ and $\beta\alpha a > 1$.

Proof. It follows from Theorem 4.1. □

For simulated data sets, various shapes for the FR of BEW distribution are shown in Figures 7a-7d.

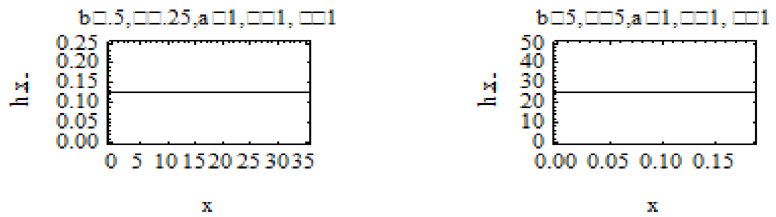

Fig. 7a Constant FR functions

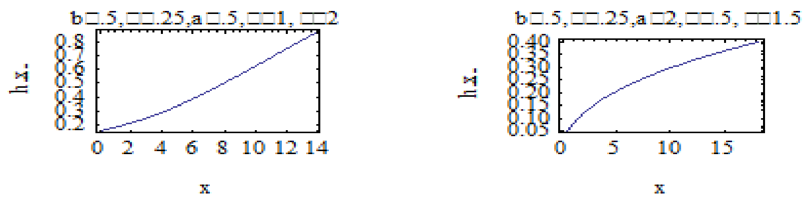

Fig. 7b Increasing FR functions

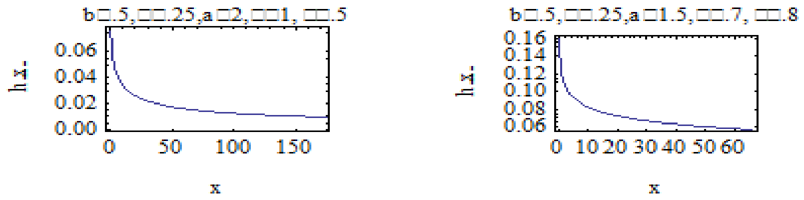

Fig. 7c Decreasing FR functions

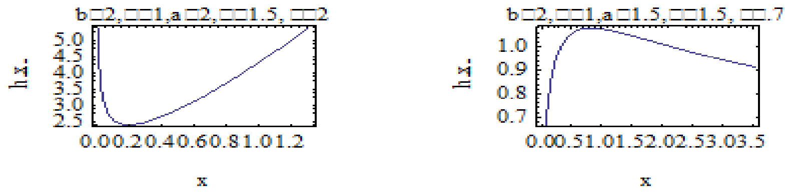

Fig. 7d BFR and UFR functions

5. Mean Residual Life Function

The mean residual life function (MRL) is an important characteristic in survival and reliability analysis. The FR function focuses only on the risk of immediate failure while the MRL function deals with the precise of the whole residual life distribution of a system. In the studies of survival analysis, the MRL function is more pertinent than the FR function. Literatures proofed that the lifetimes is of increasing MRL (IMRL), decreasing MRL (DMRL), Bathtub-shaped MRL (BMRL), or unimodal MRL (UMRL) function. The MRL function is the expected life after the present time age x. Mathematically, it is defined as

$$M\left(x\right)=\frac{1}{R(x)}\int_{x}^{\infty}R(t)dt.$$

Theorem 5.1. *For a BEW random variable, X, with finite mean μ, the MRL function is given by*

$$M\left(x\right)=\frac{\mu-g(x)}{R(x)}-x \tag{23}$$

or

$$M\left(x\right)=\frac{\mu(x)}{R(x)}-x \tag{24}$$

where

$$g(x)=\frac{\alpha}{\lambda B(a,b)}\int_{e^{-(\lambda x)^{\beta}}}^{1}\left(lnv^{-1}\right)^{1/\beta}(1-v)^{\alpha a-1}(1-(1-v)^{\alpha})^{b-1}dv$$

and

$$\mu(x)=\frac{\alpha}{\lambda B(a,b)}\int_{0}^{e^{-(\lambda x)^{\beta}}}\left(lnv^{-1}\right)^{1/\beta}(1-v)^{\alpha a-1}(1-(1-v)^{\alpha})^{b-1}dv$$

is the truncated mean. $R(x)$ is the reliability function.

Proof. We have $M\left(x\right)=E\left(T-x\mid T>x\right)=\frac{1}{R(x)}\int_{x}^{\infty}\left(t-x\right)f(t)dt,\ x\geq 0$ that can be written as

$$M\left(x\right)=J/R(x) \tag{25}$$

where $J=I_1-I_2$,

$$I_1=\int_{0}^{\infty}\left(t-x\right)f\left(t\right)dt=\mu-x,$$
$$I_2=\int_{0}^{x}\left(t-x\right)f\left(t\right)dt=g(x)-xF(x),\quad F(x)\quad \text{is the cdf of X.}$$

Substitution $f(t)$ given by (2) and using the transformation $v = e^{-(\lambda x)^{\beta}}$, we get $g(x) = \frac{\alpha}{\lambda B(a,b)} \int_{e^{-(\lambda x)^{\beta}}}^{1} (lnv^{-1})^{1/\beta} (1-v)^{\alpha a-1} (1-(1-v)^{\alpha})^{b-1} dv$. Thus, $J = \mu - x + xF(x) - g(x) = \mu - g(x) - xR(x)$. Substitution the result of J in equation (25), this compete the proof. □

The proof of the other form in equation (24) can be obtained in similar manner. The behaviors of the MRL function with respect to associate failure rates were characterized by several studies such as Mi [16], Ghai and Mi [10] and Tang et al. [28]. The results of these studies are summarized in the following theorem.

Theorem 5.2. *For a nonnegative random variable X with pdf f(x), finite mean μ, and differentiable FR function h(x), the MRL function is*

(1). constant=μ if X has an exponential distribution.

(2). DMRL (IMRL) if h(x) is IFR (DFR).

(3). BMRL (UMRL) if h(x) is UFR (BFR).

The MRL function given by Equation (23) satisfies these results as shown in Figures 8a-8e. This figure displays some curves of FR function and associate curves of MRL function for certain values of the parameters β, α and a.

Now, we characterize the behavior of the MRL function of the BEW distribution with respect to its failure rate and depending on the shape parameters β, α and a. The following theorem shows parametric characterizations for the MRL function.

Theorem 5.3. *The MRL function given by Equation (23), of the BEW distribution is characterized by the following:*

(1). $M(x) = 1/\lambda b$ iff $\beta = \alpha = a = 1$.

(2). M(x) is IMRL (DMRL) if $\beta \le 1$ and $\beta\alpha a \le 1$ ($\beta \ge 1$ and $\beta\alpha a \ge 1$).

(3). M(x) is BMRL (UMRL) if $\beta < 1$ and $\beta\alpha a > 1$ ($\beta > 1$ and $\beta\alpha a < 1$).

Proof. The proof is straightforward by using the results of Theorem 4.2 and Theorem 5.2. Figures 8a-8e also display some curves for MRL function for combinations of some values of β, α and a.

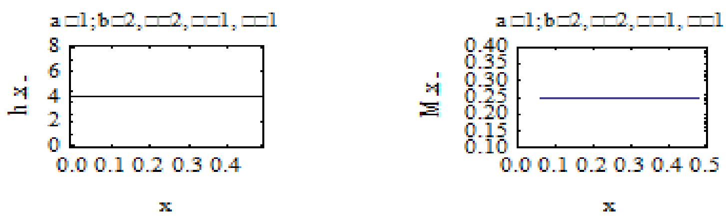

Fig. 8a Constant FR (value of λb) and associate MRL (value of $1/\lambda b = \mu$)

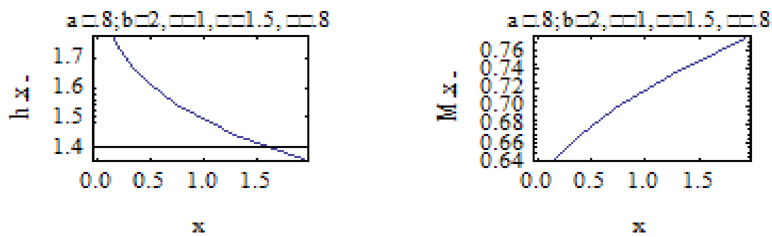

Fig. 8b DFR and associate IMRL

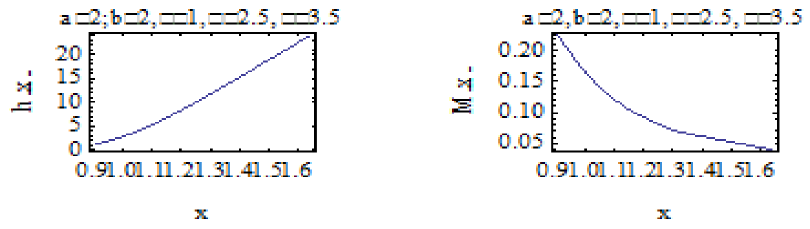

Fig. 8c IFR and associate DMRL

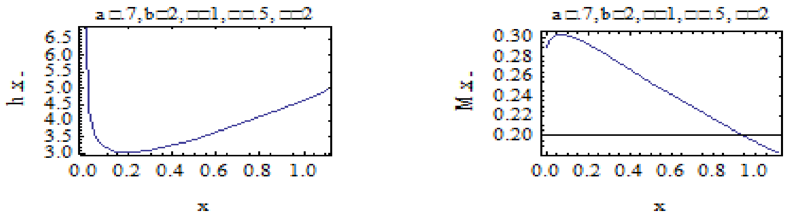

Fig. 8d BFR and associate UMRL

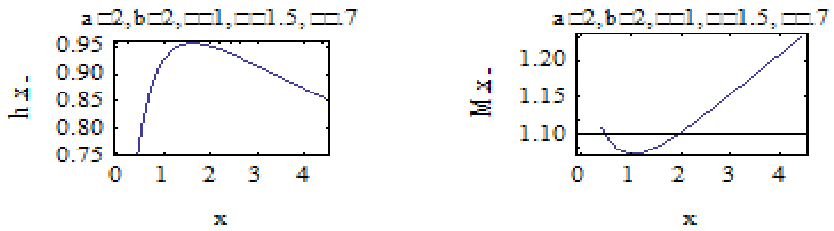

Fig. 8e UFR and associate BMRL

□

6. Application to Real Data

In this section, we use two real data sets with various shapes of failure rate and mean residual life functions to validate the theoretical results obtained in Section 4 and 5. Data set 1: We consider data set from Aarset [1], on lifetimes of 50 components. Many authors such as Mudholkar et al. [20], Wang [29], Singla et al. [26] and others studied this data set. The data set is :

0.10, 0.20, 1,1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86.

Data set 2: Quesenberry and Kent [25] previously studied this data set. The data represents a sample of failure times of 100 centimeter yarn at 2.3% strain level and it resulted in a textile experiment for testing the tensile fatigue features of yarn. The data is:

86, 146, 251, 653, 98, 249,400, 292, 131, 169, 175, 176, 76, 264, 15, 364, 195, 262, 88, 264, 157, 220, 42, 321, 180, 198, 38, 20, 61, 121, 282, 224, 149, 180, 325, 250, 196,90, 229, 166, 38, 337, 65, 151, 341, 40, 40, 135, 597, 246, 211, 180, 93, 315, 353, 571, 124, 279, 81, 186, 497, 182, 423, 185, 229, 400, 338, 290, 398, 71, 246, 185, 188, 568, 55, 55, 61, 244, 20, 284, 393, 396, 203, 829, 239, 236, 286, 194, 277, 143, 198, 264, 105, 203, 124, 137, 135, 350, 193, 188.

The two data sets are fitted to the BEW and EW distributions. The maximum likelihood estimates, the maximized log-likelihood value $(\widehat{\ell})$, Akaike Information Criterion (AIC), The Bayesian Information Criterion (BIC), the Anderson-Darling (A-D) and Pearson- χ^2 (P- χ^2) test statistics for the fitted distributions to the two data sets are reported in Table 7.

The maximum likelihood estimates for the BEW and EW parameters are obtained by maximizing the log-likelihood function,

ℓ, of $f(x)$, given in equation (2), with respect to the parameters and

$$\ell = \ell\left(a, b, \beta, \alpha, \lambda; x\right) = n\log\beta + n\log\alpha - nlog\left[B\left(a, b\right)\right] - T$$

where $T = \sum_{i=1}^{n} \left(\lambda\ x_i\right)^{\beta} - \left(\beta - 1\right)\log\left(\lambda x_i\right)\ - \left(\alpha a - 1\right)\log u_i$, $u_i = 1 - e^{-(\lambda x_i)^{\beta}}$.

The estimated values of the parameters are obtained by solving the system of estimating equations:

$$n\psi\left(a+b\right) - n\ \psi\left(a\right) + \alpha\sum_{i=1}^{n}\log u_i\ = 0,$$

$$n\ \psi\left(a+b\right) - n\ \psi\left(b\right) + \alpha\sum_{i=1}^{n}\log(1 - {u_i}^{\alpha})\ = 0,$$

$$\frac{n}{\alpha} + a\sum_{i=1}^{n}\log u_i\ - \left(b-1\right)\sum_{i=1}^{n}\left(u_i^{-\alpha} - 1\right)^{-1}\log u_i\ = 0,$$

$$\frac{n}{\beta} + n\log\lambda\ - \sum_{i=1}^{n}\left(\lambda x_i\right)^{\beta}\log\left(\lambda x_i\right)\ + \sum_{i=1}^{n}\log x_i\ + \left(\alpha a - 1\right)\sum_{i=1}^{n}\zeta_i\log\left(\lambda x_i\right) + \left(b-1\right)\alpha\sum_{i=1}^{n}\zeta_i\left(u_i^{-\alpha} - 1\right)^{-1}\log\left(\lambda x_i\right)\ = 0,$$

$$\frac{\beta}{\lambda}[n - \sum_{i=1}^{n}\left(\lambda x_i\right)^{\beta} + \left(\alpha a - 1\right)\sum_{i=1}^{n}\zeta_i - \left(b-1\right)\alpha\sum_{i=1}^{n}\zeta_i\left(u_i^{-\alpha} - 1\right)^{-1}] = 0,$$

where $\zeta_i = u_i^{-1}(\lambda x_i)^{\beta} e^{-(\lambda x_i)^{\beta}}$.

In case of EW distribution the system of estimating equations is reduced to three equations by taking $a = b = 1$. For the two data sets, the results are reported in Tables 5 and 6. The results show that the BEW model produces a better fit than the EW model particularly for Aarset data and consequently the extra shape parameters a and b are needful.

Parameter estimates for Aarset data					
Distribution	$\widehat{\beta}$	$\widehat{\alpha}$	$\widehat{\lambda}$	$\hat{a}$	$\hat{b}$
BEW	5.65	0.15	0.015	0.55	0.33
EW	4.76	0.138	0.014	1	1
Parameter estimates for data of yarn					
Distribution	$\widehat{\beta}$	$\widehat{\alpha}$	$\widehat{\lambda}$	$\hat{a}$	$\hat{b}$
BEW	1.04	1.50	0.0223	2.50	0.50
EW	1.50	1.00	0.0040	1	1

Table 5. **MLEs of the model parameters for the data sets**

Information criteria and statistics for the fitted distributions to the Aarset data							
Distribution	$\widehat{\ell}$	AIC	BIC	A-D	p-value	P- χ^2	p-value
BEW	-220.909	451.819	461.379	4.6302	0.1302	38.02	0.0310
EW	-238.184	482.368	488.105	3.2345	0.0210	44.00	0.0000
Information criteria and statistics for the fitted distributions to the data of yarn							
BEW	-625.361	1264.280	1270.501	2.5207	0.0484	14.66	0.2606
EW	-629.924	1265.850	1273.660	1.4020	0.2014	16.48	0.1702

Table 6. **Information criteria and test statistics**

It is well known that the TTT plot (Barlow and Campo [2]) is used as a tool for identification of failure rate models. The TTT plot depicts the behavior of FR function of Aarset data to be BFR. The results in Theorem 3 identify the same behavior in terms of estimated values of the parameters. Based on the Aarset data set: $\beta\alpha a\ (= 0.466) < 1$ and $\beta\,(= 5.65) > 1$ reveal that the BEW model has BFR and UMRL function. Based on the data of polyester yarn: $\beta\alpha a\ (= 3.9) > 1$ and $\beta\,(= 1.04) > 1$ reveal that the BEW model has IFR and DMRL function. Figures 9a and 9b show the FR function and associate MRL function for both data sets.

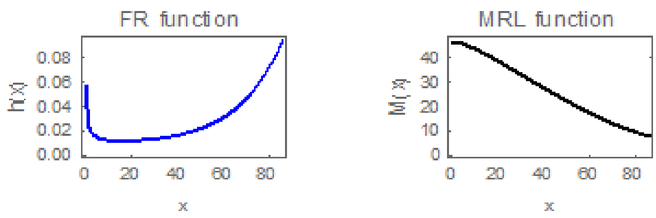

Fig. 9a The FR and MRL functions for the Aarset data

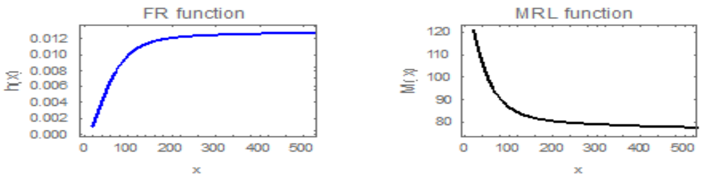

Fig. 9b The FR and MRL functions for the data of yarn

7. Conclusion

The BEW family includes most of the commonly used distributions in the lifetime literature. We provide a mathematical treatment of the properties of the BEW distribution including the density function, the failure rate function, and the mean residual function. The properties studied include the following: For pdf, deriving an explicit formula for the mode, which is an important key to understand many other behaviors of pdf of BEW family, discussing the asymptotic behaviors and the effects of the new parameters on the mean, median, mode, standard deviation, skewness and kurtosis based on extensive simulation results. For FR function, discussing asymptotic characterizations and determining the behaviors of the FR function in terms of the new parameters with combinations of the original parameters. For MRL function, deriving a closed form for the mean residual life function and constructing theorems that are determining the relations between MRL function and associated FR function in terms of the new parameters with combinations of the originals.

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Appendix

We have to proof that $\varphi(x) \to b$ as $x \to \infty$. As x is large; that is $x \to \infty$ we have $F_{EW}(x) \to 1$ and we have to obtain

$$\lim_{x\to\infty} \varphi(x) = \lim_{x\to\infty} \frac{[1-F_{EW}(x)]^b}{B_{R_{EW}(x)}(b,a)}. \tag{A.1}$$

Using L'Hospital rule, we must differentiate the numerator and denominator as follows:

$$d[1-F_{EW}(x)]^b/dx = b[1-F_{EW}(x)]^{b-1}(-f(x)). \tag{A.2}$$

To differentiate the function $B_{R_{EW}(x)}(b,a)$ we use the following rule:

$$\frac{d}{dt}\int_{\psi(t)}^{\varphi(t)} f(u)\,du = f(\varphi(t))\frac{d\varphi(t)}{dt} - f(\psi(t))\frac{d\psi(t)}{dt}$$

from which we can say

$$\frac{d}{dx}\int_0^{\varphi(x)} f\left(u\right) du = f\left(\varphi\left(x\right)\right) \frac{d\varphi\left(x\right)}{dx} \tag{A.3}$$

It is also easy to see that

$$B_{R_{EW}(x)}\left(b, a\right) = B\left(a, b\right) - B_{F_{EW}(x)}(a, b). \tag{A.4}$$

From (A.2) and (A.3), we get

$$\frac{d}{dx} B_{R_{EW}(x)}\left(b, a\right) = F_{EW}^{a-1}\left(x\right)\left[1 - F_{EW}\left(x\right)\right]^{b-1}(-f_{EW}(x)). \tag{A.5}$$

Using (A.2) and (A.3), the limit in (A.1) takes the form:

$$\lim_{x\to\infty} \varphi\left(x\right) = b \lim_{x\to\infty} \frac{1}{F_{EW}^{a-1}\left(x\right)} = b.$$

This completes the proof.