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The Hahn Sequence Space of Fuzzy Numbers Defined by a Modulus Function

Research Article

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Abstract: In this article we introduce new sequence spaces $h(F, f)$,$h_\infty(F, f)$ and $h_p(F, f, s)$ of sequence of fuzzy numbers defined by a modulus function. Further some inclusion relations regarding these spaces are studied.

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1. Introduction

In recent years there has been an increasing interest in mathematical aspects of operations defined on fuzzy sets. The concept of fuzzy sets and fuzzy set operations was first introduced by Zadeh [1] and subsequently several authors have discussed various aspects of theory and applications of fuzzy sets, such as topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events and fuzzy mathematical programming. The theory of fuzzy numbers is not only the foundation of fuzzy analysis, but it also has important applications in fuzzy optimization, fuzzy decision making etc. [2, 3]. Many authors have found interest in the study of theory of fuzzy numbers [4, 5]. Matloka [6] introduced bounded and convergent sequences of fuzzy numbers. In addition sequences of fuzzy numbers have been discussed by Aytar and Pehlian [7], Basarir and Mursaleen [8] Nanda [10] and many others.

The idea of difference sequence space of fuzzy numbers was introduced by Savas [9] and further generalized by Rifat Colak [11] and many others. Recently Talo and Basar [12] introduced and studied the space $b_p(F)$ of sequences of p-bounded variation of fuzzy numbers. The study of Hahn-sequence space was initiated by Chandrasekhara Rao [13] with certain specific purpose in Banach space theory. Indeed, he got interested in finding a semi Hahn space and proved that the intersection of all semi Hahn spaces is the Hahn space [14]. This idea motivates us to study fuzzy Hahn sequence space [18]. Talo and Basar [15] gave the idea of determining the dual of sequence space of fuzzy numbers by using the concept of convergence of a series of fuzzy numbers [16].

The present paper is devoted to the study of Hahn sequence space of fuzzy numbers defined by modulus function.

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2. Definitions and Preliminaries

We begin with giving some required definitions and statements of theorems, propositions and lemmas. A fuzzy number is a fuzzy set on the real axis i.e. a mapping $u : R \to [0,1]$ which satisfies the following four conditions.

(1). u is normal i.e. there exists an $x_0 \in R$ such that $u(x_0) = 1$.

(2). u is fuzzy convex i.e. $u[\lambda x + (1-\lambda)y] \geq \min\{u(x), u(y)\}$ for all $x, y \in R$ and for all $\lambda \in [0,1]$.

(3). u is upper semi continuous

(4). The set $[u]_0 = \overline{\{x \in R : u(x) > 0\}}$ is compact [1] where $\overline{\{x \in R : u(x) > 0\}}$ denotes the closure of the set $\{x \in R : u(x) > 0\}$ in the usual topology of R.

We denote the set of all fuzzy numbers on R by E' and called it as the space of fuzzy numbers. The λ-level set $[u]_\lambda$ of $u \in E'$ is defined by

$$[u]_\lambda = \begin{cases} \{t \in R : u(t) \geq \lambda\}, & (0 < \lambda \leq 1) \\ \overline{\{t \in R : u(t) > \lambda\}}, & (\lambda = 0). \end{cases}$$

The set $[u]_\lambda$ is a closed bounded and non-empty interval for each $\lambda \in [0,1]$ which is defined by $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$. $\mathbb{R}$ can be embedded in E'. Since each $r \in \mathbb{R}$ can be regarded as a fuzzy number $\overline{r}$ defined by

$$\overline{r} = \begin{cases} 1, & (x = r) \\ 0, & (x \neq r). \end{cases}$$

Let W be the set of all closed and bounded intervals A of real numbers with endpoints $\underline{A}$ and $\overline{A}$ i.e., $A = [\underline{A}, \overline{A}\,]$. Define the relation d on W by

$$d(A, B) = \max\{|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|\}.$$

Then it can be observed that d is a metric on W and (W, d) is a complete metric space [10]. Now we can define the metric D on E' by means of a Hausdroff metric d as

$$D(u, v) = \sup_{\lambda \in [0,1]} d([u]_\lambda, [v]_\lambda)$$

(E', D) is a complete metric space one can extend the natural order relation on the real line to intervals as follows.

$$A \leq B \quad \text{if and only if} \quad \underline{A} \leq \underline{B} \quad \text{and} \quad \overline{A} \leq \overline{B}.$$

The partial order relation on E' is defined as follows.

$$u \leq v \Leftrightarrow [u]_\lambda \leq [v]_\lambda \Leftrightarrow u^-(\lambda) \leq v^-(\lambda) \quad \text{and} \quad u^+(\lambda) \leq v^+(\lambda)$$

for all $\lambda \in [0,1]$. In the sequel, we require the following definitions and lemmas.

Definition 2.1. *A sequence $u = (u_k)$ of fuzzy numbers is a function u from the set N into the set E'. The fuzzy number u_k denotes the value of the function at $k \in \mathbb{N}$ and is called the k^{th} term of the sequence. Let $w(F)$ denote the set of all sequences of fuzzy numbers.*

Definition 2.2. *A sequence $(u_k) \in w(F)$ is called convergent with limit $u \in E'$ if and only if for every $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon) \in N$ such that*

$$D(u_k, u) < \varepsilon \quad \textit{for all } k \geq n_0.$$

Definition 2.3. *A sequence $(u_k) \in w(F)$ is called bounded if and only if the set of all fuzzy numbers consisting of the terms of the sequence (u_k) is a bounded set.*

Definition 2.4. *Let $(u_k) \in w(F)$. Then the expression $\sum u_k$ is called a series of fuzzy numbers. Denote $S_n = \sum\limits_{k=0}^{n} u_k$ for all $n \in N$, if the sequences (S_n) converges to a fuzzy number u then we say that the series $\sum u_k$ of fuzzy numbers converges to u. We say otherwise the series of fuzzy numbers diverges.*

The notion of modulus function was introduced by Nakano [19] as follows.

Definition 2.5. *A function f from $[0, \infty)$ into $[0, \infty)$ is called a modulus function if*

(1). $f(x) = 0$ if and only if $x = 0$.

(2). $f(x + y) \leq f(x) + f(y)$ for all $x, y \geq 0$.

(3). f is increasing.

(4). f is continuous from right at 0.

Hence f is continuous on the interval $[0, \infty)$.

The Hahn sequence space is the space of all sequences $x = (x_k)$ such that $\sum\limits_{k=1}^{\infty} k|x_k - x_{k-1}|$ converges and $\lim\limits_{k \to \infty} x_k = 0$.

3. Main results

Recently Balasubramanian and Pandiarani defined $h(F)$, the Hahn sequence space of fuzzy numbers. Let A denote the matrix $A = (a_{nk})$ defined by

$$a_{n_k} = \begin{cases} n(-1)^{n-k}, & n-1 \leq k \leq n \\ 0, & 1 \leq k \leq n-1 \text{ or } k > n \end{cases} \tag{1}$$

Define the sequence $y = (y_k)$ which will be frequently used as the A-transform of a sequence $x = (x_k)$,

$$\text{i.e.,} \quad y_k = (Ax)_k = k(x_k - x_{k-1}), \quad k \geq 1. \tag{2}$$

The space $h(F)$ is defined as the set of all sequences such that the A-transforms of them are in $\ell(F)$ that is

$$h(F) = \big\{u = (u_k) \in w(F) : \sum_k D[(Au)_k, \overline{0}\,] < \infty \text{ and } \lim_{k \to \infty} D[u_k, \overline{0}\,] = 0\big\}$$

$$\text{and} \quad h_\infty(F) = \big\{u = (u_k) \in w(F) : \sup_k D[(Au)_k, \overline{0}\,] < \infty\big\}.$$

In this paper we introduce the spaces $h_p(F)$,$h(F, f)$,$h_\infty(F, f)$ and $h_p(F, f, s)$.

$$h_p(F) = \big\{u = (u_k) \in w(F) : \sum_k D[(Au)_k, \overline{0}\,]^p < \infty \text{ and } \lim_{k \to \infty} D[u_k, \overline{0}\,] = 0\big\}$$

It is not hard to see that $h_p(F)$ is a complete metric space with the Hausdroff metric d_p defined by

$$d_p(u,v)=\left\{\sum_k \left(D\left[(Au)_k,(Av)_k\right]\right)^p\right\}^{\frac{1}{p}}$$

Let f be a modulus function.We define the spaces $h(F,f)$,$h_\infty(F,f)$ and $h_p(F,f,s)$ as follows.

$$h(F,f)=\left\{u=(u_k)\in w(F):\sum_k f\left(D[(Au)_k,\overline{0}\,]\right)<\infty \text{ and} \lim_{k\to\infty} D[u_k,\overline{0}\,]=0\right\}$$

$$h_\infty(F,f)=\left\{u=(u_k)\in w(F):\sup_k f\left(D[(Au)_k,\overline{0}\,]\right)<\infty\right\}.$$

$$\text{and}\quad h_p(F,f,s)=\left\{u=(u_k)\in w(F):\sum_k \frac{\left\{f\left(D[(Au)_k,\overline{0}\,]\right)\right\}^p}{k^s}<\infty\right\}$$

Theorem 3.1. *The sets $h(F,f)$, $h_\infty(F,f)$ and $h_p(F,f,s)$ are closed under the coordinatewise addition and scalar multiplication.*

Theorem 3.2. *The spaces $h(F,f)$, $h_\infty(F,f)$ and $h_p(F,f,s)$ are complete metric spaces with respect to the metrics $\overline{d}$, $\overline{d}_\infty$ and $\overline{d}_p$ defined by*

$$\overline{d}(u,v)=\sum_k f\left(D[(Au)_k,(Av)_k]\right)$$

$$\text{and}\quad \overline{d}_\infty(u,v)=\sup_{k\in N} f\left(D[(Au)_k,(Av)_k]\right)$$

$$\text{and}\quad \overline{d}_p(u,v)=\left\{\sum_k \frac{\left\{f\left(D[(Au)_k,\overline{0}\,]\right)\right\}^p}{k^s}\right\}^{1/p}$$

respectively, where $u=(u_k)$ and $v=(v_k)$ are the elements of the spaces $h(F,f)$,$h_\infty(F,f)$ and $h_p(F,f,s)$.

Proof. Since the proof is analogus for the spaces $h_\infty(F,f)$ and $h_p(F,f,s)$, we consider only the space $h(F,f)$. one can easily establish that $\overline{d}$ defines a metric on $h(F,f)$ so it remains to prove the completeness of the space $h(F,f)$. Let $\{u^i\}$ be any Cauchy sequence in the space $h(F,f)$, where $u^i=\{u_0^{(i)},u_1^{(i)},u_2^{(i)}\ldots\}$. Then for a given $\varepsilon>0$ there exists a positive integer $n_0(\varepsilon)$ such that

$$\overline{d}(u^i,u^j)=\sum_n f\left(D[(Au)_n^i,(Au)_n^j]\right)<\varepsilon \tag{3}$$

for every $i,j\geq n_0(\varepsilon)$. We obtain for each fixed $n\in\mathbb{N}$ from (3) that

$$\lim_{i,j\to\infty} f\left(D[(Au)_n^i,(Au)_n^j]\right)=0. \tag{4}$$

Since f is continuous we have from (4) that

$$f\left(\lim_{i,j\to\infty} D[(Au)_n^i,(Au)_n^j]\right)=0 \tag{5}$$

Therefore, since f is a modulus function one can derive by (5) that

$$\lim_{i,j\to\infty} D[(Au)_n^i,(Au)_n^j]=0 \tag{6}$$

This means that $(Au)_n^i$ is a cauchy sequence in E' for every fixed $n\in\mathbb{N}$. Since E' is complete, $(Au)_n^i\to(Au)_n$as $\quad i\to\infty$. Using these infinitely many limits we define the sequence $\{(Au)_1,(Au)_2,\ldots\}$. We have from (6), for each $m\in\mathbb{N}$ and for $i,j\geq n_0(\varepsilon)$ that

$$\overline{d}(u,v)=\sum_{k=0}^m f\left(D[(Au)_k,(Av)_k]\right)\leq\overline{d}(u^i,u^j)<\varepsilon. \tag{7}$$

Take any $i \geq n_0(\varepsilon)$ and taking limit as $j \to \infty$ first and next $m \to \infty$ in (3) we obtain

$$\overline{d}(u^i, u) < \varepsilon. \tag{8}$$

Finally we proceed to prove $u \in h(F, f)$. Since $\{u^i\}$ is a sequence in $h(F, f)$, we have for each $i \in \mathbb{N}$, there exist $n_1(\varepsilon)$ such that

$$\sum_k f\left(D[(Au)^i_k, \overline{0}\,]\right) \leq \varepsilon \quad \text{and} \quad \lim_{k\to\infty} f\left(D[(Au)^i_k, \overline{0}\,]\right) = 0.$$

For every $n \geq n_1(\varepsilon)$ and for each fixed $i \in \mathbb{N}$,

$$f\left(D[(Au)_k, \overline{0}]\right) \leq f\left(D[(Au)_k, (Au)^i_k]\right) + f\left(D[(Au)^i_k, (Au)^j_k]\right) + f\left(D[(Au)^j_k, \overline{0}]\right) \tag{9}$$

holds for all $i, j \in \mathbb{N}$ and for fixed $i \geq n_0(\varepsilon)$. Hence

$$\sum_k f\left(D[(Au)_k, \overline{0}\,]\right) < \varepsilon.$$

Also from (5) $\lim_{k\to\infty} f\left(D[(Au)_k, \overline{0}\,]\right) = 0$. Hence $u \in h(F, f)$. Since $\{u^i\}$ is an arbitrary Cauchy sequence, the space $h(F)$ is complete. □

Theorem 3.3. *If $f_1 and f_2$ are two modulus functions then the following inclusion relations hold.*

(a). $h(F, f_1) \cap h(F, f_2) \subseteq h(F, f_1 + f_2)$.

(b). $h(F, f_1) \subseteq h(F, f_1 \circ f_2)$.

(c). If $f_1(t) \leq f_2(t)$ for all $t \in [0, \infty)$ then $h(F, f_1) \subseteq h(F, f_2)$.

Proof.

(a). Let $u = (u_k) \in h(F, f_1) \cap h(F, f_2)$. Since

$$(f_1 + f_2)\left\{D[(Au)_k, \overline{0}\,]\right\} = (f_1)\left\{D[(Au)_k, \overline{0}\,]\right\} + (f_2)\left\{D[(Au)_k, \overline{0}\,]\right\}$$
$$\text{and} \quad (f_1 + f_2)\left\{D[u_k, \overline{0}\,]\right\} = (f_1)\left\{D[u_k, \overline{0}\,]\right\} + (f_2)\left\{D[u_k, \overline{0}\,]\right\}$$

one can see that $u \in h(F, f_1 + f_2)$.

(b). Let $u = (u_k) \in h(F, f_1)$. since f_2 is continuous there exist a $\rho > 0$ such that $f_2(\rho) = \varepsilon$ for all $\varepsilon > 0$. Since $u = (u_k) \in h(F, f_1)$, there exist an $n_0 \in \mathbb{N}$ such that

$$(f_1)\left\{D[(Au)_k, \overline{0}\,]\right\} < \rho$$
$$\text{and} \quad (f_1)\left\{D[u_k, \overline{0}\,]\right\} < \rho$$

for all $k \geq n_0$. Therefore one can derive by applying f_2 that

$$f_2\left((f_1)\left\{D[(Au)_k, \overline{0}\,]\right\}\right) < f_2(\rho) = \varepsilon$$
$$\text{and} \quad f_2\left((f_1)\left\{D[(Au)_k, \overline{0}\,]\right\}\right) < f_2(\rho) = \varepsilon$$

Since this is true for all $k \in \mathbb{N}$, we have $u \in h(F, f_1 \circ f_2)$.

(c). Since $f_1(t) \le f_2(t)$ for all $t \in [0,\infty)$, we have

$$(f_1)\left\{D[(Au)_k,\bar{0}\,]\right\} \le (f_2)\left\{D[(Au)_k,\bar{0}\,]\right\} \text{ and } (f_1)\left\{D[u_k,\bar{0}\,]\right\} \le (f_2)\left\{D[u_k,\bar{0}\,]\right\}$$

This leads to the fact that $u \in h(F,f_1)$ implies $u \in h(F,f_2)$.

□

Lemma 3.4. *Let f_1 and f_2 be two modulus functions and $0 \le \delta \le 1$. If $f_1(t) > \delta$ then $(f_1 \circ f_2)(t) \le \frac{2f_2(1)}{\delta} f_1(t)$ holds for all $t \in [0,\infty)$.*

Theorem 3.5. *If f_1 and f_2 be two modulus functions, then the following inclusion relations hold.*

(a). $h_p(F,f_1,s) \cap h(F,f_2,s) \subseteq h_p(F,f_1+f_2,s)$

(b). If $s>1$, $h_p(F,f_1,s) \subseteq h_p(F,f_1 \circ f_2,s)$

(c). If $\limsup_{t\to\infty}\left[\frac{f_1(t)}{f_2(t)}\right] < \infty, h_p(F,f_2,s) \subseteq h_p(F,f_1,s)$

(d). If $s_1 \le s_2$,$h_p(F,f_1,s_1) \subseteq h_p(F,f_1,s_2)$.

Proof.

(a). Since $\left\{(f_1+f_2)\left\{D[(Au)_k,\bar{0}\,]\right\}\right\}^p \le 2^{p-1}\left\{(f_1)\left\{D[(Au)_k,\bar{0}\,]\right\}\right\}^p + \left\{(f_2)\left\{D[(Au)_k,\bar{0}\,]\right\}\right\}^p$. This yields us by taking summation over $k \in \mathbb{N}$ that $u \in h_p(F,f_1+f_2,s)$

(b). Since f_2 is continuous from the right at 0, there exist δ with $0<\delta<1$ such that $f_2(t) < \varepsilon$ for all $\varepsilon > 0$ whenever $0 \le t \le \delta$. Define the sets N_1 and N_2 by

$$N_1 = \left\{k \in \mathbb{N} : f_1\left\{D[(Au)_k,\bar{0}\,]\right\} \le \delta\right\}, \quad N_2 = \left\{k \in \mathbb{N} : f_1\left\{D[(Au)_k,\bar{0}\,]\right\} > \delta\right\}$$

Then we obtain from Lemma 3.4 for

$$\left\{f_1\left\{D[(Au)_k,\bar{0}\,]\right\} > \delta\right\} \text{ that } (f_2 \circ f_1)\left\{\left[D(Au)_k,\bar{0}\right]\right\} \le \frac{2f_2(1)}{\delta} f_1\left\{D\left[(Au)_k,\bar{0}\right]\right\}$$

Therefore we derive for $u=(u_k) \in h_p(F,f_1,s)$ with $\delta > 1$ that

$$\begin{aligned}\sum_k \frac{\left\{(f_2\circ f_1)\left[D\left((Au)_k,\bar{0}\right)\right]\right\}^p}{k^s} &= \sum_{k\in N_1}\frac{\left\{(f_2\circ f_1)\left[D\left((Au)_k,\bar{0}\right)\right]\right\}^p}{k^s} + \sum_{k\in N_2}\frac{\left\{(f_2\circ f_1)\left[D\left((Au)_k,\bar{0}\right)\right]\right\}^p}{k^s}\\ &\le \sum_{k\in N_1}\frac{\epsilon^p}{k^s} + \sum_{k\in N_2}\frac{\left\{\frac{2f_2(1)}{\delta} f_1\left\{D\left[(Au)_k,\bar{0}\right]\right\}^p\right\}}{k^s}\\ &= \epsilon^p\sum_{k\in N_1}\frac{1}{k^s} + \left[\frac{2f_2(1)}{\delta}\right]^p \sum_{k\in N_2}\frac{f_1\left\{D\left[(Au)_k,\bar{0}\right]\right\}^p}{k^s} < \infty.\end{aligned}$$

Hence $u=(u_k) \in h_p(F,f_2\circ f_1,s)$.

(c). Suppose that $\limsup_{t\to\infty}\left[\frac{f_1(t)}{f_2(t)}\right] < \infty$. Then there is a number $M>0$ such that $\left[\frac{f_1(t)}{f_2(t)}\right] \le M$ for all $t \in [0,\infty)$. Since $D\left[(Au)_k,\bar{0}\right] \ge 0$ for all $k \in \mathbb{N}$ and for all $u=(u_k) \in h_p(F,f_2,s)$ we have $f_1\left\{D\left[(Au)_k,\bar{0}\right]\right\} \le Mf_2\left\{D\left[(Au)_k,\bar{0}\right]\right\}$ which leads us

$$\begin{aligned}\sum_k \frac{\left\{(f_1)\left[D\left((Au)_k,\bar{0}\right)\right]\right\}^p}{k^s} &\le \sum_k \frac{\left\{(Mf_2)\left[D\left((Au)_k,\bar{0}\right)\right]\right\}^p}{k^s}\\ &= M^p\sum_k \frac{\left\{(f_2)\left[D\left((Au)_k,\bar{0}\right)\right]\right\}^p}{k^s} < \infty\end{aligned}$$

Thus $u=(u_k) \in h_p(F,f_2,s)$.

(d). Let $s_1 \leq s_2$. Since $0 < k^{-1} \leq 1$ for all $k \in \mathbb{N}$, it is immediate that $k^{-s_2} \leq k^{-s_1}$. Then one can seethat

$$\sum_k \frac{\left\{(f)\left[D\left((Au)_k, \overline{0}\right)\right]\right\}^p}{k^{s_2}} \leq \sum_k \frac{\left\{(f)\left[D\left((Au)_k, \overline{0}\right)\right]\right\}^p}{k^{s_1}} < \infty$$

holds, for all $u = (u_k) \in h_p(F, f, s)$.

$\square$

Corollary 3.6. *Define the spaces $h_p(F, s)$and $h_p(F, f)$ by*

$$h_p(F, s) = \left\{u = (u_k) \in w(F) : \sum_k \frac{1}{k^s}\left[D\left[(Au)_k, \overline{0}\right]\right]^p < \infty\right\}, \quad s \geq 0.$$

$$h_p(F, f) = \left\{u = (u_k) \in w(F) : \sum_k \left\{f\left[D\left[(Au)_k, \overline{0}\right]\right]\right\}^p < \infty\right\}$$

Then we have

(a). If $s > 1$, then $h_p(F, s) \subseteq h_p(F, f, s)$.

(b). $h_p(F, f) \subseteq h_p(F, f, s)$.

Proof.

(a). follows from Theorem 3.5 (b) with $f_1(t) = t$ and $f_2 = f$.

(b). follows from taking $s_1 = 0$,$s_2 = s$ and $f_1 = f$ from Theorem 3.5 (d).

$\square$

Theorem 3.7. *Let $s > 1$. Then the following relation hold $h_\infty(F) \subseteq h_p(F, f, s)$*

Proof. Let $u = (u_k) \in h_\infty(F)$.Then there is a number $M > 0$ such that $D[(Au)_k, \overline{0}] \leq M$ for all $k \in \mathbb{N}$. Since f is continuous and increasing, there is a number $N > 0$ such that $f\left\{D[(Au)_k, \overline{0}]\right\} \leq f(M) \leq N$. Therefore we get for $s > 1$ that

$$\sum_k \frac{\left\{f\left\{D\left[(Au_k, \overline{0}\right]\right\}\right\}^p}{k^s} \leq N^p \sum_k \frac{1}{k^s} < \infty.$$

Hence $u = (u_k) \in h_p(F, f, s)$. $\square$

References

[1] L.A.Zadeh, *Fuzzy sets*, Inf. Control, 8(1965), 338-353.

[2] H-M.Hsu and C.-T.Chen, *Aggregation of fuzzy opinions under group decision making*, Fuzzy Sets and systems, 79(1996), 279-285.

[3] M.Sakawa and K.Kato, *Interactive decision making for large-scale multi-objective linear programmes with fuzzy numbers*, Fuzzy sets and systems, 88(1997), 161-172.

[4] P.Diamond and P.Kloden, *Metric spaces of fuzzy sets*, Fuzzy sets and systems, 35(1990), 241-249.

[5] D.Dubois and H.Prade, *Operations on Fuzzy numbers*, Internet. J. Systems sci., 9(1978), 613-626.

[6] M.Matloka, *Sequences of fuzzy numbers*, BUSEFAL, 28(1986), 28-37.

[7] S.Aytar and S.Pehlian, *Statistically monotonic and statistically bounded sequences of fuzzy numbers*, Inform. Sci., 176(6)(2006), 734-744.

[8] M.Mursaleen and M.Basarir, *On some sequence spaces of fuzzy numbers*, Indian J. Pure Appl. Math., 34(9)(2003), 1351-1357.

[9] E.Savas, *A note on sequences of fuzzy numbers*, Information sciences, 124(2000), 297-300.

[10] S.Nanda, *On sequences of fuzzy numbers*, Fuzzy sets and systems, 33(1989), 123-126.

[11] Rifat Colak Hifsi Altinok and Mikail Et, *Generalised difference sequences of fuzzy numbers*, Chaos Solitons and Fractals, 40(2009), 1106-1117.

[12] Ozer Talo and Feyzi Basar, *On the space* $b\vee\rho(F)$ *of sequences of* p*-bounded variation of fuzzy numbers*, Acta Mathematica Cinica English Series, 24(7)(2008), 1205-1212.

[13] K.Chandrasekhara Rao, *The Hahn sequence space*, Bull. Cal. Math. Soc., 82(1990), 72-78.

[14] K.Chandrasekhara Rao and N.Subramanian, *The Hahn sequence space-III*, Bull. Malaysian Math. Sc. Soc (Second Series), 25(2002), 163-171.

[15] Ozer Talo and Feyzi Basar, *Determination of the Duals of classical sets of sequences of Fuzzy numbers and related matrix transformations*, Computers and Mathematics with Applications, 58(2009), 717-733.

[16] M.Stojakovic and Z.Stojakovic, *Addition and series of fuzzy Numbers*, Fuzzy Sets and Systems, 83(1996), 341-346.

[17] M.H.Puri and D.A.Ralescu, *Differentials of fuzzy functions*, J. Math. Anal. Appl., 91(1983), 552-558.

[18] T.Balasubramanian and A.Pandiarani, *The Hahn sequence space of fuzzy Numbers*, Tamsui Oxford Journal of Information and Mathematical Sciences, 27(2)(2011), 213-224.

[19] H.Nakano, *Concave modulars*, J. Math. Soc. Japan, 5(1953), 29-49.