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Transformation Formulas of Lauricella's Function of the Fourth kind of Several Variables

Research Article

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Abstract: The aim of this research paper is to derive transformation formulas for Lauricella's function of the fourth kind of several variables with the help of generalized Kummer's theorem on the sum of the series ${}_2F_1(-1)$ obtained by Lavoie et al. [10]. Some special cases of these formulas are also deduced.

MSC: 33C05, 33C20, 33C65.

Keywords: Transformation formulas, Lauricella's function, Kummer's theorem, Hypergeometric functions

1. Introduction

In terms of the Pochhammer's symbol $(a)_n$ defined by [14]

$$(a)_n = \begin{cases} 1, & \text{if } n = 0; \\ a(a+1)(a+2)\dots(a+n-1), & \text{if } n = 1,2,3,\dots. \end{cases} \tag{1}$$

the Lauricella's function $F_D^{(n)}$ is defined and represented as follows [14]

$$F_D^{(n)}\,(a\,,\,b_1\,,\,\cdots\,,\,b_n\,;c;\,x_1\,,\,\cdots\,,\,x_n) = \sum_{m_1\,,\,\cdots\,,\,m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}} \frac{x_1^{m_1}}{m_1\,!}\cdots\frac{x_n^{m_n}}{m_n\,!} \tag{2}$$

$\max\{|x_1|\,,\cdots,|x_n|\} < 1$. Clearly, we have $F_D^{(2)} = F_1$, where F_1 is Appell's double hypergeometric function [14]

$$F_1\,(a\,,\,b_1\,,\;b_2\,;\,c;\,x\,,\,y) = \sum_{m\,,\,n=0}^{\infty} \frac{(a)_{m+n}\,(b_1)_m\,(b_2)_n}{(c)_{m+n}} \frac{x^m}{m\,!}\,\frac{y^n}{n\,!} \tag{3}$$

The generalized Lauricella's function of several variables is defined as follows [14]

$$F\begin{matrix} A\,:\,B'\,;\,\cdots\,;\,B^{(n)} \\ C\,:\,D'\,;\,\cdots\,;\,D^{(n)} \end{matrix}\,[z_1,\cdots,z_n]$$

$$\equiv F\begin{matrix} A\,:\,B'\,;\cdots;\,B^{(n)} \\ C\,:\,D'\,;\cdots;\,D^{(n)} \end{matrix}\left(\begin{matrix} [(a)\;:\;\theta'\;,\;\cdots\;,\;\theta^{(n)}] & : & [(b')\;:\;\varphi'] & ; & \cdots & ; & [(b^{(n)})\;:\;\varphi^{(n)}] & ; \\ [(c)\;:\;\psi'\;,\;\cdots\;,\;\psi^{(n)}] & : & [(d')\;:\;\delta'] & ; & \cdots & ; & [(d^{(n)})\;:\;\delta^{(n)}] & ; \end{matrix}\; z_1\,,\,\cdots\,,\,z_n\right)$$

$$= \sum_{m_1,\cdots,m_n=0}^{\infty} \Omega\,(m_1,\,\cdots\,,\,m_n)\,\frac{z_1^{m_1}}{m_1\,!}\;\cdots\;\frac{z_n^{m_n}}{m_n\,!}, \tag{4}$$

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where

$$\Omega\,(m_1,\cdots,m_n)=\frac{\prod_{j=1}^{A}(a_j)_{m_1\theta'_j+\cdots+m_n\theta_j^{(n)}}\prod_{j=1}^{B'}(b'_j)_{m_1\varphi'_j}\cdots\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\varphi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi'_j+\cdots+m_n\psi_j^{(n)}}\prod_{j=1}^{D'}(d'_j)_{m_1\delta'_j}\cdots\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}} \tag{5}$$

the coefficients $\theta_j^{(k)}$, $j=1,2\cdots,A$; $\varphi_j^{(k)}$, $j=1,2,\cdots,B^{(k)}$; $\psi_j^{(k)}$, $j=1,2,\cdots,C$; $\delta_j^{(k)}$, $j=1,2,\cdots,D^{(k)}$; for all $k\in\{1,2,\cdots,n\}$ are real and positive, (a) abbreviates the array of A parameters $a_1,\cdots a_A,(b^{(k)})$ abbreviates the array of $B^{(k)}$ parameters $b_j^{(k)},j=1,2,\cdots,B^{(k)}$ for all $k\in\{1,2,\cdots,n\}$ with similar interpretations for (c) and $(d^{(k)})$; $k\in 1,2,\cdots,n$; etc. Note that, when the coefficients in Equation (4) equal to 1, the generalized Lauricella function (4) reduces to the following multivariable extension of the Kamp'e de F'eriet function [14]:

$$F\begin{matrix}p:q_1;\cdots;q_n\\ l:m_1;\cdots;m_n\end{matrix}[z_1,\cdots,z_n]\equiv F\begin{matrix}p:q_1;\cdots;q_n\\ l:m_1;\cdots;m_n\end{matrix}\left(\begin{matrix}(a_p)&:&(b'_{q_1})&;&\cdots&;&(b_{q_n}^{(n)})&;\\ (c_l)&:&(d'_{m_1})&;&\cdots&;&(d_{m_n}^{(n)})&;\end{matrix}\;z_1,\cdots,z_n\right)$$

$$=\sum_{s_1,\cdots,s_n=0}^{\infty}\Omega\,(s_1,\cdots,s_n)\,\frac{z_1^{s_1}}{s_1\,!}\cdots\frac{z_n^{s_n}}{s_n\,!}, \tag{6}$$

where

$$\Omega\,(s_1,\cdots,s_n)=\frac{\prod_{j=1}^{p}(a_j)_{s_1+\cdots+s_n}\prod_{j=1}^{q_1}(b'_j)_{s_1}\cdots\prod_{j=1}^{q_n}(b_j^{(n)})_{s_n}}{\prod_{j=1}^{l}(c_j)_{s_1+\cdots+s_n}\prod_{j=1}^{m_1}(d'_j)_{s_1}\cdots\prod_{j=1}^{m_n}(d_j^{(n)})_{s_n}}. \tag{7}$$

In the theory of hypergeometric series, classical summation theorems such as Gauss, Gauss second and Kummer for the series ${}_2F_1$; Dixon, Watson and Whipple for the series ${}_3F_2$ and others play an important role. For generalizations and applications of the above mentioned theorems see for examples [1–5, 8–10] and [12]. In the present investigation, we shall require the following generalization of the classical Kummer's theorem for the series ${}_2F_1(-1)$ [10]:

$${}_2F_1\left[\begin{matrix}a,b & ; \\ 1+a-b+i & ;\end{matrix}\;-1\right]=\frac{\Gamma\,(\frac{1}{2})\Gamma\,(1+a-b+i)\,\Gamma(1-b)}{2^a\Gamma\,(1-b+\frac{1}{2}(i+|\,i\,|))}\times\left\{\frac{A_i}{\Gamma\,(\frac{1}{2}a+\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])\Gamma\,(1+\frac{1}{2}a-b+\frac{1}{2}i)}\right.$$

$$\left.+\frac{B_i}{\Gamma\,(\frac{1}{2}a+\frac{1}{2}i-[\frac{i}{2}])\,\Gamma\,(\frac{1}{2}+\frac{1}{2}a-b+\frac{1}{2}i)}\right\} \tag{8}$$

for $i=0,\pm1,\pm2,\pm3,\pm4,\pm5$, where $[x]$ denotes the greatest integer less than or equal to x and $|x|$ denotes the usual absolute value of x. The coefficients A_i and B_i are given respectively in [10]. When $i=0$, (8) reduces immediately to the classical Kummer's theorem [4], (see also [12])

$${}_2F_1\left[\begin{matrix}a,b & ; \\ 1+a-b & ;\end{matrix}\;-1\right]=\frac{\Gamma\,(1+a-b)\,\Gamma(\frac{1}{2})}{2^a\Gamma\,(1+\frac{1}{2}a-b)\Gamma\,(\frac{1}{2}a+\frac{1}{2})} \tag{9}$$

The following results will be required also [13, 14] :

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)},\quad a\neq 0,-1,-2,\cdots \tag{10}$$

$$\Gamma\,\left(\tfrac{1}{2}\right)\,\Gamma\,\left(1+a\right)\;=2^a\;\Gamma\,\left(\tfrac{1}{2}+\tfrac{1}{2}a\right)\,\Gamma\,\left(1+\tfrac{1}{2}a\right) \tag{11}$$

$$(a)_{2n}\;=\;2^{2n}\left(\tfrac{1}{2}a\right)_n\left(\tfrac{1}{2}a+\tfrac{1}{2}\right)_n \tag{12}$$

$$\frac{\Gamma\,(a-n)}{\Gamma\,(a)}\;=\frac{(-1)^n}{(1-a)_n} \tag{13}$$

$$(2n)!\;=2^{2n}\left(\tfrac{1}{2}\right)_n n\,! \tag{14}$$

$$(2n+1)!\;=2^{2n}\left(\tfrac{3}{2}\right)_n n\,! \tag{15}$$

2. Transformation Formulas

In this section, the following transformation formulas will be established:

$$F_D^{(2r)}(a\,,\ b_1-i,\,b_1,\,b_2-i,\,b_2\,,\cdots,b_r-i,\,b_r;\,c\,;x_1,-x_1,\,x_2,-x_2,\cdots,\,x_r,-x_r)$$

$$= \sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{2m_1+\cdots+2m_r}\,(b_1-i)_{2m_1}\cdots(b_r-i)_{2m_r}\,{x_1}^{2m_1}\cdots{x_r}^{2m_r}}{(c)_{2m_1+\cdots+2m_r}\,(2m_1)!\cdots(2m_r)!}$$

$$\times I_1(b_1,i\,,2m_1)\left\{\frac{A_i'}{A_1(b_1,i,\,2m_1)}+\frac{B_i'}{B_1(b_1,i,\,2m_1)}\right\}\times\cdots\times I_r(b_r,i\,,2m_r)\left\{\frac{A_i'}{A_r(b_r,i,\,2m_r)}+\frac{B_i'}{B_r(b_r,i\,,2m_r)}\right\}+\cdots$$

$$\cdots+\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{2m_1+1+\cdots+2m_r+1}\,(b_1-i)_{2m_1+1}\cdots(b_r-i)_{2m_r+1}\,{x_1}^{2m_1+1}\cdots{x_r}^{2m_r+1}}{(c)_{2m_1+1+\cdots+2m_r+1}\,(2m_1+1)!\cdots(2m_r+1)!}$$

$$\times I_1(b_1,i\,,\ 2m_1+1)\left\{\frac{A_i''}{A_1(b_1,i,\,2m_1+1)}+\frac{B_i''}{B_1(b_1,i,\,2m_1+1)}\right\}\times\cdots$$

$$\cdots\times I_r(b_r,i\,,2m_r+1)\left\{\frac{A_i''}{A_r(b_r,i,\,2m_r+1)}+\frac{B_i''}{B_r(b_r,i,\,2m_r+1)}\right\},\ \text{for } i=0,\,\pm1,\,\pm2,\,\pm3,\,\pm4,\,\pm5. \tag{16}$$

$$F_D^{(2r+1)}(a\,,\ b_1-i,\,b_1,\,b_2-i,\,b_2\,,\cdots,b_r-i,\,b_r\,,c;\,d\,;x_1,-x_1,\,x_2,-x_2,\cdots,\,x_r,-x_r,y)$$

$$= \sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\sum_{m=0}^{\infty}\frac{(a)_{2m_1+\cdots+2m_r+m}\,(b_1-i)_{2m_1}\cdots(b_r-i)_{2m_r}(c)_m\,{x_1}^{2m_1}\cdots{x_r}^{2m_r}y^m}{(d)_{2m_1+\cdots+2m_r+m}\,(2m_1)!\cdots(2m_r)!\,m!}$$

$$\times I_1(b_1,i\,,2m_1)\left\{\frac{A_i'}{A_1(b_1,i,\,2m_1)}+\frac{B_i'}{B_1(b_1,i,\,2m_1)}\right\}\times\cdots\times I_r(b_r,i\,,2m_r)\left\{\frac{A_i'}{A_r(b_r,i,\,2m_r)}+\frac{B_i'}{B_r(b_r,i\,,2m_r)}\right\}+\cdots$$

$$\cdots+\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\sum_{m=0}^{\infty}\frac{(a)_{2m_1+1+\cdots+2m_r+1+m}\,(b_1-i)_{2m_1+1}\cdots(b_r-i)_{2m_r+1}(c)_m\,{x_1}^{2m_1+1}\cdots{x_r}^{2m_r+1}y^m}{(d)_{2m_1+1+\cdots+2m_r+1+m}\,(2m_1+1)!\cdots(2m_r+1)!\,m!}$$

$$\times I_1(b_1,i\,,\ 2m_1+1)\left\{\frac{A_i''}{A_1(b_1,i,\,2m_1+1)}+\frac{B_i''}{B_1(b_1,i,\,2m_1+1)}\right\}\times\cdots$$

$$\cdots\times I_r(b_r,i\,,2m_r+1)\left\{\frac{A_i''}{A_r(b_r,i,\,2m_r+1)}+\frac{B_i''}{B_r(b_r,i,\,2m_r+1)}\right\},\ \text{for } i=0,\,\pm1,\,\pm2,\,\pm3,\,\pm4,\,\pm5. \tag{17}$$

where

$$I_r(b_r,i\,,\,m_r)=\frac{\Gamma\left(\tfrac{1}{2}\right)\Gamma\left(1-m_r-b_r+i\right)\Gamma(1-b_r)}{2^{-m_r}\Gamma\left(1-b_r+\tfrac{1}{2}(i+|\,i\,|)\right)} \tag{18}$$

$$A_r(b_r,i\,,\,m_r)=\ \Gamma\left(-\tfrac{1}{2}m+\tfrac{1}{2}i+\tfrac{1}{2}-[\tfrac{1+i}{2}]\right)\Gamma\left(1-\tfrac{1}{2}m_r-b_r+\tfrac{1}{2}i\right) \tag{19}$$

$$B_r(b_r,i\,,\,m_r)=\Gamma\left(-\tfrac{1}{2}m_r+\tfrac{1}{2}i-[\tfrac{i}{2}]\right)\Gamma\left(\tfrac{1}{2}-\tfrac{1}{2}m_r-b_r+\tfrac{1}{2}i\right) \tag{20}$$

The coefficients A_i' and B_i' can be obtained from the tables of A_i and B_i given in [10] by replacing a by $-2m_r$, also the coefficients A_i'' and B_i'' can be obtained from the same tables of A_i and B_i by replacing a by $-2m_r-1$ respectively.

Proof of (16)**:** Denoting the left hand side of (16) by S, expanding $F_D^{(2r)}$ in a power series and using the results [14]:

$$(a)_{m+n} = (a)_m (a+m)_n \tag{21}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A(n,m) = \sum_{m=0}^{\infty}\sum_{n=0}^{m} A(n,m-n) \tag{22}$$

$$(a)_{m-n} = \frac{(-1)^n (a)_m}{(1-a-m)_n}, 0 \le n \le m \ \text{ and } \ (m-n)! = \frac{(-1)^n \, m\,!}{(-m)_n}, 0 \le n \le m, \tag{23}$$

we get

$$S = \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \frac{(a)_{m_1+\cdots+m_r} (b_1-i)_{m_1} \cdots (b_r-i)_{m_r} {x_1}^{m_1} \cdots {x_r}^{m_r}}{(c)_{m_1+\cdots+m_r} \; m_1\,! \cdots m_r\,!} \times f(b_1, i\,,\, m_1) \times \cdots \times f(b_r, i,\, m_r) \tag{24}$$

where

$$f(b_r, i\,,\, m_r) = {}_2F_1\left[\begin{array}{lll} -m_r\,,\, b_r & ; & \\ & & -1 \\ 1-m_r-b_r+i & ; & \end{array}\right] \tag{25}$$

Separating (24) into its even and odd terms, we have

$$S = \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \frac{(a)_{2m_1+\cdots+2m_r} (b_1-i)_{2m_1} \cdots (b_r-i)_{2m_r} \; {x_1}^{2m_1} \cdots {x_r}^{2m_r}}{(c)_{2m_1+\cdots+2m_r} \; (2m_1)\,! \cdots (2m_r)\,!}$$

$$\times f(b_1, i,\, 2m_1) f(b_2, i,\, 2m_2) \times \cdots \times f(b_r, i,\, 2m_r)$$

$$+ \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \frac{(a)_{2m_1+1+2m_2+\cdots+2m_r} (b_1-i)_{2m_1+1} (b_2-i)_{2m_r} \cdots (b_r-i)_{2m_r} \; {x_1}^{2m_1+1} {x_2}^{2m_2} \cdots {x_r}^{2m_r}}{(c)_{2m_1+1+2m_2+\cdots+2m_r} (2m_1+1)\,! \, (2m_2)\,! \cdots (2m_r)\,!}$$

$$\times f(b_1, i,\, 2m_1+1) \times f(b_2, i,\, 2m_2) \times \cdots \times f(b_r, i,\, 2m_r) + \cdots$$

$$\cdots + \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \frac{(a)_{2m_1+2m_2+1+\cdots+2m_r+1} (b_1-i)_{2m_1} (b_2-i)_{2m_2+1} \cdots (b_r-i)_{2m_r+1} {x_1}^{2m_1} {x_2}^{2m_2+1} \cdots {x_r}^{2m_r+1}}{(c)_{2m_1+2m_2+1+\cdots+2m_r+1} (2m_1)\,! \, (2m_2+1)\,! \cdots (2m_r+1)\,!}$$

$$\times f(b_1, i,\, 2m_1) \times f(b_2, i,\, 2m_2+1) \times \cdots \times f(b_r, i,\, 2m_r+1)$$

$$+ \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \frac{(a)_{2m_1+1+\cdots+2m_r+1} \; (b_1-i)_{2m_1+1} \cdots (b_r-i)_{2m_r+1} \; {x_1}^{2m_1+1} \cdots {x_r}^{2m_r+1}}{(c)_{2m_1+1+\cdots+2m_r+1} (2m_1+1)\,! \; \cdots (2m_r+1)\,!}$$

$$\times f(b_1, i,\, 2m_1+1) \times f(b_2, i,\, 2m_2+1) \times \cdots\cdots \times f(b_r, i,\, 2m_r+1) \tag{26}$$

Finally, in (26) if we use the result (8), then we obtain the right hand side of (16). This completes the proof of (16). The result (17) can be proved by the similar manner.

3. Special Cases

In (16), if we take $i = 0$, then we have

$F_D^{(2r)}(a\,,\ b_1, b_1, b_2, b_2\,, \cdots, b_r, b_r; c\,; x_1, -x_1, x_2, -x_2, \cdots, x_r, -x_r)$

$$= \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \frac{(a)_{2m_1+\cdots+2m_r}\,(b_1)_{2m_1}\cdots(b_r)_{2m_r}\,{x_1}^{2m_1}\cdots{x_r}^{2m_r}}{(c)_{2m_1+\cdots+2m_r}\,(2m_1)!\cdots(2m_r)!}$$

$$\times \frac{I_1(b_1, i\,, 2m_1)}{A_1(b_1, i,\, 2m_1)} \times \cdots \times \frac{I_r(b_r, i\,, 2m_r)}{A_1(b_r, i,\, 2m_r)}$$

$$= \sum_{m_1=0}^{\infty} \cdots \sum_{m_r=0}^{\infty} \frac{(a)_{2m_1+\cdots+2m_r}\,(b_1)_{2m_1}\cdots(b_r)_{2m_r}\,{x_1}^{2m_1}\cdots{x_r}^{2m_r}}{(c)_{2m_1+\cdots+2m_r}\,(2m_1)!\cdots(2m_r)!}$$

$$\times \frac{2^{2m_1}\Gamma\,(\frac{1}{2})\Gamma\,(1-2m_1-b_1)}{\Gamma\,(1-m_1-b_1)\Gamma\,(\frac{1}{2}-m_1)} \times \cdots \times \frac{2^{2m_r}\Gamma\,(\frac{1}{2})\Gamma\,(1-2m_r-b_r)}{\Gamma\,(1-m_r-b_r)\Gamma\,(\frac{1}{2}-m_r)} \tag{27}$$

Now, in (27), if we use the results (10)-(15), then after some simplification we obtain the following transformation formula:

$F_D^{(2r)}(a\,,\ b_1, b_1, b_2, b_2\,, \cdots, b_r, b_r; c\,; x_1, -x_1, x_2, -x_2, \cdots, x_r, -x_r)$

$$= F\begin{matrix} 2:\ 1;\ \cdots\ ;\ 1 \\ 2:\ 0;\ \cdots\ ;\ 0 \end{matrix}\left[\begin{matrix} \frac{1}{2}a\,, \frac{1}{2}a+\frac{1}{2}\ :\ b_1\ ;\ \cdots\ ;\ b_r\ ; \\ \frac{1}{2}c, \frac{1}{2}c+\frac{1}{2}\ :\ -\ ;\ \cdots\ ;\ -\ ; \end{matrix}\ x_1^2, \cdots, x_r^2\right] \tag{28}$$

Further, in (28) if we set $x_1 = x_2 = \cdots = x_r = x$ and use the result [11], (see also [13])

$$F\begin{matrix} p\ :\ 1\ ;\ \cdots\ ;\ 1 \\ q\ :\ 0\ ;\ \cdots\ ;\ 0 \end{matrix}\left[\begin{matrix} a_1\,, \ldots, a_p\ :\ c_1\ ;\ \cdots\ ;\ c_r\ ; \\ b_1\,, \ldots, b_q\ :\ -\ ;\ \cdots\ ;\ -\ ; \end{matrix}\ x\,, \cdots, x\right] = {}_{P+1}F_q\left[\begin{matrix} a_1\,, \ldots, a_p\,, c_1+\cdots+c_r\ ; \\ b_1\,, \ldots, b_q\ ; \end{matrix}\ x\right] \tag{29}$$

we get

$$F_D^{(2r)}(a\,,\ b_1, b_1, b_2, b_2\,, \cdots, b_r, b_r; c\,; x, -x, \cdots, x, -x) = {}_3F_2\left[\begin{matrix} \frac{1}{2}a\,, \frac{1}{2}a+\frac{1}{2}\,, b_1+\cdots+b_r\ ; \\ \frac{1}{2}c, \frac{1}{2}c+\frac{1}{2}\ ; \end{matrix}\ x^2\right] \tag{30}$$

Similarly , in (17) if we take $i = 0$ and use the results (10)-(15), then we have

$F_D^{(2r+1)}(a\,, b_1, b_1, b_2, b_2\,, \cdots, b_r, b_r, c; d; x_1, -x_1, x_2, -x_2, \cdots, x_r, -x_r, y)$

$$= F\begin{matrix} 1\ :\ 1\ ;\ \cdots\ ;\ 1 \\ 1\ :\ 0\ ;\ \cdots\ ;\ 0 \end{matrix}\left[\begin{matrix} (a\ :2, \cdots, 2, 1)\ :\ (b_1:1)\ ;\ \cdots\ ;\ (b_r:1)\ ;\ (c:1)\ ; \\ (d\ :2, \cdots, 2, 1)\ :\ -\ ;\ \cdots\ ;\ -\ ;\ -\ ; \end{matrix}\ x_1^2, \cdots, x_r^2, y\right] \tag{31}$$

In (16) if we take $r = 1$, then we get the following eleven transformation formulas of Appell's function F_1:

$F_1(a\,,\ b-i, b\ ;\ d\,; x, -x)$

$$= \sum_{m=0}^{\infty} \frac{(a)_{2m}\,(b-i)_{2m}\,x^{2m}}{(d)_{2m}(2m)!} \times I(b, i\,, 2m)\left\{\frac{A_i'}{A(b, i,\, 2m)} + \frac{B_i'}{B(c, i,\, 2m)}\right\}$$

$$+ \sum_{m=0}^{\infty} \frac{(a)_{2m+1}(b-i)_{2m+1}\,x^{2m+1}}{(d)_{2m+1}(2m+1)!} \times I(b, i\,,\ 2m+1)\left\{\frac{A_i''}{A(b, i,\, 2m+1)} + \frac{B_i''}{B(b, i,\, 2m+1)}\right\}, \tag{32}$$

for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5.$

Similarly, in (17) if we take $r = 1$, then we get the following eleven transformation formulas of Lauricella's function of three variables $F_D^{(3)}$:

$$F_D^{(3)}\,(a\,,\ b-i,\,b\,,c;\,d\,;x,-x,y)$$

$$= \sum_{m\,=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{2m+n}\,(b-i)_{2m}(c)_n\,x^{2m}\,y^n}{(d)_{2m+n}\,(2m)\,!\,n\,!}\times I(b,i\,,\,2m)\left\{\frac{A_i'}{A(b,i,\,2m)}+\frac{B_i'}{B(c,i,\,2m)}\right\}$$

$$+\sum_{m\,=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{2m+n+1}\,(b-i)_{2m+1}(c)_n\,x^{2m+1}y^n}{(d)_{2m+n+1}\,(2m+1)\,!\,n\,!}\times I(b,i\,,\,\,2m+1)\left\{\frac{A_i''}{A(b,i,\,2m+1)}+\frac{B_i''}{B(b,i,\,2m+1)}\right\} \tag{33}$$

for $i\,=0,\,\pm1,\,\pm2,\,\pm3,\,\pm4,\,\pm5$.

Finally, we present the following simple special cases of (32) and (33), when $i=0$:

$$F_1\,(a\,,\ b,\,b\,;\,d\,;x,-x)\ = {}_3F_2\left[\begin{matrix}\frac{1}{2}a\,,\,\frac{1}{2}a+\frac{1}{2}\,,b & ; \\ \frac{1}{2}d\,,\,\frac{1}{2}d+\frac{1}{2} & ;\end{matrix}\ x^2\right] \tag{34}$$

which is a special case of a known result of Buschman and Srivastava [6]

$$F_D^{(3)}\ (a\,,\,b\,,\,b,c\,;d\,;\ x\ ,-x\,,y) = X\begin{matrix}1 & : & 1 & ; & 1\\ 1 & : & 0 & ; & 0\end{matrix}\left[\begin{matrix}a & : & b & ; & c & ;\\ d & : & - & ; & - & ;\end{matrix}\ \ x^2,y\right] \tag{35}$$

where $X_{E:G;H}^{A:B;D}\,[x,y]$ is double hypergeometric series of Exton [7]

$$X\begin{matrix}A: & B; & B'\\ C: & D; & D'\end{matrix}\left[\begin{matrix}(a): & (b); & (b');\\ (c): & (d); & (d');\end{matrix}\ \ x,\,y\right] = \sum_{m,\,n\,=\,0}^{\infty}\frac{((a))_{2m+n}\,((b))_m\,((b'))_n\,x^m\,y^n}{((c))_{2m\,+\,n}\,((d))_m\,((d'))_n\,\,m\,!\,n\,!} \tag{36}$$

The other special cases of (32) and (33) for $i\ =\ \pm1,\,\pm2,\,\pm3,\,\pm4,\,\pm5$ can also be obtained in a similar manner.

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