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Indefinitely Oscillating Functions-Part I

Research Article

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Abstract: This article details a class of indefinitely oscillating functions in $H^s(\mathbb{R})$. It is a class of functions of the Sobolev space $H^s(\mathbb{R})$ which have for all m integer one primitive of the order m in the same space.

Keywords: Class of indefinitely oscillating functions, Sobolev space, primitive of the order m .

1. Introduction

This work is an extension of the previous study [9]. We introduce chirp's functional spaces using Sobolev spaces. We observe that a chirp is an asymptotic signal which is of the form $s(t) = A(t)\, e^{i\lambda\Phi(t)}$, where A and Φ are two smooth functions and $\lambda \gg 1$ (actually $\Phi'(t) \to \infty$ when $t \to t_0$). The function $e^{i\cdot}$ is fundamental in the last definition. It is an indefinitely oscillating function in the L^∞-sense. It will be replaced by what we call an indefinitely oscillating function. Meyer and Xu have worked on chirps using L^p indefinitely oscillating functions defined on $\mathbb{R}^n$ (see [10]). Our contribution consists to study the behavior of the Fourier transform of indefinitely oscillating functions in H^s on $\mathbb{R}$ around 0.

The motivation for studying indefinitely oscillating functions is given by chirps. The first example here considered is the cry of a bat. The signal is given by the formula: $F(x) = e^{\frac{-i}{x}} - 1$ which is a function of the real variable x. Its Fourier transform is given on the real axis by $\widehat{F}(\xi) = J_1\left(\xi^{\frac{1}{2}}\right)^{\frac{1}{2}}/\xi$ if $\xi > 0$ and $\widehat{F}(\xi) = 0$ otherwise, where J_1 is the Bessel function of index one. $\widehat{F}$ has a discontinuity at the origin, which is obviously shown by the fact that $e^{\frac{-i}{x}} - 1 \sim \frac{-i}{x}$ at infinity. A second example is the emission of chirps when vibrating lorries are used to localize petroleum fields. It concerns signals with a large range of frequencies but with a short life. The detection is possible for a large range of objects, avoiding the interferences, thanks to short duration of these signals. The last example is given by gravitational waves. The existence of such waves follows from the theory of general relativity. The scientific world has already got indirect evidences of their existence. But the gravitational waves have never been measured by experiences. Several sources are susceptible to product these gravitational waves: coalescence of a binary star giving birth to a chirp, collapse of neutrons star and collapse of black holes. (cf. [13]).

In this work, we will consider the case where the functions defined on the whole real axis. Throughout the paper, we will systematically study the case of Sobolev spaces, that is $E = H^s(\mathbb{R})$.

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2. Indefinitely Oscillating Function Theory on the Real Axis

The general definition of an indefinitely oscillating function is the following.

Definition 2.1. *Let E a functional Banach space, with $S(\mathbb{R}) \subset E \subset S'(\mathbb{R})$, where $S(\mathbb{R})$ is the Schwartz space, assuming that the two embeddings are continuous. A function $f \in E$ is indefinitely oscillating (in the E-sense) if, for every $m \in \mathbb{N}$, there exists $f_m \in E$ such that $f = \left(\frac{df_m}{dx}\right)^m$.*

Example 2.2. *Assume that $E = L^2(\mathbb{R})$ and let us explain the nature of the difficulties which occur. The above situation can be studied using Fourier's transform. The preceding definition implies $\widehat{f}(\xi) = (i\xi)^m \widehat{f_m}(\xi)$. Then $f \in L^2(\mathbb{R})$ is indefinitely oscillating in the $L^2(\mathbb{R})$-sense if and only if $\int_{-\epsilon}^{\epsilon} \left|\widehat{f}(\xi)\right|^2 d\xi = O(\epsilon^m)$ for every $m \in \mathbb{N}$.*

2.1. Characterization Using Fourier's Transform

2.1.1. The $L^2(\mathbb{R})$ or $H^s(\mathbb{R})$ cases

Studying the behavior of the Fourier transform of the function around zero is another way to characterize indefinitely oscillating functions in the $H^s(\mathbb{R})$-sense.

Lemma 2.3. *One has the following characterizations of an indefinitely oscillating function in the $H^s(\mathbb{R})$-sense.*

(1). f is an indefinitely oscillating function in the $H^s(\mathbb{R})$-sense if and only if f belongs to $H^s(\mathbb{R})$ and one has for every $m \in \mathbb{N}$

$$\int_{-1}^{1} \frac{\left|\widehat{f}(\xi)\right|^2}{|\xi|^{2m}} d\xi < \infty,$$

where $\widehat{f}$ denotes the Fourier transform of f.

(2). f is an indefinitely oscillating function in $H^s(\mathbb{R})$ if and only if f belongs to $H^s(\mathbb{R})$ and for every integer m and every $|\xi| \leq 1$, one has $\int_{-\xi}^{\xi} \left|\widehat{f}(t)\right|^2 dt = O(|\xi|^m)$.

Proof. (1). Using the above definition of an indefinitely oscillating function 2.1, we observe that, for every $m \in \mathbb{N}$, there exists $f_m \in H^s(\mathbb{R})$ such that $f = \frac{d^m f_m}{dx^m}$. Applying Fourier's transform, one has $\widehat{f}(\xi) = \widehat{f}_m(\xi)(i\xi)^m$. A necessary and sufficient condition that f is an indefinitely oscillating function in the $H^s(\mathbb{R})$-sense is that for every $m \in \mathbb{N}$ one has the above indicated property.

(2). Assume that f is an indefinitely oscillating function in $H^s(\mathbb{R})$. We observe that

$$\int_{-\xi}^{\xi} \left|\widehat{f}(t)\right|^2 dt = \int_{-\xi}^{\xi} \frac{\left|\widehat{f}(t)\right|^2}{|t|^{2m}} |t|^{2m} dt \leq |\xi|^{2m} \int_{-\xi}^{\xi} \frac{\left|\widehat{f}(t)\right|^2}{|t|^{2m}} dt.$$

Let us prove the reverse way. A dyadic decomposition gives

$$\int_{2^{-j}}^{2^{-j+1}} \frac{\left|\widehat{f}(\xi)\right|^2}{|\xi|^{2m}} d\xi \leq 2^{2jm} \int_{2^{-j}}^{2^{-j+1}} \left|\widehat{f}(\xi)\right|^2 d\xi \leq C 2^{j(2m-q)}.$$

This is a normally convergent series for $q > 2m$. □

Remark 2.4. *The condition indicated in Lemma 2.3 does not depend on s. Hence, the characterizations of indefinitely oscillating functions either in $L^s(\mathbb{R})$ or in $H^s(\mathbb{R})$ are the same.*

2.2. Characterization Using Littlewood-Paley Analysis

2.2.1. The $L^2(\mathbb{R})$-case

Theorem 2.5. *The three following properties are equivalent:*

(1). f belongs to $L^2(\mathbb{R})$ and $\|\Delta_j(f)\|_2 \leq C_N 2^{jN}$ for $j \leq -1$

(2). f belongs to $L^2(\mathbb{R})$ and for every $n \in \mathbb{N}$ $f = \frac{d^m f_m}{dx^m}$ where $f_m \in L^2(\mathbb{R})$.

(3). For every $m \in \mathbb{N}$ $f = \frac{d^m f_m}{dx^m}$ with $f_m \in H^m(\mathbb{R})$.

Proof. (1) implies (2) The Littlewood-Paley decomposition gives $f = \sum_j \Delta_j(f)$. Let f_m be a m-th primitive defined by $f = \frac{d^m f_m}{dx^m}$. If f and f_m belong to $L^2(\mathbb{R})$ and $f = \frac{d^m f_m}{dx^m}$, one has $\widehat{f}(\xi) = (i\xi)^m \widehat{f}_m(\xi)$, with $\widehat{f}(\xi) = \sum_j \widehat{\Delta_j(f)}(\xi)$. $\widehat{f}$ and $\widehat{f}_m$ belong to $L^2(\mathbb{R})$. Then $\widehat{f}_m$ is defined almost everywhere by $\widehat{f}_m(\xi) = \frac{\widehat{f}(\xi)}{(i\xi)^m}$, and $\widehat{\Delta_j(f)}(\xi) = \widehat{f * \psi_j}(\xi) = \widehat{f}(\xi)\widehat{\psi}\left(\frac{\xi}{2^j}\right)$. $\widehat{\Delta_j(f)}$ is supported by $\alpha 2^j \leq |\xi| \leq \beta 2^j$ with $0 < \alpha < \beta$. One has $\widehat{f}_m(\xi) = \sum_j \frac{\widehat{\Delta_j(f)}(\xi)}{(i\xi)^m}$ and $\left\| \frac{\widehat{\Delta(_j f)}(\xi)}{(i\xi)^m} \right\|_2 \leq C_m 2^{-jm} \|\Delta_j f\|_2$.

(2) implies (1) Let m be a positive integer, f belong to $L^2(\mathbb{R})$. For every $m \in \mathbb{N}$ one has $f = \frac{d^m f_m}{dx^m}$ where f_m and f belong to $L^2(\mathbb{R})$. Then one has $\widehat{f}(\xi)\widehat{\psi}(2^{-j}\xi) = (i\xi)^m \widehat{f}_m(\xi)\widehat{\psi}(2^{-j}\xi)$. We deduce that

$$\|\Delta_j f\|_2 = \left\|\widehat{\Delta_j(f)}\right\|_2 = \left\|(i.)^m \widehat{f}_m \widehat{\psi}\left(2^{-j}.\right)\right\|_2 \leq C_m 2^{jm} \left\|\widehat{f}_m \widehat{\psi}\left(2^{-j}.\right)\right\|_2.$$

For $j \leq -1$ one has $\|\Delta_j f\|_2 \leq C_m 2^{jm}$ and for $j \geq 0$

$$\|\Delta_j f\|_2 \leq C_m 2^{jm} \left\|\widehat{f}_m \widehat{\psi}\left(2^{-j}.\right)\right\|_2 \leq \epsilon_{j,m},$$

with $\sum_0^{+\infty} \epsilon_{j,m}^2 < \infty$, as f belongs to $L^2(\mathbb{R})$. □

2.2.2. The $H^s(\mathbb{R})$-case

Theorem 2.6. *The two following properties are equivalent:*

(1). For every n, there exists $f_n \in H^s(\mathbb{R})$ such that $f = \frac{d^n f_n}{dx^n}$ in a distributional sense.

(2). $f \in H^s(\mathbb{R})$ and $\|\Delta_j f\|_2 \leq C_N 2^{jN}$ for every N and for every $j \leq -1$.

We say that f is indefinitely oscillating in the $H^s(\mathbb{R})$-sense. We remark that $f_n \in H^{s+n}(\mathbb{R})$.

Proof. Let us first observe that if $\widehat{f}$ has its support in $[\alpha, \beta]$ with $0 < \alpha < \beta$ then $\|f\|_2 \simeq \|f\|_{H^s}$ for every s in $\mathbb{R}$.

(1) implies (2) Take $f \in H^s(\mathbb{R})$. For every n, there exists $f_n \in H^s(\mathbb{R})$ such that $f = \frac{d^n f_n}{dx^n}$. Using Fourier's transform one has $\widehat{f}(\xi) = (i\xi)^n \widehat{f}_n(\xi)$, hence $f_n \in H^s(\mathbb{R})$ which is equivalently written as $\int_{\mathbb{R}} \left(1+\xi^2\right)^s \frac{|\widehat{f}(\xi)|^2}{\xi^{2n}} d\xi < \infty$. Hence we have $\int_{-1}^{1} \left|\widehat{f}(\xi)\right|^2 \frac{d\xi}{\xi^{2n}} < \infty$. Using Littlewood-Paley decomposition we can write $f = \sum_{-\infty}^{-1} \Delta_j f + S_0(f)$. Applying Fourier's transform we have $\widehat{f}(\xi) = \sum_{-\infty}^{+\infty} \widehat{\Delta}_j f(\xi)$. As $\widehat{\Delta_j f}$ is supported by the dyadic corona $\alpha 2^j \leq |\xi| \leq \beta 2^j$ with $0 < \alpha < \beta$. Hence the inequality $\int_{-1}^{1} \left|\widehat{f}(\xi)\right|^2 \frac{d\xi}{\xi^{2n}} < \infty$ becomes $\int_{-1}^{1} \left|\sum_{-\infty}^{+\infty} \widehat{\Delta_j f}(\xi)\right|^2 \frac{d\xi}{\xi^{2n}} < \infty$. Thanks to the quasi-orthogonality of the terms, we have $\sum_{-\infty}^{+\infty} \int_{-1}^{1} \left|\widehat{\Delta_j f}(\xi)\right|^2 \frac{d\xi}{\xi^{2n}} < \infty$. In the same way we have

$$\int_{\alpha 2^j}^{\beta 2^j} \left|\widehat{\Delta_j f}(\xi)\right|^2 \frac{d\xi}{\xi^{2n}} < \infty \Rightarrow \left\|\widehat{\Delta_j f}\right\|_2 \leq C_n 2^{jn},$$

for all n and for all $j \leq -1$.

(2) implies (1) Let f_n be such that $f = \frac{d^n f_n}{dx^n}$. Using Fourier's transform we have $\widehat{f}(\xi) = (i\xi)^n \widehat{f_n}(\xi)$. As $f \in H^s(\mathbb{R})$, we have $\int_{\mathbb{R}} (1+\xi^2)^s \left|\widehat{f}(\xi)\right|^2 d\xi < \infty$, or $\int_{\mathbb{R}} (1+\xi^2)^s |\xi|^{2n} \left|\widehat{f_n}(\xi)\right|^2 d\xi < \infty$, which implies $\int_{V(\infty)} (\xi^2)^{n+s} \left|\widehat{f_n}(\xi)\right|^2 d\xi < \infty$. Now let us verify that $\int_{-1}^{1} \left|\widehat{f_n}(\xi)\right|^2 d\xi < \infty$ which proves that $f_n \in H^{s+n}(\mathbb{R})$. The Littlewood-Paley decomposition of f_n is given by $f_n = \sum\limits_{-\infty}^{+\infty} \Delta_j f_n$. Using Fourier's transform, we have

$$\widehat{f_n}(\xi) = \sum_{-\infty}^{-1} \widehat{\Delta}_j\ f_n(\xi) + \sum_{0}^{+\infty} \widehat{\Delta}_j\ f_n(\xi).$$

We now prove that the two terms of the right hand-side belong to $L^2(-1,1)$. As the supports of the different terms are two by two disjoints, we have

$$\int_{-1}^{1} \left|\sum_{-\infty}^{-1} \widehat{\Delta}_j\ f_n(\xi)\right|^2 d\xi \leq 2\int_{-1}^{1} \sum_{-\infty}^{-1} \left|\widehat{\Delta}_j\ f_n(\xi)\right|^2 d\xi.$$

Then we have

$$\int_{-1}^{1} \left|\sum_{-\infty}^{-1} \widehat{\Delta}_j\ f_n(\xi)\right|^2 d\xi \leq 2\sum_{-\infty}^{-1} \int_{\alpha 2^j}^{\beta 2^j} \frac{\left|\widehat{\Delta}_j\ f(\xi)\right|^2}{\xi^{2n}} d\xi \leq C_{(n,N)} \sum_{-\infty}^{-1} 2^{2j(N-n)}.$$

For an appropriate choice of N, this series is normally convergent. Concerning the other term we have, again because the supports are disjoints

$$\int_{-1}^{1} \left|\sum_{0}^{+\infty} \widehat{\Delta}_j\ f_n(\xi)\right|^2 d\xi \leq 2\int_{-1}^{1} \sum_{0}^{+\infty} \left|\widehat{\Delta}_j\ f_n(\xi)\right|^2 d\xi \leq 2\int_{-1}^{1} \sum_{0}^{+\infty} \frac{\left|\widehat{\Delta}_j\ f(\xi)\right|^2}{\xi^{2n}} d\xi.$$

But $\left\|\widehat{\Delta}_j(f)(\xi)\right\|_2 \leq \epsilon_j$ for $j \geq 0$ with $\sum\limits_{0}^{+\infty} \epsilon_j^2 < \infty$ which implies

$$\int_{-1}^{1} \left|\sum_{0}^{+\infty} \widehat{\Delta}_j\ f_n(\xi)\right|^2 d\xi \leq 2\sum_{0}^{+\infty} \int_{\alpha 2^j}^{\beta 2^j} \frac{\left|\widehat{\Delta}_j\ f(\xi)\right|^2}{\xi^{2n}} d\xi \leq 2\sum_{0}^{+\infty} C_n \epsilon_j^2 2^{-2jn}.$$

□

2.3. Generalization to an Arbitrary Banach Space

Let E be a functional Banach space with $S(\mathbb{R}) \subset E \subset S'(\mathbb{R})$. We assume that the norm of E is invariant by translation and that E satisfies the following property: for every sequence $(f_j)_j$ such that $f_j \in E$, $\|f_j\|_E \leq C$ and $(f_j)_j$ converges to f in $\sigma(S', S)$-sense, then $f \in E$ and $\|f\|_E \leq C$. We observe that one has $\|f\|_E \leq \limsup_{j\to\infty} \|f_j\|_E$ when $(f_j)_j$ converges to f in the distributional sense.

Definition 2.7. *A function $f \in E$ is indefinitely oscillating relatively to E if, for every $m \geq 1$, there exists $f_m \in E$ such that $f = \left(\frac{df_m}{dx}\right)^m$.*

One has the following property.

Theorem 2.8. *The three following properties are equivalent:*

(1). f is indefinitely oscillating in the E-sense.

(2). $\|\Delta_j(f)\|_E \leq C_m 2^{jm}$ for all $m \geq 0$ and all $j \leq 0$ and $f = \sum\limits_{-\infty}^{+\infty} \Delta_j(f)$ in $\sigma(S'(\mathbb{R}), S(\mathbb{R}))$-sense.

(3). $\|S_j(f)\|_E \leq C'_m 2^{jm}$ *for all* $m \geq 0$ *and all* $j \leq 0$.

Proof. Let us first observe that for every $\omega \in L^1(\mathbb{R})$ and $f \in E$, $f * \omega \in E$ and $\|f * \omega\|_E \leq \|\omega\|_1 \|f\|_E$.

(2) implies (1) Let $\widetilde{\psi}$ be a function of the Schwartz class $S(\mathbb{R})$ whose Fourier transform is equal to 1 over $\frac{1}{4} \leq |\xi| \leq 4$ and 0 if $|\xi| \geq \frac{1}{10}$ and $|\xi| \geq 10$. The Fourier transform of ψ is taken by $\frac{1}{3} \leq |\xi| \leq 3$. One then writes $\Delta_j(f) = \widetilde{\Delta}_j(\Delta_j f) = \left(\frac{d}{dx}\right)^m 2^{-jm}\widetilde{\Delta}_{j(m)}$, or by Fourier transform $\widetilde{\psi}(2^{-j}\xi) = \left(i\xi 2^{-j}\right)^m \widetilde{\psi}_{(m)}(2^{-j}\xi)$, which means $\widetilde{\psi}(\xi) = (i\xi)^m \widetilde{\psi}_m(\xi)$. It is then evident that $\widetilde{\psi}_m \in S(\mathbb{R})$. One applies Lemma 7 and get $\Delta_j(f) = 2^{-jm}\left(\frac{d}{dx}\right)^m f_{j,m}$, where $\|f_{j,m}\|_E \leq C_N 2^{jN}$ for all integer N. Then $\sum_{-\infty}^{0} 2^{-jm} f_{j,m}$ converges in the E-norm. Let $\sigma_q(f) = \sum_{j \geq -q} \Delta_j f$. One has $\sigma_q(f) \to f$ (in the distributional sense) when $q \to +\infty$, $\sigma_q(f) = \left(\frac{d}{dx}\right)^m \sigma_{q,m}(f)$ and $\sigma_{q,m}(f) \to I_m(f)$ when $q \to +\infty$. Then $\left(\frac{dI_m(f)}{dx}\right)^m = f$.

(1) implies (3) Hence $f = \left(\frac{d}{dx}\right)^m f_m$ and $f_m \in E$. Then $S_j(f) = \left(\frac{dS_j(f_m)}{dx}\right)^m$ and $\left(\frac{dS_j}{dx}\right)^m = 2^{jm} S_j^{(m)}$. $S_j^{(m)}$ is the convolution with $2^j \varphi^{(m)}\left(2^j x\right)$ where $\varphi^{(m)} = \left(\frac{d\varphi}{dx}\right)^m$. We then apply Lemma ?.

(3) implies (2) This implication is evident since $\Delta_j = S_{j+1} - S_j$. $\square$

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