Int. J. Math. And Appl., 10(4)(2022), 1–8
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

# A Fixed Point Theorem for Generalized $(\psi - \varphi)$ Weak Contraction Mappings Involving Rational Type Expressions in Partial Metric Spaces

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**Abstract:** The aim of this paper is to establish a fixed point theorem in partial metric spaces using generalized $(\psi - \varphi)$ weak contraction mappings with rational expressions. Our result generalizes the result of Karapinar E. et al. [13]. In the last section we gave an example in support of our theorem.

**MSC:** 47H10, 54H25.

**Keywords:** Fixed point theorem, $(\psi - \varphi)$ weak contraction mapping, partial metric spaces.



## 1. Introduction

Matthews ([1, 2]) in 1992 introduced The notion of a partial metric space (PMS) as a generalizations of the notion of metric space in which the self distance is not necessarily zero and proved the partial metric version of Banach fixed point theorem. The concept of weak contraction was defined by Alber and Guerre-Delabriere ([3]) in 1997, and the generalization of $\varphi$-weak contraction was defined by Zhang and Song ([4]) in 2009. Dutta and Choudhury ([5]) in 2008 introduced $(\psi - \varphi)$ weak contraction as a extension of weak contraction in complete metric spaces. In 2009, Doric ([6]) generalized $(\psi - \varphi)$ weak contraction and proved some theorems on the existence of a fixed point.

**Theorem 1.1** ([5]). *Let $(X, d)$ be a complete metric space and $T : X \to X$ be a self-map satisfying the inequality*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)) \tag{1}$$

*for all $x, y \in X$, where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are both continuous and nondecreasing functions with $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. Then $T$ has a unique fixed point in $X$.*

In 2009, Doric ([6]) generalized Theorem 1.1 as follows:

**Theorem 1.2** ([6]). *Let $(X, d)$ be a complete metric space and $T : X \to X$ be a self-map satisfying the inequality*

$$\psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y)) \tag{2}$$

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*for all* $x, y \in X$, *where* $M$ *is given by*

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty) + (y,Tx)]\right\} \tag{3}$$

*and* $\psi : [0,\infty) \to [0,\infty)$ *is continuous monotone nondecreasing function with* $\psi(t) = 0$ *if and only if* $t = 0$ *and* $\varphi : [0,\infty) \to [0,\infty)$ *is lower semi-continuous function with* $\varphi(t) = 0$ *if and only if* $t = 0$. *Then* $T$ *has a unique fixed point in* $X$.

Recently, Karapinar et al. [13] proved some fixed point theorems for mappings satisfying rational type contractive condition using auxiliary functions as follows.

**Theorem 1.3** ([13]). *Let* $(X,\rho)$ *be a complete partial metric space and* $T : X \to X$ *be a self-map satisfying the inequality*

$$\psi(\rho(Tx,Ty)) \leq \psi(M(x,y)) - \varphi(M(x,y)) \tag{4}$$

*for all* $x, y \in X$, *where* $M$ *is given by*

$$M(x,y) = \max\left\{\rho(y,Ty)\frac{1+\rho(x,Tx)}{1+\rho(x,y)}, \rho(x,y)\right\} \tag{5}$$

*and* $\psi : [0,\infty) \to [0,\infty)$ *is continuous monotone nondecreasing function with* $\psi(t) = 0$ *if and only if* $t = 0$ *and* $\varphi : [0,\infty) \to [0,\infty)$ *is lower semi-continuous function with* $\varphi(t) = 0$ *if and only if* $t = 0$. *Then* $T$ *has a unique fixed point in* $X$.

**Definition 1.4** ([1]). *Let* $X$ *be a non-empty set. A function* $\rho : X \times X \to [0,\infty)$ *is said to be a partial metric on* $X$ *if the following conditions hold:*

*(i)* $x = y$ *if and only if* $\rho(x,x) = \rho(y,y) = \rho(x,y)$;

*(ii)* $\rho(x,x) \leq \rho(x,y)$;

*(iii)* $\rho(x,y) = \rho(y,x)$;

*(iv)* $\rho(x,y) \leq \rho(x,z) + \rho(z,y) - \rho(z,z)$ *for all* $x, y, z \in X$.

*The set* $X$ *equipped with the metric* $\rho$ *defined above is called a partial metric space and it is denoted by* $(X,\rho)$ *(in short PMS). Each partial metric* $\rho$ *on* $X$ *generates a* $T_0$ *topology* $\tau_\rho$ *on* $X$, *which has a base of the family of open* $\rho$*-balls* $\{B_\rho(x,\epsilon) : x \in X, \epsilon > 0\}$, *where* $B_\rho(x,\epsilon) = \{y \in X : \rho(x,y) < \rho(x,x) + \epsilon\}$ *for all* $x \in X$ *and* $\epsilon > 0$.

**Example 1.5** ([11]). *Let* $X = \{[a,b] : a, b \in \mathbb{R}, a \leq b\}$ *and define* $\rho([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$. *Then* $(X,\rho)$ *is a partial metric space.*

**Example 1.6** ([11]). *Let* $X = [0,\infty)$ *and define* $\rho(x,y) = \max\{x,y\}$. *Then Then* $(X,\rho)$ *is a partial metric space.*

**Lemma 1.7** ([1]). *Let* $(X,\rho)$ *be a partial metric space.*

*(a) A sequence* $\{x_n\}$ *in* $(X,\rho)$ *converges to a point* $x \in X$ *if and only if* $\rho(x,x) = \lim_{n\to\infty} \rho(x_n,x)$;

*(b) A sequence* $\{x_n\}$ *in* $(X,\rho)$ *is a Cauchy sequence if* $\lim_{m,n\to\infty} \rho(x_n,x_m)$ *exists and finite;*

*(c)* $(X,\rho)$ *is complete if every Cauchy* $\{x_n\}$ *in* $X$ *converges to a point* $x \in X$, *such that* $\rho(x,x) = \lim_{m,n\to\infty} \rho(x_m,x_n) = \lim_{n\to\infty} \rho(x_n,x) = \rho(x,x)$.

**Lemma 1.8** ([1, 2, 7]). *Let $\rho$ be a partial metric on $X$, then the functions $d_\rho, d_m : X \times X \to \mathbb{R}^+$ such that $d_\rho(x,y) = 2\rho(x,y) - \rho(x,x) - \rho(y,y)$ and*

$$\begin{aligned} d_m(x,y) =& \max\{\rho(x,y) - \rho(x,x), \rho(x,y) - \rho(y,y)\} \\ =& 2\rho(x,y) - \min\{\rho(x,x), \rho(y,y)\} \end{aligned}$$

*are metric on $X$. Furthermore $(X, d_\rho)$ and $(X, d_m)$ are metric spaces. It is clear that $d_\rho$ and $d_m$ are equivalent. Let $(X,\rho)$ be a partial metric space. Then*

*(1) A sequence $\{x_n\}$ in $(X,\rho)$ is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space $(X, d_\rho)$,*

*(2) $(X,\rho)$ is complete if and only if the metric space $(X, d_\rho)$ is complete. Moreover $\lim_{n\to\infty} d_p(x_n, x) = 0 \Leftrightarrow p(x,x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.*

**Lemma 1.9** ([8]). *Assume that $x_n \to z$ as $n \to \infty$ in a partial metric space $(X,\rho)$ such that $\rho(z,z) = 0$ Then $\lim_{n\to\infty} \rho(x_n, y) = \rho(z, y)$ for every $y \in X$.*

**Lemma 1.10** ([12]). *Let $(X,\rho)$ be a partial metric space.*

1. *if $\rho(x,y) = 0$ then $x = y$,*
2. *If $x \neq y$ then $\rho(x,y) > 0$*

**Lemma 1.11** ([9]). *If $\{x_n\}$ with $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ is not a Cauchy sequence in $(X,\rho)$, and two sequences $\{n(k)\}$ and $\{m(k)\}$ of positive integers such that $n(k) > m(k) > k$, then following four sequences*

$$\begin{aligned} &\rho(x_{m(k)}, x_{n(k)+1}), \rho(x_{m(k)}, x_{n(k)}), \\ &\rho(x_{m(k)-1}, x_{n(k)+1}), \rho(x_{m(k)-1}, x_{n(k)}) \end{aligned}$$

*tend to $\mu > 0$ when $k \to \infty$.*

**Definition 1.12** ([10]). *A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:*

*(a) $\psi$ is non decreasing and continuous.*

*(b) $\psi(t) = 0$ if and only if $t = 0$.*

**Remark 1.13.** *We denote $\Psi$ set of altering distance functions.*

## 2. Main Result

In this section, we shall prove a unique fixed point theorem in the framework of partial metric spaces with rational expressions. Also we provide an example to validate the result.

**Theorem 2.1.** *Let $(X,\rho)$ be a complete partial metric space. $T : X \to X$ be self-mapping satisfying the condition*

$$\psi(\rho(Tx,Ty)) \leq \max\left\{\psi((\rho(x,y)) - \varphi(\rho(x,y)), \psi((M(x,y)) - \varphi(M(x,y))\right\} \tag{6}$$

*For all $x, y \in X$, $\psi, \varphi \in \Psi$ and*

$$M(x,y) = \max\left\{\rho(x,y), \rho(y,Ty)\frac{1+\rho(x,Tx)}{1+\rho(x,y)}, \rho(x,Tx)\frac{1+\rho(x,Tx)}{1+\rho(x,y)}, \frac{\rho(x,Tx)}{1+\rho(x,y)}, \frac{\rho(y,Ty)}{1+\rho(Tx,Ty)}\right\}$$

*Then $T$ has a unique fixed point in $X$.*

*Proof.* Let $x_0$ be an arbitrary point. Suppose we have a sequence $\{x_n\}$ in $X$ such that $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x_n$ is a fixed point of $T$ and the existence part of the proof is finished. Suppose $x_n \neq x_{n+1}$ for every $n \in \mathbb{N}$. Then by (6) we have

$$\psi(\rho(x_n, x_{n+1})) = \psi(\rho(Tx_{n-1}, Tx_n)) \leq \max\left\{\psi((\rho(x_{n-1}, x_n)) - \varphi(\rho(x_{n-1}, x_n)), \psi((M(x_{n-1}, x_n)) - \varphi(M(x_{n-1}, x_n))\right\} \quad (7)$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\left\{\rho(x_{n-1}, x_n), \rho(x_n, Tx_n)\frac{1+\rho(x_{n-1}, Tx_{n-1})}{1+\rho(x_{n-1}, x_n)}, \rho(x_{n-1}, Tx_{n-1})\frac{1+\rho(x_{n-1}, Tx_{n-1})}{1+\rho(x_{n-1}, x_n)}, \frac{\rho(x_{n-1}, Tx_{n-1})}{1+\rho(x_{n-1}, x_n)}, \right. \\ &\quad \left. \frac{\rho(x_n, Tx_n)}{1+\rho(Tx_{n-1}, Tx_n)}\right\} \\ &= \max\left\{\rho(x_{n-1}, x_n), \rho(x_n, x_{n+1})\frac{1+\rho(x_{n-1}, x_n)}{1+\rho(x_{n-1}, x_n)}, \rho(x_{n-1}, x_n)\frac{1+\rho(x_{n-1}, x_n)}{1+\rho(x_{n-1}, x_n)}, \frac{\rho(x_{n-1}, x_n)}{1+\rho(x_{n-1}, x_n)}, \frac{\rho(x_n, x_{n+1})}{1+\rho(x_n, x_{n+1})}\right\} \\ &= \max\left\{\rho(x_n, x_{n+1}), \rho(x_{n-1}, x_n)\right\} \end{aligned}$$

Now we consider the following cases.

**Case 1:** If

$$\max\left\{\psi((\rho(x_{n-1}, x_n)) - \varphi(\rho(x_{n-1}, x_n)), \psi((M(x_{n-1}, x_n)) - \varphi(M(x_{n-1}, x_n))\right\} = \psi((\rho(x_{n-1}, x_n)) - \varphi(\rho(x_{n-1}, x_n)) \quad (8)$$

Then by (7) we have

$$\psi(\rho(x_n, x_{n+1})) \leq \psi((\rho(x_{n-1}, x_n)) - \varphi(\rho(x_{n-1}, x_n)) \leq \psi((\rho(x_{n-1}, x_n)) \quad (9)$$

**Case 2:** If

$$\max\left\{\psi((\rho(x_{n-1}, x_n)) - \varphi(\rho(x_{n-1}, x_n)), \psi((M(x_{n-1}, x_n)) - \varphi(M(x_{n-1}, x_n))\right\} = \psi((M(x_{n-1}, x_n)) - \varphi(M(x_{n-1}, x_n)) \quad (10)$$

where $M(x_{n-1}, x_n) = \max\{\rho(x_n, x_{n+1}), \rho(x_{n-1}, x_n)\}$. Then by (7) we have

$$\psi(\rho(x_n, x_{n+1})) \leq \psi(\max\{\rho(x_n, x_{n+1}), \rho(x_{n-1}, x_n)\}) - \varphi(\max\{\rho(x_n, x_{n+1}), \rho(x_{n-1}, x_n)\})$$

Now, if $\rho(x_n, x_{n+1}) > \rho(x_{n-1}, x_n)$ then $\psi(\rho(x_n, x_{n+1})) \leq \psi(\rho(x_n, x_{n+1}) - \varphi(\rho(x_n, x_{n+1}) < \psi(\rho(x_n, x_{n+1})$, which is a contradiction since $\rho(x_n, x_{n+1}) > 0$ by Lemma 1.10, therefore

$$\psi(\rho(x_n, x_{n+1})) \leq \psi(\rho(x_{n-1}, x_n)) - \varphi(\rho(x_{n-1}, x_n)) \leq \psi(\rho(x_{n-1}, x_n)) \quad (11)$$

so, in both cases we have $\psi(\rho(x_n, x_{n+1})) \leq \psi((\rho(x_{n-1}, x_n))$ i.e., $\{\rho(x_n, x_{n+1}) : n \in \mathbb{N}\}$ is a decreasing sequence of non negative real numbers. Hence it is convergent to a real number, therefore there exists $r_0 \geq 0$ such that $\lim_{n\to\infty} \rho(x_n, x_{n+1}) = r_0$. Let

$r_0 > 0$. Then taking the limit $n \to \infty$ from lemma 1.7, ( 9) and (11), we get, $\psi(r_0) \leq \psi(r_0) - \varphi(r_0) < \psi(r_0)$. This is contradiction. Hence

$$\lim_{n\to\infty} \rho(x_n, x_{n+1}) = 0 \tag{12}$$

Now by Lemma 1.8, we have

$$\begin{aligned} d_\rho(x_n, x_{n+1}) &= 2\rho(x_n, x_{n+1}) - \rho(x_n, x_n) - \rho(x_{n+1}, x_{n+1}) \\ &= 2\rho(x_n, x_{n+1}) - \min\{\rho(x_n, x_n), \rho(x_{n+1}, x_{n+1})\} \\ &\leq 2\rho(x_n, x_{n+1}) \end{aligned}$$

Therefore by lemma 1.8,

$$\lim_{n\to\infty} d_\rho(x_n, x_{n+1}) = 0 \tag{13}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in $X$ i.e., We prove that $\lim_{n,m\to\infty} \rho(x_n, x_m) = 0$. We prove it by contradiction. Let $\lim\limits_{n\to\infty} \rho(x_n, x_m) \neq 0$. Then sequences in lemma 1.11 tends to $\mu > 0$, when $k \to \infty$. So we can see that

$$\lim_{k\to\infty} \rho(x_{m(k)}, x_{n(k)}) = \mu \tag{14}$$

Further corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is smallest integer with $n(k) > m(k) > k$. Then

$$\lim_{k\to\infty} \rho(x_{n(k)-1}, x_{m(k)}) = \mu \tag{15}$$

Again, $\rho(x_{m(k)-1}, x_{n(k-1)}) \leq \rho(x_{m(k)-1}, x_{n(k)}) + \rho(x_{n(k)}, x_{n(k)-1}) - \rho(x_{n(k)}, x_{n(k)})$. Letting $k \to \infty$ and using lemma 1.11, we get,

$$\lim_{k\to\infty} \rho(x_{m(k)-1}, x_{n(k-1)}) = \mu \tag{16}$$

Now in (6) replacing $x$ by $n(k)$ and $y$ by $m(k)$ respectively, we get

$$\begin{aligned} \psi(\rho(x_{n(k)}, x_{m(k)})) = \psi(\rho(Tx_{n(k)-1}, Tx_{m(k)-1})) \leq \max\Big\{&\psi(\rho(x_{n(k)-1}, x_{m(k)-1})) - \varphi(\rho(x_{n(k)-1}, x_{m(k)-1})), \\ &\psi(M(x_{n(k)-1}, x_{m(k)-1})) - \varphi(M(x_{n(k)-1}, x_{m(k)-1}))\Big\} \end{aligned} \tag{17}$$

Where

$$\begin{aligned} M(x_{n(k)-1}, x_{m(k)-1}) = \max\Big\{&\rho(x_{n(k)-1}, x_{m(k)-1}), \rho(x_{m(k)-1}, Tx_{m(k)-1}) \frac{1 + \rho(x_{n(k)-1}, Tx_{n(k)-1})}{1 + \rho(x_{n(k)-1}, x_{m(k)-1})}, \\ &\rho(x_{n(k)-1}, Tx_{n(k)-1}) \frac{1 + \rho(x_{n(k)-1}, Tx_{n(k)-1})}{1 + \rho(x_{n(k)-1}, x_{m(k)-1})}, \frac{\rho(x_{n(k)-1}, Tx_{n(k)-1})}{1 + \rho(x_{n(k)-1}, x_{m(k)-1})}, \\ &\frac{\rho(x_{m(k)-1}, Tx_{m(k)-1})}{1 + \rho(Tx_{n(k)-1}, Tx_{m(k)-1})}\Big\} \\ = \max\Big\{&\rho(x_{n(k)-1}, x_{m(k)-1}), \rho(x_{m(k)-1}, x_{m(k)}) \frac{1 + \rho(x_{n(k)-1}, x_{n(k)})}{1 + \rho(x_{n(k)-1}, x_{m(k)-1})}, \\ &\rho(x_{n(k)-1}, x_{n(k)}) \frac{1 + \rho(x_{n(k)-1}, x_{n(k)})}{1 + \rho(x_{n(k)-1}, x_{m(k)-1})}, \frac{\rho(x_{n(k)-1}, x_{n(k)})}{1 + \rho(x_{n(k)-1}, x_{m(k)-1})}, \frac{\rho(x_{m(k)-1}, x_{m(k)})}{1 + \rho(x_{n(k)}, x_{m(k)})}\Big\} \end{aligned}$$

Letting $k \to \infty$ and using (14), (15) and (16), we get,

$$\lim_{k\to\infty} M(x_{n(k)-1}, x_{m(k)-1}) = \mu \tag{18}$$

Now, letting $k\to\infty$ in (17) and using (16) and (18), we get, $\psi(\mu)\le\max\{\psi(\mu)-\varphi(\mu),\psi(\mu)-\varphi(\mu)\}=\psi(\mu)-\varphi(\mu)<\psi(\mu)$. This is a contradiction, therefore

$$\lim_{n,m\to\infty}\rho(x_n,x_m)=0 \tag{19}$$

i.e.$\{x_n\}$ is a Cauchy sequence in complete partial metric space $(X,\rho)$.Thus by Lemma 1.8 this sequence is also Cauchy in $(X,d_\rho)$ which is complete. Therefore there exists $z\in X$ such that $\lim_{n\to\infty}\rho(x_n,z)=\rho(z,z)$. Also, from Lemma 1.7, Lemma 1.9 and (19) we get

$$0=\lim_{n,m\to\infty}\rho(x_n,x_m)=\rho(x_n,z)=\rho(z,z) \tag{20}$$

implies

$$\lim_{n\to\infty}d_\rho(x_n,z)=0 \tag{21}$$

Now, we will prove that $z$ is a fixed point of $T$ i.e., $Tz=z$. We prove it by contradiction. Let $Tz\neq z$, then from (6) we have

$$\psi(\rho(x_{n+1},Tz))=\psi(\rho(Tx_n,Tz))\le\max\left\{\psi(\rho(x_n,z))-\varphi(\rho(x_n,z),\psi(M(x_n,z))-\varphi(M(x_n,z))\right\} \tag{22}$$

Where

$$\begin{aligned}M(x_n,z)&=\max\left\{\rho(x_n,z),\rho(z,Tz)\frac{1+\rho(x_n,Tx_n)}{1+\rho(x_n,z)},\rho(x_n,Tx_n)\frac{1+\rho(x_n,Tx_n)}{1+\rho(x_n,z)},\frac{\rho(x_n,Tx_n)}{1+\rho(x_n,z)},\frac{\rho(z,Tz)}{1+\rho(Tx_n,Tz)}\right\}\\&=\max\left\{\rho(x_n,z),\rho(z,Tz)\frac{1+\rho(x_n,x_{n+1})}{1+\rho(x_n,z)},\rho(x_n,x_{n+1})\frac{1+\rho(x_n,x_{n+1})}{1+\rho(x_n,z)},\frac{\rho(x_n,x_{n+1})}{1+\rho(x_n,z)},\frac{\rho(z,Tz)}{1+\rho(x_{n+1},Tz)}\right\}\end{aligned} \tag{23}$$

**Case (i):** If $\max\left\{\psi(\rho(x_n,z))-\varphi(\rho(x_n,z),\psi(M(x_n,z)-\varphi(M(x_n,z)\right\}=\psi(\rho(x_n,z))-\varphi(\rho(x_n,z))$. Then, letting $n\to+\infty$ in (22) and using (20), we get,

$$\psi(\rho(x_{n+1},Tz))=\psi(\rho(Tx_n,Tz))\le\psi(0)-\varphi(0)\le\psi(0) \tag{24}$$

So,

$$Tx_n=Tz \tag{25}$$

**Case (ii):** If $\max\left\{\psi(\rho(x_n,z))-\varphi(\rho(x_n,z),\psi(M(x_n,z))-\varphi(M(x_n,z))\right\}=\psi(M(x_n,z))-\varphi(M(x_n,z))$. Then, letting $n\to+\infty$ in (23) and using (12), (20), we get,

$$M(x_n,z)\to\rho(z,Tz) \tag{26}$$

Now letting $n\to+\infty$ in (22) and using (26) with continuity of $\psi$ and $\varphi$, we get, $\psi(\rho(z,Tz))\le\psi(\rho(z,Tz))-\varphi(\rho(z,Tz))<\psi(\rho(z,Tz))$, which is a contradiction. Hence $z\in X$ is a fixed point. Now we are to prove that $z$ is unique. For that let $z_1$ is another fixed point such that $z\neq z_1$. Then, from (6) we have

$$\psi(\rho(z,z_1))=\psi(\rho(Tz,Tz_1))\le\max\left\{\psi(\rho(z,z_1))-\varphi(\rho(z,z_1)),\psi(M(z,z_1))-\varphi(M(z,z_1))\right\} \tag{27}$$

Where

$$\begin{aligned}M(z,z_1)&=\max\left\{\rho(z,z_1),\rho(z_1,Tz_1)\frac{1+\rho(z,Tz)}{1+\rho(z,z_1)},\rho(z,Tz)\frac{1+\rho(z,Tz)}{1+\rho(z,z_1)},\frac{\rho(z,Tz)}{1+\rho(z,z_1)},\frac{\rho(z_1,Tz_1)}{1+\rho(Tz,Tz_1)}\right\}\\&=\max\left\{\rho(z,z_1),\rho(z_1,z_1)\frac{1+\rho(z,z)}{1+\rho(z,z_1)},\rho(z,z)\frac{1+\rho(z,z)}{1+\rho(z,z_1)},\frac{\rho(z,z)}{1+\rho(z,z_1)},\frac{\rho(z_1,z_1)}{1+\rho(z,z_1)}\right\}\end{aligned}$$

$$=\rho(z,z_1) \tag{28}$$

Thus, we get,

$$\max\Big\{\psi(\rho(z,z_1))-\varphi(\rho(z,z_1)),\psi(M(z,z_1))-\varphi(M(z,z_1))\Big\}=\psi(\rho(z,z_1))-\varphi(\rho(z,z_1)) \tag{29}$$

Now from (27) using (29), we get, $\psi(\rho(z,z_1))=\psi(\rho(Tz,Tz_1))\leq\psi(\rho(z,z_1))-\varphi(\rho(z,z_1))<\psi(\rho(z,z_1))$. This is a contradiction. Hence $z=z_1$ that is, the fixed point is unique. This completes the proof. □

Following are consequences of the Theorem.

**Corollary 2.2.** *Let $(X,\rho)$ be a complete partial metric space. $T:X\to X$ be self-mapping satisfying the condition*

$$\psi(\rho(Tx,Ty))\leq\psi(\rho(x,y))-\varphi(\rho(x,y)) \tag{30}$$

*For all $x,y\in X$, $\psi,\varphi\in\Psi$. Then $T$ has a unique fixed point in $X$.*

**Corollary 2.3.** *Let $(X,\rho)$ be a complete partial metric space. $T:X\to X$ be self-mapping satisfying the condition*

$$\psi(\rho(Tx,Ty))\leq\psi((M(x,y))-\varphi(M(x,y)) \tag{31}$$

*For all $x,y\in X$, $\psi,\varphi\in\Psi$ and*

$$M(x,y)=\max\left\{\rho(x,y),\rho(y,Ty)\frac{1+\rho(x,Tx)}{1+\rho(x,y)},\rho(x,Tx)\frac{1+\rho(x,Tx)}{1+\rho(x,y)},\frac{\rho(x,Tx)}{1+\rho(x,y)},\frac{\rho(y,Ty)}{1+\rho(Tx,Ty)}\right\}$$

*Then $T$ has a unique fixed point in $X$.*

**Example 2.4.** *Let $X=[0,1]$ and $\rho(x,y)=\max\{x,y\}$. Then $(X,\rho)$ is a complete metric space. Consider the mapping $T:X\to X$ defined by $T(x)=\frac{x}{10}$ for all $x$ and $\psi(t),\varphi(t):[0,\infty)\to[0,\infty)$ $\psi(t)=t$ and $\varphi(t)=\frac{5t}{6}$. Without loss of generality, assume that $x\geq y$, we have,*

$$\rho(Tx,Ty)=\max\left\{\frac{x}{10},\frac{y}{10}\right\}=\frac{x}{10}$$

*and*

$$\psi(\rho(Tx,Ty))=\psi(\frac{x}{10})=\frac{x}{10} \tag{32}$$

*Now*

$$\begin{aligned}M(x,y)&=\max\left\{\rho(x,y),\rho(y,Ty)\frac{1+\rho(x,Tx)}{1+\rho(x,y)},\rho(x,Tx)\frac{1+\rho(x,Tx)}{1+\rho(x,y)},\frac{\rho(x,Tx)}{1+\rho(x,y)},\frac{\rho(y,Ty)}{1+\rho(Tx,Ty)}\right\}\\&=\max\left\{\rho(x,y),\rho(y,\frac{y}{10})\frac{1+\rho(x,\frac{x}{10})}{1+\rho(x,y)},\rho(x,\frac{x}{10})\frac{1+\rho(x,\frac{x}{10})}{1+\rho(x,y)},\frac{\rho(x,\frac{x}{10})}{1+\rho(x,y)},\frac{\rho(y,\frac{y}{10})}{1+\rho(\frac{x}{10},\frac{y}{10})}\right\}\\&=\max\left\{x,y\frac{1+x}{1+x},x\frac{1+x}{1+x},\frac{x}{1+x},\frac{y}{1+\frac{x}{10}}\right\}=x\end{aligned}$$

*Therefore*

$$\begin{aligned}\max\Big\{\psi((\rho(x,y))-\varphi(\rho(x,y)),\psi((M(x,y))-\varphi(M(x,y))\Big\}&=\max\Big\{\psi(x)-\varphi(x),\psi(x)-\varphi(x)\Big\}\\&=\psi(x)-\varphi(x)\\&=x-\frac{5x}{6}=\frac{x}{6}\end{aligned} \tag{33}$$

*From (32) and (33) it is clear that it satisfies all the conditions of Corollary 2.3. Hence $T$ has a unique fixed point, which in this case is 0.*

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