Int. J. Math. And Appl., 6(2–A)(2018), 13–20
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Existence Results for Katugampola-type Fractional Functional Differential Equations

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Abstract: In this paper, the initial value problem is discussed for a class of Katugampola-type fractional functional differential equations and the criteria on existence are obtained.

MSC: 26A33, 34A12.

Keywords: Katugampola fractional derivative, Fractional functional differential equations, Fractional neutral functional differential equations, Existence, Fixed point.

 Accepted on: 04.03.2018

1. Introduction

In the recent years, the investigation of fractional differential equations (FDEs) has been picking up much attention of researchers. This is due to the fact that FDEs have various applications in engineering and scientific disciplines, for example, fluid dynamics, fractal theory, diffusion in porous media, fractional biological neurons, traffic flow, polymer rheology, neural network modeling, viscoelastic panel in supersonic gas flow, real system characterized by power laws, electrodynamics of complex medium, sandwich system identification, nonlinear oscillation of earthquake, models of population growth, mathematical modeling of the diffusion of discrete particles in a turbulent fluid, nuclear reactors and theory of population dynamics. The FDEs is an important tool to describe the memory and hereditary properties of various materials and phenomena. The details on the theory and its applications may be found in books [6, 10, 12–14]. FDEs and control problems involving the Riemann-Liouville fractional derivative or the Caputo fractional derivative have been paid more and more attention. Very recently, a generalized Caputo-Katugampola derivative was proposed in [7, 8] by Katugampola, and further he proved the existence of solutions of Caputo-Katugampola FDEs in [9].

In addition, some modeling is done via functional differential equations when these processes involve hereditary phenomena such as biological and social macro systems. In [1], Agarwal, Belmekki and Benchohra obtain existence results for semilinear functional differential inclusions involving fractional derivatives. In [3], Benchohra et al. consider the IVP for a class of fractional neutral functional differential equations with infinite delay. In [4], El-Sayed discusses a class of nonlinear functional differential equations of arbitrary orders. In [11], Lakshmikantham initiates the basic theory for fractional functional differential equations. In [16–18], Zhou et al. investigated the existence and uniqueness for fractional functional differential equations with unbounded and infinite delay.

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We consider the Katugampola-type fractional functional differential equations,

$$\begin{cases} {}^{\rho}D^{\omega}_{0+}u(x) = h(x, u_x), & \text{for each } x \in \mathfrak{J} = [0, b];\ 0 < \omega < 1, \\ u(x) = \Phi(x), & x \in [-r, 0], \end{cases} \tag{1}$$

where ${}^{\rho}D^{\omega}_{0+}$ is the Katugampola-type fractional derivative in Caputo sense, $h : \mathfrak{J} \times \mathfrak{C}([-r, 0], \mathbb{R}) \to \mathbb{R}$ is a given function and $\Phi \in \mathfrak{C}([-r, 0], \mathbb{R})$ with $\Phi(0) = 0$. For any function u defined on $[-r, b]$ and any $x \in \mathfrak{J}$, we denote by u_x the element of $\mathfrak{C}([-r, 0], \mathbb{R})$ and is defined by

$$u_x(\theta) = u(x + \theta),\ \theta \in [-r, 0].$$

Here $u_x(.)$ represents the history of state from time $x - r$ upto the present time x. Now, the second problem is considered to the study of Katugampola-type fractional neutral functional differential equations,

$$\begin{cases} {}^{\rho}D^{\omega}_{0+}[u(x) - f(x, u_x)] = h(x, u_x), & x \in \mathfrak{J}, \\ u(x) = \Phi(x), & x \in [-r, 0], \end{cases} \tag{2}$$

where h and Φ are same as in problem (1), and $f : \mathfrak{J} \times \mathfrak{C}([-r, 0], \mathbb{R}) \to \mathbb{R}$ is a given function such that $f(0, \Phi) = 0$. The organization of the paper is as follows: In Section 2, we present some necessary definitions and preliminary results that will be used to prove our main results. In Section 3, the proofs of our main results are given for problem (1), while the existence results for the problem (2) are presented in Section 4. Finally, we will give an example to demonstrate our main results.

2. Prerequisites

In this section, we introduce background definitions and lemmas that are needed for the proof of the main results. We refer the readers to [2, 9, 15]. Let $\mathfrak{C}(\mathfrak{J}, \mathbb{R})$ be the Banach space of all continuous functions from $\mathfrak{J}$ into $\mathbb{R}$ with the norm

$$\|u\|_{\infty} := \sup\{|u(x)| : x \in \mathfrak{J}\},$$

and $\mathfrak{C}([-r, 0], \mathbb{R})$ is endowed with the norm

$$\|\Phi\|_{\mathfrak{C}} := \sup\{|\Phi(\theta)| : -r \le \theta < 0\}.$$

Definition 2.1. *The Riemann-Liouville fractional integral and derivative of order $\omega \in \mathbb{C}$, $Re(\omega) \ge 0$ are given by*

$$(I^{\omega}_{0+}h)(x) = \frac{1}{\Gamma(\omega)} \int_0^x (x - s)^{\omega - 1} h(s) \mathrm{d}s,$$

and

$$(D^{\omega}_{0+}h)(x) = \left(\frac{d}{dx}\right)^n \left(I^{n-\omega}_{0+}h\right)(x),\ x > 0,$$

respectively, where $n = [Re(\omega)]$ and $\Gamma(\omega)$ is the Gamma function.

Definition 2.2. *The generalized left-sided fractional integral ${}^{\rho}I^{\omega}_{0+}h$ of order $\omega \in \mathbb{C}(Re(\omega))$ is defined by*

$$({}^{\rho}I^{\omega}_{0+}h)(x) = \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_0^x (x^{\rho} - s^{\rho})^{\omega - 1} s^{\rho - 1} h(s) \mathrm{d}s, \tag{3}$$

for $x > 0$, if the integral exists.

Definition 2.3. *The generalized fractional derivative, corresponding to the generalized fractional integral* (3)*, is defined by*

$$({}^{\rho}D^{\omega}_{0+}h)(x) = \frac{\rho^{\omega-n+1}}{\Gamma(n-1)}\left(x^{1-\rho}\frac{d}{dx}\right)^n \int_0^x (x^{\rho}-s^{\rho})^{n-\omega-1}s^{\rho-1}h(s)\mathrm{d}s, \tag{4}$$

if the integral exists.

3. Functional Differential Equations

Definition 3.1. *A function $u \in \mathfrak{C}([-r,b],\mathbb{R})$, is said to be a solution of* (1)*, if u satisfies the equation ${}^{\rho}D^{\omega}_{0+}u(x) = h(x,u_x)$ on $\mathfrak{J}$, and the condition $u(x) = \Phi(x)$ on $[-r,0]$.*

The Banach contraction principle is used to prove the existence results for equation (1).

Theorem 3.2. *Let $h : \mathfrak{J}\times\mathfrak{C}([-r,0],\mathbb{R}) \to \mathbb{R}$. Assume that*

(A1) there exists a constant $k > 0$ such that $|h(x,u_1) - h(x,u_2)| \leq k\|u_1 - u_2\|_{\mathfrak{C}}$, for $x \in \mathfrak{J}$ and every $u_1, u_2 \in \mathfrak{C}([-r,0],\mathbb{R})$.

If

$$\frac{kT^{\rho\omega}}{\rho^{\omega}\Gamma(\omega+1)} < 1,$$

then there exists a unique solution for equation (1) *on the interval $[-r,b]$.*

Proof. Transform the problem (1) into a fixed point problem. Consider the operator $M : \mathfrak{C}([-r,b],\mathbb{R}) \to \mathfrak{C}([-r,b],\mathbb{R})$ defined by

$$M(u)(x) = \begin{cases} \Phi(x), & \text{if } x \in [-r,0], \\ \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_0^x (x^{\rho}-s^{\rho})^{\omega-1}s^{\rho-1}h(s,u_s)\mathrm{d}s, & \text{if } x \in [0,b]. \end{cases} \tag{5}$$

Let $u, v \in \mathfrak{C}([-r,b],\mathbb{R})$. Then, for each $x \in \mathfrak{J}$,

$$\begin{aligned} |M(u)(x) - M(v)(x)| &\leq \frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_0^x (x^{\rho}-s^{\rho})^{\omega-1}s^{\rho-1}\left|h(s,u_s) - h(s,v_s)\right|\mathrm{d}s \\ &\leq \frac{k\rho^{1-\omega}}{\Gamma(\omega)}\int_0^x (x^{\rho}-s^{\rho})^{\omega-1}s^{\rho-1}\|u_s - v_s\|_{\mathfrak{C}}\mathrm{d}s \\ &\leq \frac{k\rho^{1-\omega}}{\Gamma(\omega)}\|u-v\|_{[-r,b]}\int_0^x (x^{\rho}-s^{\rho})^{\omega-1}s^{\rho-1}\mathrm{d}s \\ &\leq \frac{kT^{\rho\omega}}{\rho^{\omega}\Gamma(\omega+1)}\|u-v\|_{[-r,b]}. \end{aligned}$$

Consequently,

$$\|M(u) - M(v)\|_{[-r,b]} \leq \frac{kT^{\rho\omega}}{\rho^{\omega}\Gamma(\omega+1)}\|u-v\|_{[-r,b]},$$

which implies that M is a contraction, and hence M has a unique fixed point by Banach's contraction principle. □

Now, the nonlinear alternative of Leray-Schauder type is used to prove the existence results for equation (1).

Lemma 3.3 (Nonlinear alternative for single valued maps [5])**.** *Let E be a Banach space, $\mathcal{C}$ a closed, convex subset of E, $\mathfrak{U}$ an opensubset of $\mathcal{C}$ on $0 \in \mathfrak{U}$. Suppose that $G : \overline{\mathfrak{U}} \to \mathcal{C}$ is a continuous, compact(that is, $G(\overline{\mathfrak{U}})$ is a relatively compact subset of $\mathcal{C}$) map. Then either*

(i) G has a fixed point in $\overline{\mathfrak{U}}$, or

(ii) there is a $u \in \partial\mathfrak{U}$(the boundary of $\mathfrak{U}$ in $\mathcal{C}$) and $\mu \in (0,1)$ with $u = \mu G(u)$.

Theorem 3.4. *Assume that the following hypotheses holds:*

(A2) $h : \mathfrak{J} \times \mathfrak{C}([-r,0],\mathbb{R}) \to \mathbb{R}$ is a continuous function;

(A3) there exist a continuous nondecreasing function $\Psi : [0,\infty) \to (0,\infty)$ and a function $q \in \mathfrak{C}([0,b],\mathbb{R}^+)$ such that

$$|h(x,u)| \leq q(x)\Psi(\|u\|_{\mathfrak{C}}), \text{ for each } (x,u) \in [0,b] \times \mathfrak{C}([-r,0],\mathbb{R});$$

(A4) there exists a constant $N > 0$ such that

$$\frac{N}{\Psi(N)\|q\|_\infty \frac{T^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)}} > 1.$$

Then the equation (1) has at least one solution on $[-r,b]$.

Proof. We consider the operator $M : \mathfrak{C}([-r,b],\mathbb{R}) \to \mathfrak{C}([-r,b],\mathbb{R})$ defined by (5) . We shall prove that the operator M is continuous and completely continuous.

Step 1: M is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $\mathfrak{C}([-r,b],\mathbb{R})$. Let $\Omega > 0$ such that $\|u_n\|_\infty \leq \Omega$. Then,

$$\begin{aligned}
|M(u_n)(x) - M(u)(x)| &\leq \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_0^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} \left|h(s,u_{n s}) - h(s,u_s)\right| \mathrm{d}s \\
&\leq \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_0^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} \sup_{s\in[0,b]} \left|h(s,u_{n s}) - h(s,u_s)\right| \mathrm{d}s \\
&\leq \frac{\rho^{1-\omega}\|h(.,u_n) - h(.,u)\|_\infty}{\Gamma(\omega)} \int_0^b (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} \mathrm{d}s \\
&\leq \frac{T^{\rho\omega}\|h(.,u_{n.}) - h(.,u)\|_\infty}{\rho^\omega\omega\Gamma(\omega)}.
\end{aligned}$$

Since h is a continuous function, we have

$$\|M(u_n) - M(u)\|_\infty \leq \frac{T^{\rho\omega}\|h(.,u_{n.}) - h(.,u)\|_\infty}{\rho^\omega\Gamma(\omega+1)} \to 0, \text{ as } n \to \infty.$$

Step 2: M maps bounded sets into bounded sets in $\mathfrak{C}([-r,b],\mathbb{R})$. It is enough to show that for any $\Omega^* > 0$, there exists a positive constant $\tilde{k}$ such that for each

$$u \in B_{\Omega^*} = \left\{u \in \mathfrak{C}([-r,b],\mathbb{R}) : \|u\|_\infty \leq \Omega^*\right\},$$

we have $\|M(u)\|_\infty \leq \tilde{k}$. By $(A3)$, for each $x \in [0,b]$, we get

$$\begin{aligned}
|M(u)(x)| &\leq \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_0^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} \left|h(s,u_s)\right| \mathrm{d}s \\
&\leq \frac{\rho^{1-\omega}\Psi(\|u\|_{[-r,b]})\|q\|_\infty}{\Gamma(\omega)} \int_0^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} \mathrm{d}s \\
&\leq \frac{T^{\rho\omega}\Psi(\|u\|_{[-r,b]})\|q\|_\infty}{\rho^\omega\Gamma(\omega+1)}
\end{aligned}$$

Thus,

$$\|M(u)\|_\infty \leq \frac{T^{\rho\omega}\Psi(\Omega^*)\|q\|_\infty}{\rho^\omega\Gamma(\omega+1)} := \tilde{k}$$

Step 3: M maps bounded sets into equicontinuous sets of $\mathfrak{C}([-r,b],\mathbb{R})$. Let $x_1, x_2 \in [0,b]$, $x_1 < x_2$, B_{Ω^*} be a bounded set of $\mathfrak{C}([-r,b],\mathbb{R})$ as in Step 2, and let $u \in B_{\Omega^*}$. Then

$$\begin{aligned}|M(u)(x_2) - M(u)(x_1)| \leq & \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_0^{x_1} \left[({x_2}^\rho - s^\rho)^{\omega-1} - ({x_1}^\rho - s^\rho)^{\omega-1} \right] s^{\rho-1} |h(s,u_s)| \mathrm{d}s \\ & + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_1}^{x_2} ({x_2}^\rho - s^\rho)^{\omega-1} s^{\rho-1} |h(s,u_s)| \mathrm{d}s \\ \leq & \frac{\rho^{1-\omega}\Psi(\Omega^*)\|q\|_\infty}{\Gamma(\omega)} \int_0^{x_1} \left[({x_2}^\rho - s^\rho)^{\omega-1} - ({x_1}^\rho - s^\rho)^{\omega-1} \right] s^{\rho-1} \mathrm{d}s \\ & + \frac{\rho^{1-\omega}\Psi(\Omega^*)\|q\|_\infty}{\Gamma(\omega)} \int_{x_1}^{x_2} ({x_2}^\rho - s^\rho)^{\omega-1} s^{\rho-1} \mathrm{d}s.\end{aligned}$$

As $x_1 \to x_2$, the right hand side of the above inequality tends to zero. The equicontinuity for the cases $x_1 < x_2 \leq 0$ and $x_1 \leq 0 \leq x_2$ is obvious. In consequence of Steps 1 to 3, it follows by the Arzelá-Ascoli theorem that $M : \mathfrak{C}([-r,b],\mathbb{R}) \to \mathfrak{C}([-r,b],\mathbb{R})$ is continuous and completely continuous.

Step 4: We prove that there exists an open set $\mathfrak{U} \subseteq \mathfrak{C}([-r,b],\mathbb{R})$ with $u \neq \mu M(u)$ for $\mu \in (0,1)$ and $u \in \partial\mathfrak{U}$. Let $u \in \mathfrak{C}([-r,b],\mathbb{R})$ and $u = \mu M(u)$ for some $0 < \mu < 1$. Thus, for each $x \in [0,b]$,

$$u(x) = \mu \left(\frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_0^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} h(s,u_s) \mathrm{d}s \right).$$

By assumption $(A3)$, for each $x \in \mathfrak{J}$, we get

$$\begin{aligned}|u(x)| &\leq \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_0^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} q(s) \Psi(\|u_s\|_{\mathfrak{C}}) \mathrm{d}s \\ &\leq \frac{T^{\rho\omega}\|q\|_\infty \Psi(\|u\|_{[-r,b]})}{\rho^\omega \Gamma(\omega+1)},\end{aligned}$$

which can be written as,

$$\frac{\|u\|_{[-r,b]}}{\Psi(\|u\|_{[-r,b]})\|q\|_\infty \frac{T^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)}} \leq 1.$$

From the Hypothesis $(A4)$, there exists N such that $\|u\|_{[-r,b]} \neq N$. Let us assume,

$$\mathfrak{U} = \left\{ u \in \mathfrak{C}([-r,b],\mathbb{R}) : \|u\|_{[-r,b]} < N \right\}.$$

Note that the operator $M : \overline{\mathfrak{U}} \to \mathfrak{C}([-r,b],\mathbb{R})$ is continuous and completely continuous. From the choice of $\mathfrak{U}$, there is no $u \in \partial\mathfrak{U}$ such that $u = \mu M(u)$ for some $\mu \in (0,1)$. Consequently, by Lemma 3.3, we deduce that M has a fixed point $u \in \overline{\mathfrak{U}}$, which is a solution of (1). This completes the proof. □

4. Neutral Functional Differential Equations

In this section, we prove the existence results for problem (2).

Definition 4.1. *A function $u \in \mathfrak{C}([-r,b],\mathbb{R})$, is said to be a solution of (2), if u satisfies the equation ${}^\rho D^\omega_{0+}[u(x) - f(x,u_x)] = h(x,u_x)$ on $\mathfrak{J}$, and the condition $u(x) = \Phi(x)$ on $[-r,0]$.*

Theorem 4.2 (Uniqueness Result)**.** *Assume that $(A1)$ and the following conditions holds:*

(H1) there exists a nonnegative constant k_1 such that $|f(x,u_1) - f(x,u_2)| \leq k_1\|u_1 - u_2\|_{\mathfrak{C}}$, for every $u_1, u_2 \in \mathfrak{C}([-r,0],\mathbb{R})$.

If

$$k_1 + \frac{kT^{\rho\omega}}{\rho^{\omega}\Gamma(\omega+1)} < 1, \tag{6}$$

then there exists a unique solution for equation (2) *on the interval* $[-r,b]$.

Proof. Consider the operator $M_1 : \mathfrak{C}([-r,b],\mathbb{R}) \to \mathfrak{C}([-r,b],\mathbb{R})$ defined by

$$M_1(u)(x) = \begin{cases} \Phi(x), & \text{if } x \in [-r,0], \\ f(x,u_x) + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_0^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} h(s,u_s)\mathrm{d}s, & \text{if } x \in [0,b]. \end{cases} \tag{7}$$

To show that the operator M_1 is a contraction, let $u, v \in \mathfrak{C}([-r,b],\mathbb{R})$, then we have

$$\begin{aligned} |M_1(u)(x) - M_1(v)(x)| &\leq |f(x,u_x) - f(x,v_x)| + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_0^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} |h(s,u_s) - h(s,v_s)| \,\mathrm{d}s \\ &\leq k_1 \|u_x - v_x\|_{\mathfrak{C}} + \frac{k\rho^{1-\omega}}{\Gamma(\omega)} \int_0^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} \|u_s - v_s\|_{\mathfrak{C}} \mathrm{d}s \\ &\leq k_1 \|u - v\|_{[-r,b]} + \frac{k\rho^{1-\omega} \|u-v\|_{[-r,b]}}{\Gamma(\omega)} \int_0^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} \mathrm{d}s \\ &\leq k_1 \|u - v\|_{[-r,b]} + \frac{kT^{\rho\omega}}{\rho^{\omega}\Gamma(\omega+1)} \|u - v\|_{[-r,b]}. \end{aligned}$$

Thus, we get

$$\|M_1(u) - M_1(v)\|_{[-r,b]} \leq \left[k_1 + \frac{kT^{\rho\omega}}{\rho^{\omega}\Gamma(\omega+1)}\right] \|u - v\|_{[-r,b]},$$

from (6), M_1 is a contraction. Hence M_1 has a unique fixed point by Banach's contraction principle. From this result, we get that the equation (2) has a unique solution on $[-r,b]$. □

Theorem 4.3. *Assume that* $(A2)$ - $(A3)$ *holds. Further we suppose that*

(A5) the function f *is continuous and completely continuous, and for any bounded set* B *in* $\mathfrak{C}([-r,b],\mathbb{R})$*, the set* $\{x \to f(x,u_x) : u \in B\}$ *is equicontinous in* $\mathfrak{C}([0,b],\mathbb{R})$*, and there exist constants* $0 \leq l_1 < 1$*,* $l_2 \geq 0$ *such that*

$$|f(x,u)| \leq l_1 \|u\|_{\mathfrak{C}} + l_2, \ \ x \in [0,b], \ \ u \in \mathfrak{C}([-r,0],\mathbb{R}).$$

(A6) there exists a constant $N > 0$ *such that*

$$\frac{(1-l_1)N}{l_2 + \frac{T^{\rho\omega}\|q\|_\infty \Psi(N)}{\rho^\omega \Gamma(\omega+1)}} > 1,$$

then the equation (2) *has at least one solution on* $[-r,b]$.

Proof. We consider the operator $M_1 : \mathfrak{C}([-r,b],\mathbb{R}) \to \mathfrak{C}([-r,b],\mathbb{R})$ defined by (7) and prove that the operator M_1 is continuous and completely continuous. Using $(A4)$, it is sufficient to prove that the operator $M_2 : \mathfrak{C}([-r,b],\mathbb{R}) \to \mathfrak{C}([-r,b],\mathbb{R})$ defined by

$$M_2(u)(x) = \begin{cases} \Phi(x), & \text{if } x \in [-r,0], \\ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_0^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} h(s,u_s)\mathrm{d}s, & \text{if } x \in [0,b], \end{cases}$$

is continuous and completely continuous. The proof is similar to that of Theorem 3.4. So we omit the details. We now prove that there exists an open set $\mathfrak{U} \subseteq \mathfrak{C}([-r,b],\mathbb{R})$ with $u \neq \mu M_1(u)$ for $\mu \in (0,1)$ and $u \in \partial\mathfrak{U}$. Let $u \in \mathfrak{C}([-r,b],\mathbb{R})$ and $u = \mu M_1(u)$ for some $0 < \mu < 1$. Thus, for each $x \in [0,b]$, we have

$$u(x) = \lambda \left(f(x,u_x) + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_0^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} h(s,u_s) \mathrm{d}s \right).$$

For each $x \in \mathfrak{J}$, it follows by $(A3)$ and $(A4)$ that,

$$\begin{aligned} |u(x)| &\leq l_1 \|u_x\|_{\mathfrak{C}} + l_2 + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_0^x (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} q(s) \Psi(\|u_s\|_{\mathfrak{C}}) \mathrm{d}s \\ &\leq l_1 \|u_x\|_{\mathfrak{C}} + l_2 + \frac{T^{\rho\omega} \|q\|_\infty \Psi(\|u\|_{[-r,b]})}{\rho^\omega \Gamma(\omega+1)} \end{aligned}$$

which yields

$$(1-l_1)\|u\|_{[-r,b]} \leq l_2 + \frac{T^{\rho\omega} \|q\|_\infty \Psi(\|u\|_{[-r,b]})}{\rho^\omega \Gamma(\omega+1)}$$

From this, we obtain

$$\frac{(1-l_1)\|u\|_{[-r,b]}}{l_2 + \frac{T^{\rho\omega} \|q\|_\infty \Psi(\|u\|_{[-r,b]})}{\rho^\omega \Gamma(\omega+1)}} \leq 1.$$

In view of $(A5)$, there exists N such that $\|u\|_{[-r,b]} \neq N$. Let us set

$$\mathfrak{U} = \left\{ u \in \mathfrak{C}([-r,b],\mathbb{R}) : \|u\|_{[-r,b]} < N \right\}.$$

Note that the operator $M_1 : \overline{\mathfrak{U}} \to \mathfrak{C}([-r,b],\mathbb{R})$ is continuous and completely continuous. From the choice of $\mathfrak{U}$, there is no $u \in \partial\mathfrak{U}$ such that $u = \mu M_1(u)$ for some $\mu \in (0,1)$. Thus, by Lemma 3.3, we deduce that M_1 has a fixed point $u \in \overline{\mathfrak{U}}$, which is a solution of the equation (2). This completes the proof. □

5. An Example

In this section, an example is given to illustrate our main results. Consider the Katugampola-type fractional functional differential equations,

$$\begin{cases} {}^\rho D_{0+}^{\frac{1}{2}} u(x) = \frac{\|u_x\|_{\mathfrak{C}}}{2\left(1+\|u_x\|_{\mathfrak{C}}\right)}, & x \in \mathfrak{J} := [0,e], \\ u(x) = \Phi(x), & x \in [-r,0] \end{cases} \tag{8}$$

Let $h(x,t) = \frac{t}{2(1+t)}$, $(x,t) \in [0,e] \times [0,\infty)$, also let us assume the values $\rho = 1$, $\omega = \frac{1}{2}$. For $t_1, t_2 \in [0,\infty)$, and $x \in \mathfrak{J}$, we get

$$|h(x,t_1) - h(x,t_2)| \leq \frac{|t_1 - t_2|}{2\,|1+t_1|\,|1+t_2|} \leq \frac{|t_1 - t_2|}{2}.$$

Hence the condition $(A1)$ holds with $k = \frac{1}{2}$. Since $\frac{kT^{\rho\omega}}{\rho^\omega \Gamma(\omega+1)} = \frac{1}{\sqrt{\pi}} < 1$, comparing this with the Theorem 3.2, we get that the given problem (8) has a unique solution on $[-r,e]$.

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