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Analytic Study on Nonlocal Initial Value Problems for Pantograph Equations with Hilfer-Hadamard Fractional Derivative

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Abstract: This paper is concerned with the existence and stability of solutions for a new class of nonlocal initial value problems involving Hilfer-Hadamrad type fractional pantograph equations. The existence results are obtained with the aid of some classical fixed point theorems, while the uniqueness of solutions is established by means of classical contraction mapping principle. We also discuss Ulam-Hyers stability for proposed problem.

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1. Introduction

There are some works on the existence results for initial problems of first order nonlinear differential systems with different nonlocal conditions [3, 21]. With the rapid development of fractional calculus in modern times, fractional calculus arise naturally in various areas of mechanics, electricity, biology, control theory and signal processing, etc. [15, 22]. In fact, fractional order differential equations (DEs) have attracted the great attention of several researchers and they have been proved to be effective tools in real world applications. For more recent development of such interesting branch, one can see the monographs [5, 16, 20]. There are several definitions of fractional integrals and derivatives in the literature, but the most popular definitions are in the sense of the Riemann-Liouville (R-L) and Caputo. Recently, Hilfer has introduced a generalized form of the R-L fractional derivative. In short, Hilfer fractional derivative is an interpolation between the R-L and Caputo fractional derivatives. This set of parameters gives an extra degree of freedom on the initial conditions and produces more types of stationary states. For some recent results and applications of Hilfer fractional derivative, we refer the reader to a series of papers [1, 2, 6, 7, 9–11, 31–33] and the references cited therein. In addition, there is another kind of fractional derivative in the literature that is due to Hadamard; it is known as the Hadamard and differs from the preceding ones in the sense that is definiton involves the logarithmic function with an arbitary exponent.

Recently, many studies about the delay differential equations (DDEs) have appeared in science literature. The pantograph equation is one of the most important kinds of DDEs that arise in a variety of applications in physics and engineering. Pantograph type always has the delay term fall after the initial value but before the desire approximation being calculated.

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When the delay term of pantograph type involved with the derivative(s), the equation is named as neutral DDE of pantograph type. Very recently, Vivek and his co-authors studied about the initial value problems for fractional pantograph equations; one can refer to [27–30].

The study of stability of functional equations has been developed at a high rate in the last three decades. This subject dates back to the talk given by the Polish-American mathematician Ulam at the University of Wisconsin in 1940 (see [24]). In that talk, Ulam asked whether an approximate solution of a functional equation must be near an exact solution of that equation. One year later, a partial answer to this question was given by Hyers [12]. Since then, a functional equation is said to have the Ulam-Hyers (U-H) stability if the corresponding Ulam's question has an affirmative answer. Ulam's stability problem has been attracted by many famous researchers, for example [12, 13, 23]. For more recent contribution on such hot topic, see [4, 17–19] and references therein .

2. Prerequisites

In this section, we mainly recall some definitions and lemmas which will be used later.

Definition 2.1 ([34]). *The Hadamard fractional integral of order α for a continuous function $\mathscr{F}$ is defined as*

$$_H\mathscr{I}^{\alpha}_{a+}\mathscr{F}(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{\alpha-1}\mathscr{F}(s)\frac{ds}{s},\quad \alpha>0,$$

provided the integral exists.

Similar to the R-L fractional calculus, the Hadamard fractional derivative is defined in expressions of the Hadamard fractional integral in the following way:

Definition 2.2 ([34]). *The Hadamard derivative of fractional order α for a continuous function $\mathscr{F}:[a,\infty)\to\mathscr{R}$ is defined as*

$$_H\mathscr{D}^{\alpha}_{a+}\mathscr{F}(t)=\frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n\int_a^t\left(\log\frac{t}{s}\right)^{n-\alpha-1}\mathscr{F}(s)\frac{ds}{s},\ n-1<\alpha<n,\ n=\lceil\alpha\rceil+1,$$

where $\lceil\alpha\rceil$ denotes the integer part of real number α and $\log(\cdot)=\log_e(\cdot)$.

From the Hadamard fractional integral, the Hilfer-Hadamard fractional derivative (introduced for the first time in [14]) is defined in the following approach:

Definition 2.3. *The Hilfer-Hadamard fractional derivative of order $0<\alpha<1$ and $0\le\beta\le1$ of function $\mathscr{F}(t)$ is defined by*

$$_H\mathscr{D}^{\alpha,\beta}_{a+}\mathscr{F}(t)=\left(_H\mathscr{I}^{\beta(1-\alpha)}_{a+}\mathscr{D}\left(_H\mathscr{I}^{(1-\beta)(1-\alpha)}_{a+}\mathscr{F}\right)\right)(t),$$

where $\mathscr{D}:=\frac{d}{dt}$.

This type of fractional derivative interpolates the R-L Hadamard fractional derivative ($\beta=0$) and the Caputo-Hadamard fractional derivative ($\beta=1$). In this part, we take $\mathscr{J}'=(a,b]$.

Definition 2.4. *For $0\le\gamma<1$, we denote the space $\mathscr{C}_{\gamma,\log}[\mathscr{J}',\mathscr{R}]$ as*

$$\mathscr{C}_{\gamma,\log}[\mathscr{J}',\mathscr{R}]:=\left\{\mathscr{F}(t):(a,b]\to\mathscr{R}|\left(\log\frac{t}{a}\right)^{\gamma}\mathscr{F}(t)\in\mathscr{C}[\mathscr{J}',\mathscr{R}]\right\},$$

where $\mathscr{C}_{\gamma,\log}[\mathscr{J}',\mathscr{R}]$ is the weighted space of the continuous functions $\mathscr{F}$ on the finite interval $\mathscr{J}'$.

Obviously, $\mathscr{C}_{\gamma,\log}[\mathscr{J}',\mathscr{R}]$ *is the Banach space with the norm*

$$\|\mathscr{F}\|_{\mathscr{C}_{\gamma,\log}} = \left\|\left(\log\frac{t}{a}\right)^{\gamma}\mathscr{F}(t)\right\|_{\mathscr{C}}.$$

Meanwhile, $\mathscr{C}^n_{\gamma,\log}[\mathscr{J}',\mathscr{R}] := \left\{\mathscr{F}\in\mathscr{C}^{n-1}[\mathscr{J}',\mathscr{R}] : \mathscr{F}^{(n)}\in\mathscr{C}_{\gamma,\log}[\mathscr{J}',\mathscr{R}]\right\}$ *is the Banach space with the norm*

$$\|\mathscr{F}\|_{\mathscr{C}^n_{\gamma,\log}} = \sum_{i=0}^{n-1}\left\|\mathscr{F}^{(k)}\right\|_{\mathscr{C}} + \left\|\mathscr{F}^{(n)}\right\|_{\mathscr{C}_{\gamma,\log}},\ n\in\mathbb{N}.$$

Moreover, $\mathscr{C}^0_{\gamma,\log}[\mathscr{J}',\mathscr{R}] := \mathscr{C}_{\gamma,\log}[\mathscr{J}',\mathscr{R}]$.

We consider the underlying spaces defined by

$$\mathscr{C}^{\alpha,\beta}_{1-\gamma,\log} = \left\{\mathscr{F}\in\mathscr{C}_{1-\gamma,\log}[\mathscr{J}',\mathscr{R}], {}_H\mathscr{D}^{\alpha,\beta}_{a^+}\mathscr{F}\in\mathscr{C}_{1-\gamma,\log}[\mathscr{J}',\mathscr{R}]\right\},$$

and

$$\mathscr{C}^{\gamma}_{1-\gamma,\log} = \left\{\mathscr{F}\in\mathscr{C}_{1-\gamma,\log}[\mathscr{J}',\mathscr{R}], {}_H\mathscr{D}^{\gamma}_{a^+}\mathscr{F}\in\mathscr{C}_{1-\gamma,\log}[\mathscr{J}',\mathscr{R}]\right\}.$$

It is clear that

$$\mathscr{C}^{\gamma}_{1-\gamma,\log}[\mathscr{J}',\mathscr{R}]\subset\mathscr{C}^{\alpha,\beta}_{1-\gamma,\log}[\mathscr{J}',\mathscr{R}].$$

The following generalized Gronwall inequalities will be used to deal with our systems in the sequence.

Lemma 2.5 ([34])**.** *Let* $\mathscr{V},\mathscr{W}:[1,b]\to[1,+\infty)$ *be continuous functions. If* $\mathscr{W}$ *is nondecreasing and there are constants* $k\geq 0$ *and* $0<\alpha<1$ *such that*

$$\mathscr{V}(t)\leq\mathscr{W}(t)+k\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\mathscr{V}(s)\frac{ds}{s},\quad t\in[1,b],$$

then

$$\mathscr{V}(t)\leq\mathscr{W}(t)+\int_1^t\left[\sum_{n=1}^{\infty}\frac{(k\Gamma(\alpha))^n}{\Gamma(n\alpha)}\left(\log\frac{t}{s}\right)^{n\alpha-1}\mathscr{W}(s)\right]\frac{ds}{s},\quad t\in[1,b].$$

Remark 2.6. *Under the assumptions of Lemma 2.5, let* $\mathscr{W}(t)$ *be a nondecreasing function on* $[1,b]$*. Then we have*

$$\mathscr{V}(t)\leq\mathscr{W}(t)E_{\alpha,1}\left(k\Gamma(\alpha)(\log t)^{\alpha}\right),$$

where $E_{\alpha,1}$ *is the Mittag-leffler function defined by*

$$E_{\alpha,1}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)},\quad z\in\mathbb{C}.$$

Theorem 2.7 (Banach's fixed point theorem [8])**.** *Let* C *be a non-empty closed subset of a Banach space* X*, then any contraction mapping* T *of* C *into itself has a unique fixed point.*

Theorem 2.8 (Schaefer's fixed point theorem [8])**.** *Let* X *be a Banach space, and* $N:X\to X$ *completely continuous operator.*

If the set $\omega=\{\mathscr{U}\in X:\ \mathscr{U}=\delta N\mathscr{U},\ \text{for some}\ \delta\in(0,1)\}$ *is bounded, then* N *has fixed points.*

Theorem 2.9 (Krasnoselskii's fixed point theorem [8])**.** *Let* M *be a closed convex and nonempty subset of a Banach space* X*. Let* A,B *be the operators such that (i)* $Ax+By\in M$ *whenever* $x,y\in M$ *(ii)* A *is compact and continuous (iii)* B *is a contraction mapping. Then there exists* $z\in M$ *such that* $z=Az+Bz$*.*

3. Existence and Uniqueness Results

In this section, we discuss nonlocal initial value problem for the Hilfer-Hadamard fractional pantograph equation of the form

$$ {}_H\mathscr{D}^{\alpha,\beta}_{1^+}\mathscr{U}(t) = \mathscr{F}(t,\mathscr{U}(t),\mathscr{U}(\lambda t)),\ t\in \mathscr{J}^{'} := [1,b], \tag{1}$$

$$ {}_H\mathscr{I}^{1-\gamma}_{1^+}\mathscr{U}(1) = \sum_{i=1}^{m} c_i\mathscr{U}(\tau_i),\ \alpha\leq\gamma=\alpha+\beta-\alpha\beta<1,\ \tau_i\in\mathscr{J}^{'}, \tag{2}$$

where the two parameter family of fractional derivative ${}_H\mathscr{D}^{\alpha,\beta}_{1^+}$ denote the left-sided Hilfer-Hadamard fractional derivative of order α and type β, $(0<\alpha<1,\ 0\leq\beta\leq 1)$, $0<\lambda<1$. The nonlinear term $\mathscr{F}$ is a given function, τ_i, $i=1,2,...,m$ are pre-fixed points satisfying $1<\tau_i\leq ...\leq\tau_m<b$ and c_i are real numbers. As argued in [33], the above system (1)-(2) is equivalent to the following mixed type integral equation:

$$ \mathscr{U}(t) = \begin{cases} \frac{Z(\log t)^{\gamma-1}}{\Gamma(\alpha)}\sum_{i=1}^{m} c_i\int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1}\mathscr{F}(s,\mathscr{U}(s),\mathscr{U}(\lambda s))\frac{ds}{s} \\ \quad +\frac{1}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\mathscr{F}(s,\mathscr{U}(s),\mathscr{U}(\lambda s))\frac{ds}{s}, \end{cases} \tag{3}$$

where

$$ Z := \frac{1}{\Gamma(\gamma)-\sum_{i=1}^{m}c_i(\log\tau_i)^{\gamma-1}},\quad \text{if}\quad \Gamma(\gamma)\neq\sum_{i=1}^{m}c_i(\log\tau_i)^{\gamma-1}. \tag{4}$$

In passing, we remark that the applications of nonlinear condition ${}_H\mathscr{I}^{1-\gamma}_{1^+}\mathscr{U}(1)=\sum_{i=1}^{m}c_i\mathscr{U}(\tau_i)$ in physical problems yields better effect than the initial condition ${}_H\mathscr{I}^{1-\gamma}_{1^+}\mathscr{U}(1)=\mathscr{U}_1$.
We need the following lemma to prove the main results.

Lemma 3.1. *Let $\mathscr{F}:\mathscr{J}^{'}\times\mathscr{R}\times\mathscr{R}\to\mathscr{R}$ be a function such that $\mathscr{F}\in\mathscr{C}_{1-\gamma,\log}(\mathscr{J}^{'},\mathscr{R})$ for any $\mathscr{U}\in\mathscr{C}_{1-\gamma,\log}(\mathscr{J}^{'},\mathscr{R})$. A function $\mathscr{U}\in\mathscr{C}^{\gamma}_{1-\gamma}(\mathscr{J},\mathscr{R})$ is a solution of the problem (1)-(2) if and only if $\mathscr{U}$ satisfies the mixed type integral (3).*

To establish our main result concerning existence and stability of solutions of (1)-(2), we impose the following hypotheses:

(H1) The function $\mathscr{F}:\mathscr{J}^{'}\times\mathscr{R}\times\mathscr{R}\to\mathscr{R}$ is continuous.

(H2) There exists a positive constants $\mathscr{L}>0$ such that

$$ \left|\mathscr{F}(t,\mathscr{U},\mathscr{V})-\mathscr{F}(t,\overline{\mathscr{U}},\overline{\mathscr{V}})\right|\leq\mathscr{L}\left(\left|\mathscr{U}-\overline{\mathscr{U}}\right|+\left|\mathscr{V}-\overline{\mathscr{V}}\right|\right)$$

for each $t\in\mathscr{J}^{'}$ and $\mathscr{U},\mathscr{V},\overline{\mathscr{U}},\overline{\mathscr{V}}\in\mathscr{R}$.

(H3) The function $\mathscr{F}:\mathscr{J}^{'}\times\mathscr{R}\times\mathscr{R}\to\mathscr{R}$ is completely continuous and there exists a function $\mu\in\mathscr{L}^1(\mathscr{J}^{'})$ such that

$$ |\mathscr{F}(t,\mathscr{U},\mathscr{V})|\leq\mu(t),\quad t\in\mathscr{J}^{'},\ \mathscr{U},\mathscr{V}\in\mathscr{R}.$$

(H4) there exists an increasing function $\varphi\in\mathscr{C}_{1-\gamma,\log}(\mathscr{J}^{'},\mathscr{R})$ and there exists $\lambda_\varphi>0$ such that for $t\in\mathscr{J}^{'}$,

$$ {}_H\mathscr{I}^{\alpha}_{1^+}\varphi(t)\leq\lambda_\varphi\varphi(t)$$

.

The following existence result is based on Krasnoselskii fixed point theorem.

Theorem 3.2. *Assume that (H1), (H2) and the constant*

$$\mathfrak{Q} := \frac{2\mathscr{L}B(\gamma,\alpha)}{\Gamma(\alpha)}\left(|Z|\sum_{i=1}^{m} c_i(\log\lambda_i)^{\alpha+\gamma-1} + (\log b)^{\alpha}\right) < 1 \tag{5}$$

are satisfied. Then problem (1)-(2) has at least one solution in $\mathscr{C}^{\gamma}_{1-\gamma,\log}(\mathscr{J}^{'},\mathscr{R}) \subset \mathscr{C}^{\alpha,\beta}_{1-\gamma,\log}(\mathscr{J}^{'},\mathscr{R})$.

Proof. Consider the operator $\mathscr{N} : \mathscr{C}_{1-\gamma,\log}(\mathscr{J}^{'},\mathscr{R}) \to \mathscr{C}_{1-\gamma,\log}(\mathscr{J}^{\log},\mathscr{R})$.

$$(\mathscr{N}\mathscr{U})(t) = \begin{cases} \frac{Z(\log t)^{\gamma-1}}{\Gamma(\alpha)}\sum_{i=1}^{m} c_i \int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1}\mathscr{F}(s,\mathscr{U}(s),\mathscr{U}(\lambda s))\frac{ds}{s} \\ \quad + \frac{1}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\mathscr{F}(s,\mathscr{U}(s),\mathscr{U}(\lambda s))\frac{ds}{s}, \end{cases} \tag{6}$$

It is obvious that the operator $\mathscr{N}$ is well defined. Set $\overline{\mathscr{F}}(s) := \mathscr{F}(s,0,0)$ and

$$\varpi := \frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(|Z|\sum_{i=1}^{m} c_i(\log\tau_i)^{\alpha+\gamma-1} + (\log b)^{\alpha}\right)\|\overline{\mathscr{F}}\|_{\mathscr{C}_{1-\gamma,\log}}.$$

Consider a ball

$$\mathscr{B}_q := \left\{\mathscr{U} \in \mathscr{C}_{1-\gamma,\log}(\mathscr{J}^{'},\mathscr{R}) : \ \|\mathscr{U}\|_{\mathscr{C}_{1-\gamma,\log}} \le q\right\}$$

with $q \ge \frac{\varpi}{1-\mathfrak{Q}}$, $(\mathfrak{Q} < 1)$. We subdivide the operator $\mathscr{N}$ into two operator P and Q on $\mathscr{B}_q$ as follows:

$$(P\mathscr{U})(t) = \frac{Z}{\Gamma(\alpha)}t^{\gamma-1}\sum_{i=1}^{m} c_i\int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1}\mathscr{F}(s,\mathscr{U}(s),\mathscr{U}(\lambda s))\frac{ds}{s},$$
$$(Q\mathscr{U})(t) = \frac{1}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\mathscr{F}(s,\mathscr{U}(s),\mathscr{U}(\lambda s))\frac{ds}{s}.$$

Claim 1. $P\mathscr{U} + Q\mathscr{U} \in \mathscr{B}_q$ for every $\mathscr{U},\mathscr{V} \in \mathscr{B}_q$. For the operator P,

$$\begin{aligned} \left|(P\mathscr{U})(t)(\log t)^{1-\gamma}\right| &\le \frac{|Z|}{\Gamma(\alpha)}\sum_{i=1}^{m} c_i\int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1}|\mathscr{F}(s,\mathscr{U}(s),\mathscr{U}(\lambda s))|\frac{ds}{s} \\ &\le \frac{|Z|}{\Gamma(\alpha)}\sum_{i=1}^{m} c_i\int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1}\left(|\mathscr{F}(s,\mathscr{U}(s),\mathscr{U}(\lambda s)) - \mathscr{F}(s,0,0)| + |\mathscr{F}(s,0,0)|\right)\frac{ds}{s} \\ &\le \frac{|Z|}{\Gamma(\alpha)}\sum_{i=1}^{m} c_i\left[(\log\tau_i)^{\alpha+\gamma-1}B(\gamma,\alpha)\left(2\mathscr{L}\|\mathscr{U}\|_{\mathscr{C}_{1-\gamma,\log}} + \|\overline{\mathscr{F}}\|_{\mathscr{C}_{1-\gamma,\log}}\right)\right]. \end{aligned}$$

This gives

$$\|P\mathscr{U}\|_{\mathscr{C}_{1-\gamma}} \le \frac{|Z|}{\Gamma(\alpha)}\sum_{i=1}^{m} c_i\left[(\log\tau_i)^{\alpha+\gamma-1}B(\gamma,\alpha)\left(2\mathscr{L}\|\mathscr{U}\|_{\mathscr{C}_{1-\gamma,\log}} + \|\overline{\mathscr{F}}\|_{\mathscr{C}_{1-\gamma,\log}}\right)\right]. \tag{7}$$

For the operator Q

$$\begin{aligned} \left|(Q\mathscr{U})(t)(\log t)^{1-\gamma}\right| &\le \frac{(\log t)^{1-\gamma}}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\left(2\mathscr{L}|\mathscr{U}(s)| - |\overline{\mathscr{F}}(s)|\right)\frac{ds}{s} \\ &\le \frac{B(\gamma,\alpha)}{\Gamma(\alpha)}(\log b)^{\alpha}\left(2\mathscr{L}\|\mathscr{U}\|_{\mathscr{C}_{1-\gamma,\log}} + \|\overline{\mathscr{F}}\|_{\mathscr{C}_{1-\gamma,\log}}\right). \end{aligned}$$

Thus, we obtain

$$\|Q\mathscr{U}\|_{C_{1-\gamma,\log}} \le \frac{B(\gamma,\alpha)}{\Gamma(\alpha)}(\log b)^{\alpha}\left(2\mathscr{L}\|\mathscr{U}\|_{\mathscr{C}_{1-\gamma,\log}} + \|\overline{\mathscr{F}}\|_{\mathscr{C}_{1-\gamma,\log}}\right). \tag{8}$$

Linking (7) and (8), for every $\mathscr{U},\mathscr{V}\in\mathscr{B}_q$, one has

$$\|P\mathscr{U}+Q\mathscr{V}\|_{\mathscr{C}_{1-\gamma,\log}}\le\|P\mathscr{U}\|_{\mathscr{C}_{1-\gamma,\log}}+\|Q\mathscr{V}\|_{\mathscr{C}_{1-\gamma,\log}},\ \mathfrak{Q}q+\overline{\omega}\le q,$$

which yields that $P\mathscr{U}+Q\mathscr{V}\in\mathscr{B}_q$.

Claim 2. The operator P is contraction mapping. For the operator P, any $\mathscr{U},\mathscr{V}\in\mathscr{B}_q$,

$$\begin{aligned}\left|((P\mathscr{U})(t)-(P\mathscr{V})(t))(\log t)^{1-\gamma}\right|&\le\frac{|Z|}{\Gamma(\alpha)}\sum_{i=1}^{m}c_i\int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1}|\mathscr{F}(s,\mathscr{U}(s),\mathscr{U}(\lambda s))-\mathscr{F}(s,\mathscr{V}(s),\mathscr{V}(\lambda s))|\frac{ds}{s}\\&\le\frac{2\mathscr{L}|Z|B(\gamma,\alpha)}{\Gamma(\alpha)}\sum_{i=1}^{m}c_i(\log\tau_i)^{\alpha+\gamma-1}\|\mathscr{U}-\mathscr{V}\|_{\mathscr{C}_{1-\gamma,\log}}.\end{aligned}$$

This gives

$$\begin{aligned}\|P\mathscr{U}-P\mathscr{V}\|_{\mathscr{C}_{1-\gamma,\log}}&\le\frac{2\mathscr{L}|Z|B(\gamma,\alpha)}{\Gamma(\alpha)}\sum_{i=1}^{m}c_i(\log\tau_i)^{\alpha+\gamma-1}\|\mathscr{U}-\mathscr{V}\|_{\mathscr{C}_{1-\gamma,\log}}\\&\le\mathfrak{Q}\|\mathscr{U}-\mathscr{V}\|_{\mathscr{C}_{1-\gamma,\log}}.\end{aligned}$$

The operator P is contraction mapping due to the condition (5).

Claim 3. The operator Q is compact and continuous. Since the function $\mathscr{F}\in\mathscr{C}_{1-\gamma,\log}(\mathscr{J}',\mathscr{R})$, the operator Q is continuous following the definition of $\mathscr{C}_{1-\gamma,\log}(\mathscr{J},\mathscr{R})$. According to Claim 1, we know that

$$\|Q\mathscr{U}\|_{C_{1-\gamma,\log}}\le\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}(\log b)^{\alpha}\left(2\mathscr{L}\|\mathscr{U}\|_{\mathscr{C}_{1-\gamma,\log}}+\|\overline{\mathscr{F}}\|_{\mathscr{C}_{1-\gamma,\log}}\right).$$

So Q is uniformly bounded on $\mathscr{B}_q$. Now we prove the compactness of the operator Q. For any $1<t_1<t_2\le T$, we have

$$|(Q\mathscr{U})(t_1)-(Q\mathscr{U})(t_2)|\le\frac{\|\mathscr{F}\|_{\mathscr{C}_{1-\gamma,\log}}B(\gamma,\alpha)}{\Gamma(\alpha)}\left|(\log t_1)^{\alpha+\gamma-1}-(\log t_2)^{\alpha+\gamma-1}\right|$$

tending to zero as $t_2\to t_1$, whether $\alpha+\gamma\ge1$. Thus, Q is equicontinuous. Hence, the operator Q is compact on $\mathscr{B}_q$ by the Arzëla-Ascoli Theorem. □

Next, we will use Schaefer fixed point theorem to derive the existence result.

Theorem 3.1. *Assume that (H1),(H3) are satisfied. Then, the problem* (1)-(2) *has at least one solution in* $\mathscr{C}^{\gamma}_{1-\gamma,\log}(\mathscr{J}',\mathscr{R})\subset\mathscr{C}^{\alpha,\beta}_{1-\gamma,\log}(\mathscr{J}',\mathscr{R})$.

Proof. Consider the operator $\mathscr{N}:\mathscr{C}_{1-\gamma,\log}(\mathscr{J}',\mathscr{R})\to\mathscr{C}_{1-\gamma,\log}(\mathscr{J}',\mathscr{R})$ as defined in Theorem 3.2. The proof will be given in several steps.

Claim 1. The operator $\mathscr{N}$ is continuous. Let $\mathscr{U}_n$ be a sequence such that $\mathscr{U}_n\to\mathscr{U}$ in $\mathscr{C}_{1-\gamma,\log}(\mathscr{J}',\mathscr{R})$. Then for each $t\in\mathscr{J}'$,

$$\begin{aligned}&\left|((\mathscr{N}\mathscr{U}_n)(t)-(\mathscr{N}\mathscr{U})(t))(\log t)^{1-\gamma}\right|\\&\le\frac{|Z|}{\Gamma(\alpha)}\sum_{i=1}^{m}c_i\int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1}|\mathscr{F}(s,\mathscr{U}_n(s),\mathscr{U}_n(\lambda s))-\mathscr{F}(s,\mathscr{U}(s),\mathscr{U}(\lambda s))|\frac{ds}{s}\\&\quad+\frac{(\log t)^{1-\gamma}}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}|\mathscr{F}(s,\mathscr{U}_n(s),\mathscr{U}_n(\lambda s))-\mathscr{F}(s,\mathscr{U}(s),\mathscr{U}(\lambda s))|\frac{ds}{s}\\&\le\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(|Z|\sum_{i=1}^{m}c_i(\log\tau_i)^{\alpha+\gamma-1}+(\log b)^{\alpha}\right)\|\mathscr{F}(\cdot,\mathscr{U}_n(\cdot),\mathscr{U}_n(\cdot))-\mathscr{F}(\cdot,\mathscr{U}(\cdot),\mathscr{U}(\cdot))\|_{\mathscr{C}_{1-\gamma,\log}}.\end{aligned}$$

Since $\mathscr{F}$ is continuous, then the Lebesgue Dominated Convergence Theorem implies that

$$\|(\mathscr{N}\mathscr{U}_n) - (\mathscr{N}\mathscr{U})\|_{\mathscr{C}_{1-\gamma,\log}} \to 0 \quad \text{as} \quad n \to \infty.$$

Claim 2. The operator $\mathscr{N}$ maps bounded sets into bounded sets in $\mathscr{C}_{1-\gamma,\log}(\mathscr{J}',\mathscr{R})$. Indeed, it is enough to show that for $q > 0$, there exists a positive constant l such that $\mathscr{U} \in \mathscr{B}_q = \left\{\mathscr{U} \in \mathscr{C}_{1-\gamma,\log}(\mathscr{J}',\mathscr{R}) : \|\mathscr{U}\| \leq q\right\}$, we have $\|(\mathscr{N}\mathscr{U})\|_{\mathscr{C}_{1-\gamma,\log}} \leq l$.

$$\begin{aligned}\left|(\mathscr{N}\mathscr{U})(t)(\log t)^{1-\gamma}\right| &\leq \frac{|Z|}{\Gamma(\alpha)}\sum_{i=1}^{m} c_i \int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1}|\mathscr{F}(s,\mathscr{U}(s),\mathscr{U}(\lambda s))|\frac{ds}{s} \\ &\quad + \frac{(\log t)^{1-\gamma}}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}|\mathscr{F}(s,\mathscr{U}(s),\mathscr{U}(\lambda s))|\frac{ds}{s}, \\ &\leq \frac{1}{\Gamma(\alpha)}B(\gamma,\alpha)\left(|Z|\sum_{i=1}^{m} c_i(\log\tau_i)^{\alpha+\gamma-1} + (\log b)^{\alpha}\right)\|\mu\|_{\mathscr{C}_{1-\gamma,\log}} \\ &:= l.\end{aligned}$$

Claim 3. The operator $\mathscr{N}$ maps bounded sets into equicontinuous set of $\mathscr{C}_{1-\gamma,\log}(\mathscr{J}',\mathscr{R})$. Let $t_1, t_2 \in \mathscr{J}', t_2 \leq t_1$, $\mathscr{B}_q$ be a bounded set of $\mathscr{C}_{1-\gamma,\log}(\mathscr{J}',\mathscr{R})$ as in Claim 2, and let $\mathscr{U} \in \mathscr{B}_q$. Then

$$\begin{aligned}&\left|(\log t_1)^{1-\gamma}(\mathscr{N}\mathscr{U})(t_1) - (\log t_2)^{1-\gamma}(\mathscr{N}\mathscr{U})(t_2)\right| \\ &\leq \left|\frac{(\log t_1)^{1-\gamma}}{\Gamma(\alpha)}\int_1^{t_1}\left(\log\frac{t_1}{s}\right)^{\alpha-1}\mathscr{F}(s,\mathscr{U}(s),\mathscr{U}(\lambda s))\frac{ds}{s}\right. \\ &\quad \left.-\frac{(\log t_2)^{1-\gamma}}{\Gamma(\alpha)}\int_1^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\mathscr{F}(s,\mathscr{U}(s),\mathscr{U}(\lambda s))\frac{ds}{s}\right| \\ &\leq \left|\frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left[(\log t_1)^{1-\gamma}\left(\log\frac{t_1}{s}\right)^{\alpha-1} - (\log t_2)^{1-\gamma}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\right]\mathscr{F}(s,\mathscr{U}(s),\mathscr{U}(\lambda s))\frac{ds}{s}\right| \\ &\quad + \left|\frac{(\log t_2)^{1-\gamma}}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\mathscr{F}(s,\mathscr{U}(s),\mathscr{U}(\lambda s))\frac{ds}{s}\right|.\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. As a consequence of Claim 1 to 3, together with Arzëla-Ascoli theorem, we can conclude that $\mathscr{N} : \mathscr{C}_{1-\gamma,\log}(\mathscr{J}',\mathscr{R}) \to \mathscr{C}_{1-\gamma,\log}(\mathscr{J}',\mathscr{R})$ is continuous and completely continuous.

Claim 4. A priori bounds. Now, it remains to show that the set

$$\omega = \left\{\mathscr{U} \in \mathscr{C}_{1-\gamma,\log}(\mathscr{J}',\mathscr{R}) : \mathscr{U} = \delta(\mathscr{N}\mathscr{U}),\ 0 < \delta < 1\right\}$$

is bounded set. Let $\mathscr{U} \in \omega$, $\mathscr{U} = \delta(\mathscr{N}\mathscr{U})$ for some $0 < \delta < 1$. Thus for each $t \in \mathscr{J}'$, we have

$$\mathscr{U}(t) = \omega\left[\frac{Z(\log t)^{\gamma-1}}{\Gamma(\alpha)}\sum_{i=1}^{m} c_i\int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1}\mathscr{F}(s,\mathscr{U}(s),\mathscr{U}(\lambda s))\frac{ds}{s} + \frac{1}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}\mathscr{F}(s,\mathscr{U}(s),\mathscr{U}(\lambda s))\frac{ds}{s}\right]$$

We can complete this step by considering the estimation in Claim 2. As a consequence of Schaefer's fixed point theorem, we conclude that $\mathscr{N}$ has fixed point which is solution of the problem (1)-(2). □

The next results is based on the Banach fixed point theorem.

Theorem 3.3. *Assume that (H1)-(H2) are satisfied. If*

$$\frac{2\mathscr{L}}{\Gamma(\alpha)}B(\gamma,\alpha)\left(|Z|\sum_{i=1}^{m} c_i(\log\tau_i)^{\alpha+\gamma-1} + (\log b)^{\alpha}\right) < 1, \tag{9}$$

then, the problem (1)-(2) *has a unique solution.*

Proof. We consider the operator $\mathscr{N}$ is defined as in Thoerem 3.1. By Lemma 3.1, it is clear that the fixed point of $\mathscr{N}$ are solutions of (1)-(2). Let $\mathscr{U}_1, \mathscr{U}_2 \in \mathscr{C}_{1-\gamma,\log}(\mathscr{J}', \mathscr{R})$ and $t \in \mathscr{J}'$, then we have

$$\begin{aligned}\left|((\mathscr{N}\mathscr{U}_1)(t) - (\mathscr{N}\mathscr{U}_2)(t))(\log t)^{\gamma-1}\right| &\leq \frac{|Z|}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} |\mathscr{F}(s, \mathscr{U}_1(s), \mathscr{U}_1(\lambda s)) - \mathscr{F}(s, \mathscr{U}_2(s), \mathscr{U}_2(\lambda s))| \frac{ds}{s} \\ &\quad + \frac{(\log t)^{1-\gamma}}{\Gamma(\alpha)} \int_1^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} |\mathscr{F}(s, \mathscr{U}_1(s), \mathscr{U}_1(\lambda s)) - \mathscr{F}(s, \mathscr{U}_2(s), \mathscr{U}_2(\lambda s))| \frac{ds}{s} \\ &\leq \frac{2\mathscr{L}}{\Gamma(\alpha)} B(\gamma, \alpha) \left(|Z| \sum_{i=1}^{m} c_i (\log \tau_i)^{\alpha+\gamma-1} + (\log b)^{\alpha} \right).\end{aligned}$$

From (9), it follows that $\mathscr{N}$ has a unique fixed point which is solution of problem (1)-(2). ☐

4. Stability Analysis

In this section, we study U-H stability for the solutions of our proposed problem.

Definition 4.1. *Equation (1) is U-H stable if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for solution $\mathscr{Z} \in \mathscr{C}^{\gamma}_{1-\gamma,\log}(\mathscr{J}', \mathscr{R})$ of the inequality*

$$\left| {}_H\mathscr{D}^{\alpha,\beta}_{1^+} \mathscr{Z}(t) - \mathscr{F}(t, \mathscr{Z}(t), \mathscr{Z}(\lambda t)) \right| \leq \epsilon, \quad t \in \mathscr{J}', \tag{10}$$

there exists a solution $\mathscr{U} \in \mathscr{C}^{\gamma}_{1-\gamma,\log}(\mathscr{J}', \mathscr{R})$ of equation (1) with

$$|\mathscr{Z}(t) - \mathscr{U}(t)| \leq C_f \epsilon, \quad t \in \mathscr{J}'.$$

Definition 4.2. *Equation (1) is generalized U-H stable if there exists $\psi_f \in \mathscr{C}(\mathscr{R}_+, \mathscr{R}_+)$, $\psi_f(0) = 0$, such that for each solution $\mathscr{Z} \in \mathscr{C}^{\gamma}_{1-\gamma,\log}(\mathscr{J}', \mathscr{R})$ of the inequality (10) there exists a solution $\mathscr{U} \in \mathscr{C}^{\gamma}_{1-\gamma,\log}(\mathscr{J}', \mathscr{R})$ of equation (1) with*

$$|\mathscr{Z}(t) - \mathscr{U}(t)| \leq \psi_f(\epsilon), \quad t \in \mathscr{J}'.$$

Definition 4.3. *Equation (1) is U-H-Rassias stable with respect to $\varphi \in \mathscr{C}_{1-\gamma,\log}(\mathscr{J}', \mathscr{R})$ if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $\mathscr{Z} \in \mathscr{C}^{\gamma}_{1-\gamma,\log}(\mathscr{J}', \mathscr{R})$ of the inequality*

$$\left| {}_H\mathscr{D}^{\alpha,\beta}_{1^+} \mathscr{Z}(t) - \mathscr{F}(t, \mathscr{Z}(t), \mathscr{Z}(\lambda t)) \right| \leq \epsilon\varphi(t), \quad t \in \mathscr{J}', \tag{11}$$

there exists a solution $\mathscr{U} \in \mathscr{C}^{\gamma}_{1-\gamma,\log}(\mathscr{J}', \mathscr{R})$ of equation (1) with

$$|\mathscr{Z}(t) - \mathscr{U}(t)| \leq C_f \epsilon\varphi(t), \quad t \in \mathscr{J}'.$$

Definition 4.4. *Equation (1) is generalized U-H-Rassias stable with respect to $\varphi \in \mathscr{C}_{1-\gamma,\log}(\mathscr{J}', \mathscr{R})$ if there exists a real number $C_{f,\varphi} > 0$ such that for each solution $\mathscr{Z} \in \mathscr{C}^{\gamma}_{1-\gamma,\log}(\mathscr{J}', \mathscr{R})$ of the inequality*

$$\left| {}_H\mathscr{D}^{\alpha,\beta}_{1^+} \mathscr{Z}(t) - \mathscr{F}(t, \mathscr{Z}(t), \mathscr{Z}(\lambda t)) \right| \leq \varphi(t), \quad t \in \mathscr{J}', \tag{12}$$

there exists a solution $\mathscr{U} \in \mathscr{C}^{\gamma}_{1-\gamma,\log}(\mathscr{J}', \mathscr{R})$ of equation (1) with

$$|\mathscr{Z}(t) - \mathscr{U}(t)| \leq C_{f,\varphi}\varphi(t), \quad t \in \mathscr{J}'.$$

Remark 4.5. *A function $\mathscr{Z} \in \mathscr{C}^{\gamma}_{1-\gamma,\log}(\mathscr{J}',\mathscr{R})$ is a solution of the inequality (10) if and only if there exist a function $\mathscr{G} \in \mathscr{C}^{\gamma}_{1-\gamma,\log}(\mathscr{J}',\mathscr{R})$ such that*

(1). $|\mathscr{G}(t)| \le \epsilon$, $t \in \mathscr{J}'$;

(2). ${}_H\mathscr{D}^{\alpha,\beta}_{1^+}\mathscr{Z}(t) = \mathscr{F}(t,\mathscr{Z}(t),\mathscr{Z}(\lambda t)) + \mathscr{G}(t)$, $t \in \mathscr{J}'$.

For more details, one can refer to [25, 26].

Lemma 4.6. *Let $0<\alpha<1$, $0\le\beta\le1$, if a function $\mathscr{Z} \in \mathscr{C}^{\gamma}_{1-\gamma,\log}(\mathscr{J}',\mathscr{R})$ is a solution of the inequality (10), then $\mathscr{Z}$ is a solution of the following inequality*

$$\left|\mathscr{Z}(t) - \mathscr{A}_{\mathscr{Z}} - \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\mathscr{F}(s,\mathscr{Z}(s),\mathscr{Z}(\lambda s))\frac{ds}{s}\right| \le \left(\frac{|\mathscr{Z}|\,(mc)(\log b)^{\gamma+\alpha+1}}{\Gamma(\alpha+1)} + \frac{(\log b)^{\alpha}}{\Gamma(\alpha+1)}\right)\epsilon,$$

where $\mathscr{A}_{\mathscr{Z}} = \frac{Z(\log t)^{\gamma-1}}{\Gamma(\alpha)}\sum_{i=1}^{m} c_i \int_1^{\tau_i}\left(\log\frac{t}{s}\right)^{\alpha-1}\mathscr{F}(s,\mathscr{Z}(s),\mathscr{Z}(\lambda s))\frac{ds}{s}$.

Lemma 4.7. *Let $0<\alpha<1$, $0\le\beta\le1$, if a function $\mathscr{Z} \in \mathscr{C}^{\gamma}_{1-\gamma,\log}(\mathscr{J}',\mathscr{R})$ is a solution of the inequality (12), then $\mathscr{Z}$ is a solution of the following inequality*

$$\left|\mathscr{Z}(t) - \mathscr{A}_{\mathscr{Z}} - \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\mathscr{F}(s,\mathscr{Z}(s),\mathscr{Z}(\lambda s))\frac{ds}{s}\right| \le \left(|Z|\,(\log t)^{\gamma-1}(mc)+1\right)\epsilon\lambda_{\varphi}\varphi(t).$$

Theorem 4.8. *If the hypotheses (H1), (H2) and (9) are satisfied, then the system (1)-(2) is U-H stable.*

Proof. Let $\epsilon>0$ and let $\mathscr{Z} \in \mathscr{C}^{\gamma}_{1-\gamma,\log}(\mathscr{J}',\mathscr{R})$ be a function which satisfies the inequality (10) and let $\mathscr{U} \in \mathscr{C}^{\gamma}_{1-\gamma,\log}(\mathscr{J}',\mathscr{R})$ the unique solution of the following system

$$\begin{cases} {}_H\mathscr{D}^{\alpha,\beta}_{1^+}\mathscr{U}(t) = \mathscr{F}(t,\mathscr{U}(t),\mathscr{U}(\lambda t)),\ 0<\lambda<1,\ t\in\mathscr{J}' := [1,b],\\ {}_H\mathscr{I}^{1-\gamma}_{1^+}\mathscr{Z}(1) = {}_H\mathscr{I}^{1-\gamma}_{1^+}\mathscr{U}(1) = \sum_{i=1}^m c_i\mathscr{U}(\tau_i),\ \alpha\le\gamma=\alpha+\beta-\alpha\beta<1,\ \tau_i\in\mathscr{J}'. \end{cases}$$

Using Lemma 3.1, we obtain

$$\mathscr{U}(t) = \mathscr{A}_{\mathscr{U}} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\mathscr{F}(s,\mathscr{U}(s),\mathscr{U}(\lambda s))\frac{ds}{s}.$$

On the other hand, if $\mathscr{U}(\tau_i) = \mathscr{Z}(\tau_i)$, and ${}_H\mathscr{I}^{1-\gamma}_{1^+}\mathscr{U}(1) = {}_H\mathscr{I}^{1-\gamma}_{1^+}\mathscr{Z}(1)$, then $\mathscr{A}_{\mathscr{U}} = \mathscr{A}_{\mathscr{Z}}$. Indeed,

$$\begin{aligned} |\mathscr{A}_{\mathscr{U}} - \mathscr{A}_{\mathscr{Z}}| &\le \frac{|Z|}{\Gamma(\alpha)}(\log t)^{\gamma-1}\sum_{i=1}^m c_i\int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1}|\mathscr{F}(s,\mathscr{U}(s),\mathscr{U}(\lambda s)) - \mathscr{F}(s,\mathscr{Z}(s),\mathscr{Z}(\lambda s))|\frac{ds}{s}\\ &\le 2\mathscr{L}\frac{|Z|}{\Gamma(\alpha)}(\log t)^{\gamma-1}\sum_{i=1}^m c_i\ {}_H\mathscr{I}^{\alpha}_{1^+}|\mathscr{U}(\tau_i) - \mathscr{Z}(\tau_i)|\\ &= 0. \end{aligned}$$

Thus, $\mathscr{A}_{\mathscr{U}} = \mathscr{A}_{\mathscr{Z}}$. Then, we have

$$\mathscr{U}(t) = \mathscr{A}_{\mathscr{Z}} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\mathscr{F}(s,\mathscr{U}(s),\mathscr{U}(\lambda s))\frac{ds}{s}.$$

By integration of the inequality (10) and applying Lemma 4.6, we obtain

$$\left|\mathscr{Z}(t) - \mathscr{A}_{\mathscr{Z}} - \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\mathscr{F}(s,\mathscr{Z}(s),\mathscr{Z}(\lambda s))\frac{ds}{s}\right| \le \left(\frac{|\mathscr{Z}|\,(mc)(\log b)^{\gamma+\alpha+1}}{\Gamma(\alpha+1)} + \frac{(\log b)^{\alpha}}{\Gamma(\alpha+1)}\right)\epsilon. \tag{13}$$

We have for any $t \in \mathscr{J}'$

$$
\begin{aligned}
|\mathscr{Z}(t) - \mathscr{U}(t)| \leq & \left| \mathscr{Z}(t) - \mathscr{A}_{\mathscr{Z}} - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \mathscr{F}(s, \mathscr{Z}(s), \mathscr{Z}(\lambda s)) \frac{ds}{s} \right| \\
& + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} |\mathscr{F}(s, \mathscr{Z}(s), \mathscr{Z}(\lambda s)) - \mathscr{F}(s, \mathscr{U}(s), \mathscr{U}(\lambda s))| \frac{ds}{s} \\
\leq & \left| \mathscr{Z}(t) - \mathscr{A}_{\mathscr{Z}} - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \mathscr{F}(s, \mathscr{Z}(s), \mathscr{Z}(\lambda s)) \frac{ds}{s} \right| \\
& + 2\mathscr{L} \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} |\mathscr{Z}(s) - \mathscr{U}(s)| \frac{ds}{s} \\
\leq & \left(\frac{|\mathscr{Z}|\,(mc)(\log b)^{\gamma+\alpha+1}}{\Gamma(\alpha+1)} + \frac{(\log b)^{\alpha}}{\Gamma(\alpha+1)} \right) \epsilon + \frac{2\mathscr{L}}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} |\mathscr{Z}(s) - \mathscr{U}(s)|\, ds.
\end{aligned}
$$

and applying Lemma 2.5 and Remark 2.6, we obtain

$$
\begin{aligned}
|\mathscr{Z}(t) - \mathscr{U}(t)| \leq & \left(\frac{|\mathscr{Z}|\,(mc)(\log b)^{\gamma+\alpha+1}}{\Gamma(\alpha+1)} + \frac{(\log b)^{\alpha}}{\Gamma(\alpha+1)} \right) E_{\alpha,1}\left(2\mathscr{L}(\log b)^{\alpha}\right) \cdot \epsilon, \\
:= & C_f \epsilon.
\end{aligned}
$$

where $\nu = \nu(\alpha)$ is a constant, which completes the proof of the theorem. □

Theorem 4.9. *Assume that (H1), (H2), (H4) and* (9) *are satisfied, then the problem* (1)-(2) *is U-H-Rassias stable.*

Proof. The proof of the theorem directly follows from the Lemma 4.7 and the process of Theorem 4.8. □

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