Int. J. Math. And Appl., 6(2–A)(2018), 33–42
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Oscillation of Fractional Nonlinear Partial Differential Equations with Continuous Distributed Deviating Arguments

Vadivel Sadhasivam[1,*], Nagamanickam Nagajothi[1] and Muthusamy Deepa[1]

P.G & Research Department of Mathematics, Thiruvalluvar Government Arts College, Rasipuram, Namakkal, Tamil Nadu, India.

Abstract: In this article, we establish some oscillation criteria for the fractional order partial differential equation with continuous distributed deviating arguments of the form

$$\frac{\partial}{\partial t}\left[r(t)D^{\alpha}_{+,t}\left(u(x,t)-\int_{\gamma}^{\delta}q_0(t,\zeta)u(x,\rho(t,\zeta))d\eta(\zeta)\right)\right]=a(t)\Delta u(x,t)+\int_c^d p(t,\xi)\Delta u[x,\tau(t,\xi)]d\omega(\xi)$$
$$-\int_c^d q(x,t,\xi)g\left(u[x,\sigma(t,\xi)]\right)d\omega(\xi)+f(x,t),\quad (x,t)\in G=\Omega\times\mathbb{R}_+,$$

with subject to the boundary conditions

$$\frac{\partial u(x,t)}{\partial \nu}+\mu(x,t)u(x,t)=\psi(x,t),\qquad (x,t)\in\partial\Omega\times\mathbb{R}_+$$

and $u=\chi(x,t)$, $(x,t)\in\partial\Omega\times\mathbb{R}_+$. Using the generalized Riccati technique and integral averaging method, new oscillation criteria are obtained.

MSC: 35R11, 34K11.

Keywords: Fractional partial differential equation, continuous deviating arguments, oscillation.

Accepted on: 21.03.2018

1. Introduction

For the past few decades, the problem of oscillation and non oscillation of solutions of partial differential equations is one of the active area of research in the qualitative theory of partial differential equations. See the monograph Yoshida [32]. In [14] Kreith et al., and Yoshida [33] were investigated the oscillation properties of hyperbolic equations without functional arguments by using the averaging techniques. In [18], Mishev et al., introduced the oscillation criteria for solutions of hyperbolic equations with delay. In the recent times, there has been an increasing interest in studying the oscillation of hyperbolic equation with continuous distributed deviating arguments. For linear hyperbolic equation with continuous distributed deviating arguments, one can refer [7, 8, 27, 30] and for nonlinear hyperbolic continuous distributed deviating arguments, one can refer [6, 11, 12, 25, 26].

One can find a number of applications, for example, population ecology, climate models, control theory, coupled oscillators,

* *E-mail: ovsadha@gmail.com*

viscoelastic materials, and structured population models studied distributed delay with boundary conditions Dirichlet, Neumann and Robin in [31]. Distributed delay models appears in logistics [4], traffic flow [24], micro organism growth [20], and hematopoiesis [2, 3].

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the field of physics, chemistry, aerodynamics, electro dynamics of complex medium, polymer rheology, etc., involves derivatives of fractional order [5, 13, 17, 23, 28, 29].

Fractional - order differential equations also serve better for the discription of hereditary properties of various materials and processes than integer - order differential equations. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives and the fluid dynamic traffic model with fractional derivatives can eliminate the deficiency arising from the assumption of continumum traffic flow. Also, the use of fractional partial differential equations in mathematical models become increasingly popular in recent years. One of the simplest example is fractional order diffusion equation which is the generalization of classical diffusion equations treating super diffusive flow processes. For the basic theory one can refer [1, 9, 10, 15, 19, 21, 22, 34] and references cited therein.

In [16], Liu et al., studied oscillation properties of nonlinear inhomogeneous hyperbolic equations with distributed deviating arguments. To the best of our knowledge, there exists almost no literature on the oscillation of fractional nonlinear partial differential equations with inhomogeneous boundary conditions using generalized Riccati technique. Motivated by this gap we propose to initiate the following fractional nonlinear partial differential equations with continuous distributed deviating arguments of the form,

$$\frac{\partial}{\partial t}\left[r(t)D^{\alpha}_{+,t}\left(u(x,t)-\int_{\gamma}^{\delta}q_0(t,\zeta)u(x,\rho(t,\zeta))d\eta(\zeta)\right)\right]=a(t)\Delta u(x,t)$$
$$+\int_c^d p(t,\xi)\Delta u[x,\tau(t,\xi)]d\omega(\xi)-\int_c^d q(x,t,\xi)g\left(u[x,\sigma(t,\xi)]\right)d\omega(\xi)+f(x,t),\quad (x,t)\in G=\Omega\times\mathbb{R}_+, \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with piecewise smooth boundary $\partial\Omega$; $\alpha\in(0,1)$ is a constant; $G=\Omega\times\mathbb{R}_+$, $\mathbb{R}_+=(0,\infty)$, $D^{\alpha}_{+,t}u$ is the Riemann- Liouville fractional derivative of order α of u with respect to t; Δ is the Laplacian operator in the Euclidean N-space $\mathbb{R}^N$. Equation (1) subject to the boundary conditions

$$\frac{\partial u(x,t)}{\partial\nu}+\mu(x,t)u(x,t)=\psi(x,t),\quad (x,t)\in\partial\Omega\times\mathbb{R}_+ \tag{2}$$

and

$$u=\chi(x,t),\quad (x,t)\in\partial\Omega\times\mathbb{R}_+, \tag{3}$$

where ν is the unit outward normal vector to $\partial\Omega$ and $\mu\in C\left(\partial\Omega\times\mathbb{R}_+,[0,\infty)\right)$, $\psi,\chi\in C\left(\partial\Omega\times\mathbb{R}_+,\mathbb{R}\right)$.

Throughout this paper, we will assume that the following conditions hold:

(A_1) $r(t)\in C^1\left([0,\infty);\mathbb{R}_+\right), a(t)\in C\left([0,\infty);[0,\infty)\right), \omega(\xi)\in([c,d],\mathbb{R})$ and $\eta(\zeta)\in C\left([\gamma,\delta],\mathbb{R}\right)$ are nondecreasing functions on $[c,d]$ and $[\gamma,\delta]$ respectively, the integrals of the equation (1) are Stieltjes integrals;

(A_2) $q_0\in C\left([0,\infty)\times[\gamma,\delta];[0,\infty)\right)$, there exists a positive constant k_0 satisfying $\int_\gamma^\delta q_0(t,\zeta)d\eta(\zeta)\le k_0<1$, $p\in C\left(\mathbb{R}_+\times[c,d],\mathbb{R}_+\right), q\in C\left(\overline{G}\times[c,d],\mathbb{R}_+\right), g\in C(\mathbb{R},\mathbb{R})$ is convex in $\mathbb{R}_+, ug(u)>0$ and $\frac{g(u)}{u}\ge k>0$ for $u\ne 0$ and $-g(-u)=g(u)>0, u\in\mathbb{R}_+$;

(A_3) $\rho\in C\left([0,\infty)\times[\gamma,\delta],\mathbb{R}\right)$, $\rho(t,\zeta)\le t$ for $(t,\zeta)\in(0,\infty)\times[\gamma,\delta]$, $\tau,\sigma\in C\left(\mathbb{R}_+\times[c,d],\mathbb{R}\right), \frac{d}{dt}\sigma(t,c)=\sigma'(t,c)$ exists and $\sigma(t,\xi)\le t,\tau(t,\xi)\le t$ for $\xi\in[c,d]$, τ and σ are nondecreasing with t and ξ respectively and $\lim_{t\to\infty}\min_{\xi\in[c,d]}\sigma(t,\xi)=+\infty$, $\lim_{t\to\infty}\min_{\xi\in[c,d]}\tau(t,\xi)=+\infty$;

(A_4) $f \in C\left(\overline{G};\mathbb{R}\right)$.

This paper is organized as follows: In section 2, we present the definitions and notations that will be needed in the sequel. In section 3, we discuss the oscillation of the problems (1)-(2) and (1)-(3). In section 4, we illustrate the main results with a suitable example. The results obtained in this current article generalize and improve numerous findings in the earlier works.

2. Preliminaries

First let us introduce, the following notations will be used for our convenience.

$$v(t) = \frac{1}{|\Omega|}\int_{\Omega} u(x,t)dx, \quad where\ |\Omega| = \int_{\Omega} dx. \tag{4}$$

$$Q(t,\xi) = \min\left\{q(x,t,\xi) : x \in \overline{\Omega}\right\}, \tag{5}$$

$$\overline{\psi}(t) = \frac{1}{|\Omega|}\int_{\partial\Omega} \psi(x,t)dx, \tag{6}$$

$$\overline{F}(t) = \frac{1}{|\Omega|}\int_{\Omega} f(x,t)dx, \tag{7}$$

$$\overline{R}(t) = a(t)\overline{\psi}(t) + \overline{F}(t) + \int_c^d p(t,\xi)\overline{\psi}\left[\tau(t,\xi)\right] d\sigma(\xi). \tag{8}$$

Definition 2.1. *By a solution of (1) we mean a function* $u \in C^{1+\alpha}\left(\overline{\Omega}\times[t_{-1},\infty);\mathbb{R}\right) \cap C\left(\overline{\Omega}\times[\tilde{t}_{-1},\infty);\mathbb{R}\right)$ *that satisfies (1), where*

$$t_{-1} = \min\left\{0, \min_{\zeta\in[\gamma,\delta]}\left\{\inf_{t\geq 0}\rho(t,\zeta)\right\}, \min_{\xi\in[c,d]}\left\{\inf_{t\geq 0}\tau(t,\xi)\right\}\right\},$$

$$\tilde{t}_{-1} = \min\left\{0, \min_{\xi\in[c,d]}\left\{\inf_{t\geq 0} g(t,\xi)\right\}\right\}.$$

Definition 2.2. *The solution u(x,t) of the problem (1) and (2) (or (1) and(3)) is said to be oscillatory in the domain G if for any positive number* μ *there exists a point* $(x_0,t_0)\in\Omega\times[\mu,\infty)$ *such that* $u(x_0,t_0)=0$ *holds.*

Definition 2.3. *The Riemann-Liouville fractional partial derivative of order* $0<\alpha<1$ *with respect to* t *of a function* $u(x,t)$ *is given by*

$$(D^{\alpha}_{+,t}u)(x,t) := \frac{\partial}{\partial t}\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\xi)^{-\alpha}u(x,\xi)d\xi \tag{9}$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$, *where* Γ *is the gamma function.*

Definition 2.4. *The Riemann-Liouville fractional integral of order* $\alpha>0$ *of a function* $y:\mathbb{R}_+\to\mathbb{R}$ *on the half-axis* $\mathbb{R}_+$ *is given by*

$$(I^{\alpha}_{+}y)(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}y(\xi)d\xi \quad for \quad t>0 \tag{10}$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$.

Definition 2.5. *The Riemann-Liouville fractional derivative of order* $\alpha>0$ *of a function* $y:\mathbb{R}_+\to\mathbb{R}$ *on the half-axis* $\mathbb{R}_+$ *is given by*

$$(D^{\alpha}_{+}y)(t) := \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}\left(I^{\lceil\alpha\rceil-\alpha}_{+}y\right)(t) \quad for \quad t>0 \tag{11}$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$, *where* $\lceil\alpha\rceil$ *is the ceiling function of* α.

3. Main Results

In this section we will employ some integral means of solutions to reduce the multi dimensional oscillation problem to one dimensional problem. Also, we will give some oscillation criteria of (1) with the inhomogeneous boundary conditions (2) and (3).

Theorem 3.1. *Assume that the conditions* $(A_1)-(A_4)$ *hold and let the fractional differential inequality*

$$\left[r(t)D_+^{\alpha}\left(y(t)-\int_{\gamma}^{\delta}q_0(t,\zeta)y(\rho(t,\zeta))d\eta(\zeta)\right)\right]'+\int_c^d Q(t,\xi)g(y[\sigma(t,\xi)])d\omega(\xi)\leq \pm\bar{R}(t) \tag{12}$$

has no eventually positive unbounded solutions, then every solution of (1), (2) *is oscillatory in* G.

Proof. Assume to the contrary that there is a nonoscillatory solution $u(x,t)$ to the problem (1), (2) with the property that $v(t)$ is unbounded . Without loss of generality, we may assume that $u(x,t)>0, (x,t)\in\Omega\times[t_0,\infty),\ t_0>0$. By the assumption (A_3), there exists a $t_1\geq t_0$ such that $\sigma(t,\xi)\geq t_0$ and $\tau(t,\xi)\geq t_0$ for $(t,\xi)\in[t_1,\infty)\times[c,d]$ then

$$u(x,\tau(t,\xi))>0 \ and \ u(x,\sigma(t,\xi))>0 \ for \ (x,t,\xi)\in\Omega\times[t_1,\infty)\times[c,d]. \tag{13}$$

Integrating (1) with respect to x over the domain Ω, using (5) and (7) we have

$$\begin{aligned}\frac{d}{dt}\left[r(t)D_+^{\alpha}\left(v(t)-\int_{\gamma}^{\delta}q_0(t,\zeta)v(\rho(t,\zeta))d\eta(\zeta)\right)\right]&=a(t)\frac{1}{|\Omega|}\int_{\Omega}\Delta u(x,t)dx+\int_c^d p(t,\xi)\frac{1}{|\Omega|}\int_{\Omega}\Delta u[x,\tau(t,\xi)]d\omega(\xi)dx\\&-\int_c^d Q(t,\xi)\frac{1}{|\Omega|}\int_{\Omega}g\left(u[x,\sigma(t,\xi)]\right)d\omega(\xi)dx+\bar{F}(t),\ t\geq t_1.\end{aligned}$$

Using Green's formula and (2) we obtain

$$\begin{aligned}\frac{1}{|\Omega|}\int_{\Omega}\Delta u(x,t)dx&=\frac{1}{|\Omega|}\int_{\partial\Omega}\frac{\partial u(x,t)}{\partial\nu}dS\\&=-\frac{1}{|\Omega|}\int_{\partial\Omega}\left[\mu(x,t)u(x,t)-\psi(x,t)\right]dS\\&\leq\overline{\psi}(t),\quad t\geq t_1,\end{aligned} \tag{14}$$

where dS is surface element on $\partial\Omega$. Similarly, we have

$$\frac{1}{|\Omega|}\int_{\Omega}\Delta u(x,\tau(t,\xi))dx\leq\overline{\psi}(\tau(t,\xi)),\quad t\geq t_1.$$

Moreover, using Jensen's inequality, it follows that

$$\int_c^d Q(t,\xi)\frac{1}{|\Omega|}\int_{\Omega}g\left(u[x,\sigma(t,\xi)]\right)dx\ d\omega(\xi)\geq\int_c^d Q(t,\xi)g\left(v[\sigma(t,\xi)]\right)d\omega(\xi),\ t\geq t_1. \tag{15}$$

Combining $(14)-(15)$ and using (8), we have

$$\left[r(t)D_+^{\alpha}\left(v(t)-\int_{\gamma}^{\delta}q_0(t,\zeta)v(\rho(t,\zeta))d\eta(\zeta)\right)\right]'+\int_c^d Q(t,\xi)g(v[\sigma(t,\xi)])d\omega(\xi)\leq\bar{R}(t),\quad t\geq t_1. \tag{16}$$

It is clear that $v(t)>0$ on $[t_1,\infty)$. Hence, $v(t)$ is an eventually positive unbounded solution of (12) with $+\bar{R}(t)$. This contradicts the hypothesis. If $u<0$ in $\Omega\times[t_0,\infty)$ for some $t_0>0$, we observe that $v^*(t)=-v(t)$ is an eventually positive unbounded solution of (12) with $-\bar{R}(t)$. This also contradicts the hypothesis. The proof is complete. □

Theorem 3.2. *Assume that the conditions* $(A_1) - (A_4)$ *hold and let the fractional differential inequality*

$$\left[r(t)D_+^\alpha\left(y(t) - \int_\gamma^\delta q_0(t,\zeta)y(\rho(t,\zeta))d\eta(\zeta)\right)\right]' + \int_c^d Q(t,\xi)g(y[\sigma(t,\xi)])d\omega(\xi) \le \pm\tilde{R}(t), \tag{17}$$

$t \ge t_1$, *has no eventually positive unbounded solutions, then every solution of* $(1),(3)$ *is oscillatory in* G.

Theorem 3.3. *Assume that the hypothesis* $(A_1) - (A_4)$ *hold, sufficient condition for the fractional differential inequality*

$$\left[r(t)D_+^\alpha\left(y(t) - \int_\gamma^\delta q_0(t,\zeta)y(\rho(t,\zeta))d\eta(\zeta)\right)\right]' + \int_c^d Q(t,\xi)g(y[\sigma(t,\xi)])d\omega(\xi) \le R(t), \tag{18}$$

$t \ge t_1$, *where R(t) is a continuous function and that the following hypothesis is satisfied*

(A_5) *there is a* $c^{1+\alpha}$ *- function* $\theta(t)$ *such that* $\theta(t)$ *is bounded and* $(r(t)D_+^\alpha\theta(t))' = R(t)$. *If the following condition is satisfied:*

$$\int_b^\infty \left[\int_c^d Q(t,\xi)d\omega(\xi)\right]dt = +\infty, \tag{19}$$

for some $b > 0$, *then* (18) *has no eventually positive unbounded solution.*

Proof. Assume that (18) has an eventually positive unbounded solution $y(t)$. Letting

$$z(t) = y(t) - \int_\gamma^\delta q_0(t,\zeta)y(\rho(t,\zeta))d\eta(\zeta) - \theta(t) \tag{20}$$

and taking into account (A_5), we find that

$$(r(t)D_+^\alpha z(t))' \le -\int_c^d Q(t,\xi)g(y[\sigma(t,\xi)])d\omega(\xi) \le 0. \tag{21}$$

Therefore, $r(t)D_+^\alpha z(t) \ge 0$ or $r(t)D_+^\alpha z(t) < 0$ eventually. Since $r(t) > 0$, we see that $D_+^\alpha z(t) \ge 0$ or $D_+^\alpha z(t) < 0$. Hence, $z(t)$ is a monotone function, and $z(t) > 0$ or $z(t) \le 0$ eventually. We prove that $\lim\limits_{t\to\infty} z(t) = \infty$. Hence, $z(t) > 0$ eventually. Since $y(t)$ is unbounded from above, there exists a sequence $\{t_n\}_{n=1}^\infty$ satisfying $\lim_{n\to\infty} t_n = \infty$, $\lim\limits_{n\to\infty} y(t_n) = \infty$ and $\max_{t_0\le t\le t_n} y(t) = y(t_n)$. The hypothesis (A_2) and (A_3) implies that

$$\begin{aligned} z(t_n) &= y(t_n) - \int_\gamma^\delta q_0(t_n,\zeta)y(\rho(t_n,\zeta))d\eta(\zeta) - \theta(t_n) \\ &\ge y(t_n) - y(t_n)\int_\gamma^\delta q_0(t_n,\zeta)d\eta(\zeta) - \theta(t_n) \\ &= \left(1 - \int_\gamma^\delta q_0(t_n,\zeta)d\eta(\zeta)\right)y(t_n) - \theta(t_n) \\ &\ge (1-k_0)\,y(t_n) - \theta(t_n) \end{aligned}$$

for sufficiently large n. Since $\theta(t)$ is bounded and $\lim_{n\to\infty}(1-k_0)y(t_n) = \infty$, we find that $\lim_{t\to\infty} z(t_n) = \infty$. This combined with the monotonicity property of $z(t)$ implies that $\lim_{t\to\infty} z(t) = \infty$. In this case it is easy to seen that $D_+^\alpha z(t) \ge 0$. Since $\theta(t)$ is bounded and $\lim_{t\to\infty} z(t) = \infty$, for any $\varepsilon > 0$ there is a sufficiently large number T such that $\theta(t) \ge -\varepsilon z(t)$ $(t \ge T)$. Hence

$$y(t) \ge z(t) + \theta(t) \ge (1-\varepsilon)z(t)$$

and

$$y(\sigma(t,\xi)) \geq (1-\varepsilon)z(\sigma(t,\xi)).$$

Using (A_3), the inequality (21) implies that

$$\begin{aligned}(r(t)D_+^\alpha z(t))' &\leq -\int_c^d Q(t,\xi)g((1-\varepsilon)z(\sigma(t,\xi)))d\omega(\xi)\\ &\leq -g((1-\varepsilon)z(\sigma(t))\int_c^d Q(t,\xi)d\omega(\xi)\\ &= -b_0\int_c^d Q(t,\xi)d\omega(\xi),\ t\geq T,\end{aligned} \tag{22}$$

where $T>0$ sufficiently large and $b_0>0$ by (A_2). Integrating (22) over $[T,t]$, we get

$$r(t)D_+^\alpha z(t) - r(T)D_+^\alpha z(T) \leq -b_0\int_T^t\left[\int_c^d Q(s,\xi)d\omega(\xi)\right]ds,$$

which implies that

$$r(T)D_+^\alpha z(T) \geq b_0\int_T^t\left[\int_c^d Q(s,\xi)d\omega(\xi)\right]ds.$$

Taking $t\to\infty$ in the above inequality, we obtain

$$\int_T^\infty\left[\int_c^d Q(s,\xi)d\omega(\xi)\right]ds \leq \frac{1}{b_0}r(T)D_+^\alpha z(T) < \infty,$$

which contradicts the hypothesis (19). The proof is complete. □

Theorem 3.4. *Assume that the hypothesis $(A_1)-(A_4)$ hold, and that there exists a $C^{1+\alpha}$- function $\theta(t)$ such that $\theta(t)$ is bounded and $(r(t)D_+^\alpha\theta(t))' = \bar{R}(t)$. If the condition (19) is satisfied, then every solution u of the boundary value problem (1), (2) with unbounded solution is oscillatory in G.*

Proof. A combination of Theorem 3.1 and Theorem 3.3 yields the conclusion. □

Theorem 3.5. *Assume that the hypothesis $(A_1)-(A_4)$ hold, and that there exists a $C^{1+\alpha}$- function $\theta(t)$ such that $\theta(t)$ is bounded and $(r(t)D_+^\alpha\theta(t))' = \tilde{R}(t)$. If the condition (19) is satisfied, then every solution u of the boundary value problem (1), (3) with unbounded solution is oscillatory in G.*

Proof. The conclusion follows by combining Theorem 3.2 with Theorem 3.3. □

Next, we discussed some new oscillation criteria for (1), (2) by using integral average conditions of Philo's type. Let $D_0 = \{(t,s): t>s\geq t_0\}$ and $D=\{(t,s): t\geq s\geq t_0\}$. The function $H\in C(D;\mathbb{R})$ is said to belong to the class P if

(T_1) $H(t,t)=0$ for $t\geq t_0, H(t,s)>0$ on D_0,

(T_2) H has a continuous and nonpositive partial derivative on D_0 with respect to the second variable and there exists a function $h\in C(D_0,\mathbb{R})$ such that

$$-\frac{\partial H(t,s)}{\partial s} = h(t,s)\sqrt{H(t,s)}$$

for all $(t,s)\in D_0$.

Theorem 3.6. *Suppose that* $(A_1)-(A_4)$ *hold. If there exists a function* $\lambda \in C^1([t_0,\infty),\mathbb{R}_+)$ *and let* H *belong to the class* P *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(H(t,s)\varphi(s) - \frac{1}{4}\left[\frac{h(t,s)}{\sqrt{\Theta(s)}} + \frac{\sqrt{H(t,s)}\pi(s)}{\sqrt{\Theta(s)}}\right]^2 \right) ds = \infty, \tag{23}$$

where

$$\varPhi(s) = \exp(-2\int^{t} \lambda(s)ds),\ \ Q(s) = \int_c^d kQ(s,\xi)d\sigma(\xi), \varphi(t) = \varPhi(t)(Q(t)-\lambda'(t))$$
$$\Theta(t) = \frac{z'[\sigma(t,c)]\sigma'(t,c)}{\Phi(t)r(t)D_+^{\alpha}(z(t)+\theta(t))},\ \Pi(t) = 2\lambda(t) - \frac{z'[\sigma(t,c)]\sigma'(t,c)2\lambda(t)}{r(t)D_+^{\alpha}(z(t)+\theta(t))} - \frac{R(t)}{r(t)D_+^{\alpha}(z(t)+\theta(t))},$$

then each solution of (1) and (2) is oscillatory in G.

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (1), (2). Without loss of generality, we may assume that $u(x,t)>0$, $u(x,t)\in\Omega\times[t_0,\infty)$. By (A_3), there exists a $t_1\geq t_0$ such that (13) holds. From the inequality (18) and by (A_2), we obtain,

$$[r(t)D_+^{\alpha}(z(t)+\theta(t))]' + \int_c^d kQ(t,\xi)(z[\sigma(t,\xi)])d\omega(\xi) \leq R(t),\ t\geq t_1,$$

where $Q(t,\xi)$ and $z(t)$ are defined by (5), (6). It is easy to calculate $z(t)>0, D_+^{\alpha}z(t)>0$ for $t\geq t_1$ and $\sigma(t,\xi)$ is nondecreasing in ξ, we have

$$[r(t)D_+^{\alpha}(z(t)+\theta(t))]' + Q(t)z(\sigma(t,c)) \leq R(t) \tag{24}$$

for $t\geq t_1$. Set

$$W(t) = \Phi(t)\left[\frac{r(t)D_+^{\alpha}(z(t)+\theta(t))}{z(\sigma(t,c))} + \lambda(t)\right], \tag{25}$$

then $W(t)>0$ for $t\geq t_1$. From (24),(25) and (A_3),

$$W'(t) \leq -\Theta(t)W^2(t) - \Pi(t)W(t) - \varphi(t),\ t\geq t_1. \tag{26}$$

From (26) for all $t\geq t_1$ we have

$$\begin{aligned}\int_{t_1}^{t} H(t,s)\varphi(s)ds &\leq -\int_{t_1}^{t} H(t,s)W'(s)ds - \int_{t_1}^{t}\Big[H(t,s)W^2(s)\Theta(s) + H(t,s)\Pi(s)W(s)\Big]ds\\ &\leq H(t,t_1)W(t_1) - \int_{t_1}^{t}\left(H(t,s)W^2(s)\Theta(s) + \Big[h(t,s)\sqrt{H(t,s)} + H(t,s)\Pi(s)\Big]W(s)\right)ds\end{aligned} \tag{27}$$

$$\begin{aligned}\int_{t_1}^{t}\left[H(t,s)\varphi(s) - \frac{1}{4}\left(\frac{h(t,s)}{\sqrt{\Theta(s)}} + \frac{\sqrt{H(t,s)}\Pi(s)}{\sqrt{\Theta(s)}}\right)^2\right]ds &\leq H(t,t_1)W(t_1)\\ &- \int_{t_1}^{t}\left(\sqrt{H(t,s)\Theta(s)} + \left[\frac{h(t,s)}{2\sqrt{\Theta(s)}} + \frac{\sqrt{H(t,s)}\Pi(s)}{\sqrt{\Theta(s)}}\right]\right)^2 ds\end{aligned}$$

$$\int_{t_1}^{t}\left[H(t,s)\varphi(s)-\frac{1}{4}\left(\frac{h(t,s)}{\sqrt{\Theta(s)}}+\frac{\sqrt{H(t,s)}\Pi(s)}{\sqrt{\Theta(s)}}\right)^2\right]ds \le H(t,t_0)|W(t_1)|$$
$$+\int_{t_0}^{t}\left[H(t,s)\varphi(s)-\frac{1}{4}\left(\frac{h(t,s)}{\sqrt{\Theta(s)}}+\frac{\sqrt{H(t,s)}\Pi(s)}{\sqrt{\Theta(s)}}\right)^2\right]ds$$
$$\le \int_{t_0}^{t} H(t,s)\varphi(s)ds + H(t,t_0)|W(t_1)|$$
$$\le H(t,t_0)\int_{t_0}^{t_1}|\varphi(s)|ds + H(t,t_0)|W(t_1)|.$$

which implies

$$\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left(H(t,s)\varphi(s)-\frac{1}{4}\left[\frac{h(t,s)}{\sqrt{\Theta(s)}}+\frac{\sqrt{H(t,s)}\pi(s)}{\sqrt{\Theta(s)}}\right]^2\right)ds \le \int_{t_0}^{t_1}|\varphi(s)|ds + |W(t_1)|,$$

which contradicts (23). If $u(x,t)<0$ for $\Omega\times[t_0,\infty)$, then $-u(x,t)$ is a positive solution of (1) and (2) and the proof is similar. □

Theorem 3.7. *Let all the conditions of Theorem 3.6 hold. Then every solution of the problem (1) and (3) oscillates in G.*

4. Example

Example 4.1. *Consider the fractional partial differential equation*

$$\frac{\partial}{\partial t}\left[e^{-t}D^{\frac{1}{2}}_{+,t}\left(u(x,t)-\frac{1}{4}\int_0^{\pi}u(x,t-2\pi+\zeta)d\zeta\right)\right]=e^{-t}\Delta u(x,t)+\int_0^{\frac{\pi}{2}}\frac{e^{\pi}}{e^{t}}\Delta u[x,t-\frac{5\pi}{2}+\xi]d\xi$$
$$-\int_0^{\frac{\pi}{2}}\left(u[x,t-\frac{3\pi}{2}+\xi]\right)d\xi-\frac{\sqrt{2}+1}{\sqrt{2}}\sin x\sin t+\frac{\sqrt{2}-1}{\sqrt{2}}\sin x\cos t+\frac{(e^{\pi}+1)e^{-2\pi}}{8}\sin x(\cos t-\sin t(\sqrt{2}+1)),$$
$$(x,t)\in(0,\pi)\times(0,\infty). \tag{28}$$

Here $\alpha=\frac{1}{2}, r(t)=a(t)=e^{-t}, q_0(t,\zeta)=\frac{1}{4}, [\gamma,\delta]=[0,\pi], \rho(t,\zeta)=t-2\pi+\zeta, \eta(\zeta)=\zeta, [c,d]=[0,\frac{\pi}{2}], p(t,\xi)=\frac{e^{\pi}}{e^{t}}, \tau(t,\xi)=t-\frac{5\pi}{2}+\xi, \omega(\xi)=\xi, q(x,t,\xi)=Q(t,\xi)=1, g(u)=u, \sigma(t,\xi)=t-\frac{3\pi}{2}+\xi$ *and* $f(x,t)=-\frac{\sqrt{2}+1}{\sqrt{2}}\sin x\sin t+\frac{\sqrt{2}-1}{\sqrt{2}}\sin x\cos t+\frac{(e^{\pi}+1)e^{-2\pi}}{8}\sin x(\cos t-\sin t(\sqrt{2}+1))$. *Thus all the conditions of Theorem 3.3 hold. Therefore every solution of (28) is oscillatory.*

5. Conclusion

In this paper, we have obtained some new oscillation criteria for fractional nonlinear partial differential equations with continuous distributed deviating arguments supplemented with the effect of inhomogeneous boundary conditions by using generalized Riccati transformation method and inequality technique. These results extends and compliments of some of the existing litrature.

References

[1] S. Abbas, M. Benchohra and G. M. NGuerekata, *Topics in fractional differential equations*, Springer, Newyork, (2012).

[2] M. Adimy and F. Crauste, Global stability of a partial differential equation with distributed delay due to cellular replication, *Nonlinear analysis*, 54(2003), 1469-1491.

[3] M. Adimy and F. Crauste, *Modelling and asymptotic stability of a growth factor dependent stem cell dynamics model with distributed delay*, Discrete Contin Dyn Syst, 8(1)(2007), 19-38.

[4] L. Berezansky and E. Braverman, *Oscillation properties of a logistic equation with distributed delay*, Nonlinear Anal Real World Appl., 4(2003), 1-19.

[5] S. Das, *Functional fractional calculus for system identification and controls*, Springer, Berlin, (2012).

[6] L. H. Deng, *Oscillation criteria for certain hyperbolic functional differential equations with Robin boundary condition*, Indian J. Pure App. Math., 33(2002), 1137-1146.

[7] L. H. Deng and W.G. Ge, *Oscillation for certain delay hyperbolic equations satisfying the Robin boundary condition*, Indian J. Pure App. Math., 32(2001), 1269-1274.

[8] L. H. Deng, W.G. Ge and P.G. Wang, *Oscillation of hyperbolic equations with continuous deviating argument under the Robin boundary condition*, Soochow J. Math., 29(2003), 1-6.

[9] K. Diethelm, *The analysis of fractional differential equations*, Springer, Berlin, (2010).

[10] Q. Feng and F. Meng, *Oscillation of solutions to nonlinear forced fractional differential equations*, Electron. J. Differential Equations, 2013(169)(2013), 1-10.

[11] Z. Y. Han and Y. H. Yu, *Oscillation criteria of hyperbolic equations with continuous deviating arguments*, Kyungpook Math. J., 47(2007), 347-356.

[12] S. Harikrishnan and P. Prakash, *Oscillation of neutral hyperbolic differential equation with deviating arguments and damping term*, J. Differ. Equations Appl., (2014), 114-119.

[13] A. A. Kilbas, H. M. Srivastava, and J.J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier science B. V., Amsterdam, The Netherlands, 204(2006).

[14] K. Kreith, T. Kusano and N. Yoshida, *Oscillation properties of linear hyperbolic equations*, SIAM J. Math. Anal., 29(2003), 1-6.

[15] W. N. Li and W. Sheng, *Oscillation properties for solutions of a kind of partial fractional differential equations with damping term*, J. Nonlinear Sci. Appl., (2016), 1600-1608.

[16] X. Liu and X. Fu, *Oscillation criteria for nonlinear inhomogeneous hyperbolic equations with distributed deviating arguments*, Journal of Applied Mathematics and Stochastic Analysis, 9(1)(1996), 21-31.

[17] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, John Wiley and Sons, New York, (1993).

[18] D. P. Mishev and D.D. Bainov, *Oscillation properties of the solutions of hyperbolic equations of neutral type*, Colloq. Math. Soc. Jnos Bolyai, 47(1984), Differential equations, Szeged (Hungary), 771-780.

[19] I. Podlubny, *Fractional differential equations*, Academic press, San Diego, Calif, USA, (1999).

[20] V. S. H. Rao and P. R. S. Rao, *Global stability in chemostat models involving time delays and wall growth*, Nonlinear Anal Real World Appl., 5(2004), 141-158.

[21] V. Sadhasivam and J. Kavitha, *Forced oscillation of solutions of a fractional neutral partial functional differential equations*, Applied Mathematics, 6(2015), 1302-1317.

[22] V. Sadhasivam, J. Kavitha and N. Nagajothi, *Oscillation of neutral fractional order partial differential equations with damping term*, Int. J. Pure Appl Math., 115(9)(2017), 47-64.

[23] S. G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional integrals and derivatives, Theory and applications*, Gordan and Breach Science publishers, Singapore, (1993).

[24] R. Sipahi, F. M. Atay, and S.I Nicukscu, *Stability of traffic flow behavior with distributed delays modelling the memory effects of the drivers*, SIAM J. Appl. Math., 68(3)(2007), 738-759.

[25] S.Tanaka and N.Yoshida, *Forced oscillation of certain hyperbolic equations with continuous distributed deviating arguments*, Ann. Pol. Math., 81(2)(2005), 37-54.

[26] Y. Tao and N. Yoshida, *Oscillation of nonlinear hyperbolic equations with distributed deviating arguments*, Toyama Math. J., 28(2005), 27-40.

[27] Y. Tao and N. Yoshida, *Oscillation criteria for hyperbolic equations with distributed deviating arguments*, Indian J. Pure Appl. Math., 37(5)(2006), 291-305.

[28] Varsha Daftardar-Gejji, *Fractional calculus theory and applications*, Narosa publishing house pvt.Ltd, (2014).

[29] E. Vasily Tarasov, *Fractional dynamics*, Springer, (2009).

[30] P. G. Wang, *Oscillation of certain neutral hyperbolic equations*, Indian J. Pure Appl. Math., 31(2000), 949-956.

[31] J. H. Wu, *Theory and applications of partial functional differential equations*, Springer, Newyork, (1996).

[32] N. Yoshida, *Oscillation theory of partial differential equations*, World scientific publishing, Singapore, (2008).

[33] N. Yoshida, *An oscillation theorem for characteristic initial value problems for nonlinear hyperbolic equations*, Proc. Amer. Math. Soc., (1979), 95-100.

[34] Y. Zhou, *Basic theory of fractional differential equations*, World scientific, Singapore, (2014).