Int. J. Math. And Appl., 6(2–A)(2018), 43–51
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of *Mathematics And its Applications*

Existence of Best Proximity Points for Proximal C-Contractions Satisfying Rational Expression on Ordered Metric Spaces

D. Balraj[1,*], V. Piramananthami[1] and A. H. Ansari[2]

1 Department of Mathematics, Bharathidasan University, Trichy, Tamil Nadu, India.

2 Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.

Abstract: The aim of this paper is to present in the partial order metric space setting,the concept of C-class functions has been introduced and existence of best proximity point has been proved. Our rational type mapping generalisations in both contraction mapping and also generalisations the results of Luong and Thuan [10]. Moreover, some examples are given to support the main results.

MSC: Primary 47H10; Secondary 54H25.

Keywords: Best proximity point, fixed point, C-class function.

Accepted on: 17.03.2018

1. Introduction

Let A and B are two non-empty closed subsets of a metric space and T be a non-selfmap from A to B. The natural question is whether one can find an element $x_0 \in A$ such that $d(x_0, Tx_0) = \min\{d(x, Tx) : x \in A\}$. Since $d(x, Tx) \geq d(A, B)$, the optimal solution to the problem of minimizing the real valued function $x \to d(x, Tx)$ over the domain A of the mapping T will be the one for which the value $d(A, B)$ is attained. A point $x_0 \in A$ is called a best proximity point of T if $d(x_0, Tx_0) = d(A, B)$. Note that if $d(A, B) = 0$, then the best proximity point is nothing but a fixed point of T. After several generalisations of Banach's fixed point theorem, in 2004, Ran and Reuring[13] proved the existence of fixed point in partially ordered metric space setting. The result of Ran and Reuring is very much interesting due to its application to linear and non-linear matrix equations. So, existence of fixed point on partially ordered metric space is proved by many authors for weak contraction mappings. Amoung all those generalisations, one of the interesting result is due to Luong and Thuan's [10], in which they have proved the existence of fixed point for generalised weak contraction satisfying rational expression. In this paper, our attempt is to give a generalisation to the results of Luong and Thuan [10], by considering a non-self-map T. The existence and convergence of best proximity points is an interesting topic of optimization theory which recently attracted the attention of many authors [2, 4–6, 14, 16]. Also one can find the existence of best proximity point in the setting of partially order metric space in [11, 12].

The purpose of this article is to present best proximity point theorems for non-self mappings in the setting of partially ordered metric spaces, thereby producing optimal approximate solutions for $Tx = x$, where T is a non-self mapping.

* *E-mail: duraisamybalraj@gmail.com*

2. Preliminaries

Given non-empty subsets A and B of a metric space X, the following notions are used subsequently:

$$d(A,B) := \inf\{d(x,y) : x \in A \text{ and } y \in B\},$$
$$A_0 = \{x \in A : d(x,y) = d(A,B) \text{ for some } y \in B\},$$
$$B_0 = \{y \in B : d(x,y) = d(A,B) \text{ for some } x \in A\}.$$

In [9], the authors discussed sufficient conditions which guarantee the non-emptiness of A_0 and B_0. Moreover, in [14], the authors proved that A_0 is contained in the boundary of A in the setting of normed linear spaces. In [10] Luong and Thuan proved the following theorem.

Theorem 2.1 ([10]). *Let $(X, \preceq)$ be an ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Let $T : X \to X$ be a non-decreasing mapping such that*

$$d(Tx,Ty) \leq m(x,y) - \phi(m(x,y)) \text{ for } x,y \in X, x \succeq y, x \neq y, \tag{1}$$

where $\phi : [0,\infty) \to [0,\infty)$ is a lower semi-continuous function with $\phi(t) = 0$ if and only if $t = 0$, and $m(x,y) = \max\left\{\frac{d(x,Tx)d(y,Ty)}{d(x,y)}, d(x,y)\right\}$. Also, assume either T is continuous or X has the property that

$$\{x_n\} \text{ is a nondecreasing sequence in } X \text{ such that } x_n \to x, \text{then } x = \sup\{x_n\}. \tag{2}$$

If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point.

Definition 2.2 ([15]). *A mapping $T : A \to B$ is said to be proximally increasing if it satisfies the condition that*

$$\left.\begin{array}{r} y_1 \preceq y_2 \\ d(x_1, Ty_1) = d(A,B) \\ d(x_2, Ty_2) = d(A,B) \end{array}\right\} \implies x_1 \preceq x_2$$

where $x_1, x_2, y_1, y_2 \in A$.

One can see that, for a self-mapping, the notion of proximally increasing mapping reduces to that of increasing mapping. In 2014, Ansari defined the concept of C-class functions and presented new fixed point results which improve and extend some results in the literature.

Definition 2.3 ([1]). *A continuous function $F : [0,\infty)^2 \to \mathbb{R}$ is called C-class function if for any $s,t \in [0,\infty)$, the following conditions hold:*

(1). $F(s,t) \leq s$;

(2). $F(s,t) = s$ implies that either $s = 0$ or $t = 0$.

Example 2.4 ([1]). *Following examples show that the class $\mathcal{C}$ is nonempty:*

(1). $F(s,t) = s - t$.

(2). $F(s,t) = ms$, for some $m \in (0,1)$.

(3). $F(s,t) = \frac{s}{(1+t)^r}$ *for some* $r \in (0,\infty)$.

(4). $F(s,t) = \log(t + a^s)/(1+t)$, *for some* $a > 1$.

(5). $F(s,t) = \ln(1 + a^s)/2$, *for* $e > a > 1$. *Indeed* $F(s,t) = s$ *implies that* $s = 0$.

(6). $F(s,t) = (s + l)^{(1/(1+t)^r)} - l$, $l > 1$, *for* $r \in (0,\infty)$.

(7). $F(s,t) = s \log_{t+a} a$, *for* $a > 1$.

(8). $F(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$.

(9). $F(s,t) = s\beta(s)$, *where* $\beta : [0,\infty) \to [0,1)$ *and continuous*

(10). $F(s,t) = s - \frac{t}{k+t}$.

(11). $F(s,t) = s - \varphi(s)$, *where* $\varphi : [0,\infty) \to [0,\infty)$ *is a continuous function such that* $\varphi(t) = 0$ *if and only if* $t = 0$.

(12). $F(s,t) = sh(s,t)$, *where* $h : [0,\infty) \times [0,\infty) \to [0,\infty)$ *is a continuous function such that* $h(t,s) < 1$ *for all* $t, s > 0$.

(13). $F(s,t) = s - (\frac{2+t}{1+t})t$.

(14). $F(s,t) = \sqrt[n]{\ln(1 + s^n)}$.

(15). $F(s,t) = \phi(s)$, *where* $\phi : [0,\infty) \to [0,\infty)$ *is a upper semicontinuous function such that* $\phi(0) = 0$ *and* $\phi(t) < t$ *for* $t > 0$.

(16). $F(s,t) = \frac{s}{\Gamma(1/2)} \int_0^\infty \frac{e^{-x}}{\sqrt{x}+t}\, dx$, *where* Γ *is the Euler Gamma function.*

Definition 2.5 ([8]). *A function* $\psi : [0,\infty) \to [0,\infty)$ *is called an altering distance function if the following properties are satisfied:*

(1). ψ *is non-decreasing and continuous,*

(2). $\psi(t) = 0$ *if and only if* $t = 0$.

Definition 2.6. *An ultra altering distance function is a continuous, nondecreasing mapping* $\varphi : [0,\infty) \to [0,\infty)$ *such that* $\varphi(t) > 0$, $t > 0$ *and* $\varphi(0) \geq 0$.

Remark 2.7. *We denote* Ψ *for set of all altering distance functions and* Φ_u *for set of all ultra altering distance functions.*

Lemma 2.8 ([3]). *Suppose* (X,d) *is a metric space. Let* $\{x_n\}$ *be a sequence in* X *such that* $d(x_n, x_{n+1}) \to 0$ *as* $n \to \infty$. *If* $\{x_n\}$ *is not a Cauchy sequence then there exist an* $\varepsilon > 0$ *and sequences of positive integers* $\{m(k)\}$ *and* $\{n(k)\}$ *with* $m(k) > n(k) > k$ *such that* $d(x_{m(k)}, x_{n(k)}) \geq \varepsilon$, $d(x_{m(k)-1}, x_{n(k)}) < \varepsilon$ *and*

(1). $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \varepsilon$;

(2). $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon$;

(3). $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \varepsilon$

It also follows that $\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \varepsilon$ *and* $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)-1}) = \varepsilon$.

In this paper, using the concept of C-class function, we prove the existence of proximity point for proximal C-contraction of rational-type proximal maps. We also give an example to show that our result is a proper extension of the result in [10].

3. Main Results

Definition 3.1. *Let A and B are two non-empty closed subsets of the partially ordered metric space (X, d). A mapping $T : A \to B$ is said to be proximal C-contraction of rational type if, for $u_1, u_2, x, y \in A$, it satisfies the condition that*

$$\left.\begin{array}{r} x \preceq y, x \neq y \\ d(u_1, Tx) = d(A, B) \\ d(u_2, Ty) = d(A, B) \end{array}\right\} \implies \psi(d(u_1, u_2)) \leq F(\psi(m(x, y)), \phi(m(x, y))) \tag{3}$$

where, $\psi \in \Psi, \phi \in \Phi_u, F \in C$ and $m(x, y) = \max\left\{\dfrac{d(x, u_1)d(y, u_2)}{d(x, y)}, d(x, y)\right\}$.

One can see that, when T is a self-mapping, ψ is defined as identity function and $F(s, t) = s - t$, the notion of proximal $C-$contraction of rational type reduces to generalised weak contraction of rational type.

Theorem 3.2. *Let X be a non-empty set such that $(X, \preceq)$ is a partially ordered set and (X, d) is a complete metric space. Let A and B be non-empty closed subsets of the metric space (X, d) such that $A_0 \neq \emptyset$. Let $T : A \to B$ satisfy the following conditions.*

(1). T is continuous, proximally increasing and proximal C-contraction of rational type such that $T(A_0) \subseteq B_0$.

(2). There exist x_0 and x_1 in A_0 such that $d(x_1, Tx_0) = d(A, B)$ and $x_0 \preceq x_1$.

Then, there exists an element x in A such that

$$d(x, Tx) = d(A, B).$$

Further, the sequence $\{x_n\}$, defined by

$$d(x_{n+1}, Tx_n) = d(A, B) \text{ for } n \geq 1,$$

converges to the element x.

Proof. By hypothesis there exist elements x_0 and x_1 in A_0 such that $d(x_1, Tx_0) = d(A, B)$ and $x_0 \preceq x_1$. Because of the fact that $T(A_0) \subseteq B_0$, there exists an element x_2 in A_0 such that $d(x_2, Tx_1) = d(A, B)$. Since T is a proximally increasing, we get $x_1 \preceq x_2$. Continuing this process, we can construct a sequence, $\{x_n\}$ in A_0 such that

$$d(x_{n+1}, Tx_n) = d(A, B) \text{ for all } n \in \mathbb{N} \tag{4}$$

with $x_0 \preceq x_1 \preceq x_2 \preceq \cdots x_n \preceq x_{n+1} \ldots$. If there exist n_0 such that $x_{n_0} = x_{n_0+1}$, then $d(x_{n_0+1}, Tx_{n_0}) = d(x_{n_0}, Tx_{n_0}) = d(A, B)$. This means that x_{n_0} is a best proximity point of T and the proof is finished. Thus, we can suppose that $x_n \neq x_{n+1}$ for all n. Since $x_{n-1} \neq x_n$ for all n, we get

$$\psi\Big(d(x_n, x_{n+1})\Big) \leq F\left(\psi\Big(\max\Big\{\frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{d(x_{n-1}, x_n)}, d(x_{n-1}, x_n)\Big\}\Big), \phi\Big(\max\Big\{\frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{d(x_{n-1}, x_n)}, d(x_{n-1}, x_n)\Big\}\Big)\right) \tag{5}$$

$$= F\left(\psi\Big(\max\Big\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\Big\}\Big), \phi\Big(\max\Big\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\Big\}\Big)\right) \tag{6}$$

Suppose that there exists m_0 such that $d(x_{m_0}, x_{m_0+1}) > d(x_{m_0-1}, x_{m_0})$, from (5), we have

$$\psi\Big(d(x_{m_0}, x_{m_0+1})\Big) \leq F\left(\psi\Big(d(x_{m_0}, x_{m_0+1})\Big), \phi\Big(d(x_{m_0}, x_{m_0+1})\Big)\right).$$

So, $\psi\left(d(x_{m_0}, x_{m_0+1})\right) = 0$ or $\phi(d(x_{m_0}, x_{m_0+1})) = 0$. Hence $d(x_{m_0}, x_{m_0+1}) = 0$, that is $x_{m_0} = x_{m_0+1}$, which is a contradiction. Hence, the sequence $\{d(x_n, x_{n+1})\}$ is monotone non-increasing and bounded. Thus, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r \geq 0. \tag{7}$$

Since, $\{d(x_n, x_{n+1})\}$ is a non-increasing sequence, from (5), we get

$$\psi\Big(d(x_n, x_{n+1})\Big) \leq F\Big(\psi\Big(d(x_{n-1}, x_n)\Big), \phi\Big(d(x_{n-1}, x_n)\Big)\Big), \;\; \forall\, x_{n-1} < x_n, n \geq 1 \tag{8}$$

Suppose that, $\lim_{n\to\infty} d(x_n, x_{n+1}) = r > 0$, then using inequality (8)

$$\psi(\lim_{n\to\infty} d(x_n, x_{n+1})) \leq F\Big(\psi\Big(\lim_{n\to\infty} d(x_{n-1}, x_n)\Big), \phi\Big(\lim_{n\to\infty} d(x_{n-1}, x_n)\Big)\Big)$$

that is

$$\psi(r) \leq F\Big(\psi(r), \phi(r)\Big) \tag{9}$$

So, $\psi\,(r) = 0$ or $\phi(r) = 0$. Hence, $r = 0$ and therefore,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{10}$$

Next, the claim is that $\{x_n\}$ is a Cauchy sequence. Suppose, $\{x_n\}$ is not a Cauchy sequence, then by lemma 2.8, there exists $\epsilon > 0$ for which we can find subsequences $\{x_{n_k}\}$and $\{x_{m_k}\}$ of $\{x_n\}$with $n_k > m_k > k$ such that

$$\begin{aligned} \epsilon &= \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) \\ &= \lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)}) = \lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) \end{aligned} \tag{11}$$

By choosing, $u_1 = x_{m(k)+1}, u_2 = x_{n(k)+1}, x = x_{m(k)}, y = x_{n(k)}$ from (3), we have

$$\begin{aligned} \psi\Big(d(x_{m(k)+1}, x_{n(k)+1})\Big) \leq F\Big(&\psi\Big(\max\Big\{\frac{d(x_{m(k)}, x_{m(k)+1})d(x_{n(k)}, x_{n(k)+1})}{d(x_{m(k)}, x_{n(k)})}, d(x_{m(k)}, x_{n(k)})\Big\}\Big), \\ &\phi\Big(\max\Big\{\frac{d(x_{m(k)}, x_{m(k)+1})d(x_{n(k)}, x_{n(k)+1})}{d(x_{m(k)}, x_{n(k)})}, d(x_{m(k)}, x_{n(k)})\Big\}\Big)\Big) \end{aligned}$$

Using (10) and (11) with $k \to \infty$ we obtain

$$\psi(\epsilon) \leq F\Big(\psi(\epsilon), \phi(\epsilon)\Big)$$

So, $\psi\,(\epsilon) = 0$ or $\phi(\epsilon) = 0$. Hence $\epsilon = 0$ which contradicts that $\epsilon > 0$. Thus, $\{x_n\}$ is a Cauchy sequence in A and hence converges to some element x in A. Since T is a continuous, we have $Tx_n \to Tx$. Hence, the continuity of the metric function d implies that $d(x_{n+1}, Tx_n) \to d(x, Tx)$. But (4) shows that the sequence $d(x_{n+1}, Tx_n)$ is a constant sequence with the value $d(A, B)$. Therefore, $d(x, Tx) = d(A, B)$. This completes the proof. □

Corollary 3.3. *Let X be a non-empty set such that $(X, \leq)$ is a partially ordered set and (X, d) is a complete metric space. Let A be a non-empty closed subset of the metric space (X, d). Let $T : A \to A$ satisfy the following conditions.*

(1). T is continuous, proximally increasing and proximal C-contraction of rational type.

(2). There exist elements x_0 and x_1 in A such that $d(x_1, Tx_0) = 0$ with $x_0 \leq x_1$.

Then, there exist an element x in A such that $d(x, Tx) = 0$.

Next, we prove that Theorem-(3.2) is still valid for T not necessarily continuous, assuming the following hypothesis in A.

$$\{x_n\} \text{ is a nondecreasing sequence in } A \text{ such that } x_n \to x, \text{ then } x = \sup\{x_n\}. \tag{12}$$

Theorem 3.4. *Assume the condition-(12) and A_0 is closed in X instead of continuity of T in the Theorem-(3.2), then the conclusion of Theorem-(3.2) holds.*

Proof. Following the proof of Theorem-(3.2), there exists a sequence $\{x_n\}$ in A satisfying the following condition

$$d(x_{n+1}, Tx_n) = d(A, B) \text{ for all } n \in \mathbb{N} \tag{13}$$

with $x_0 \preceq x_1 \preceq x_2 \preceq \cdots x_n \preceq x_{n+1} \ldots$ and x_n converges to x in A. Note that the sequence $\{x_n\}$ in A_0 and A_0 is closed. Therefore, $x \in A_0$. Since $T(A_0) \subseteq B_0$, we get $Tx \in B_0$. Since, $Tx \in B_0$, there exist $y_1 \in A$ such that

$$d(y_1, Tx) = d(A, B). \tag{14}$$

Since, $\{x_n\}$ is a non-decreasing sequence and $x_n \to x$, then $x = \sup\{x_n\}$. Particularly, $x_n \preceq x$ for all n. Since, T is a proximally increasing and from (13) and (14), we obtain $x_{n+1} \preceq y_1$. But $x = \sup\{x_n\}$ which implies $x \preceq y_1$. Therefore, we get there exist elements x and y_1 in A_0 such that

$$d(y_1, Tx) = d(A, B) \text{and } x \preceq y_1. \tag{15}$$

Consider the sequence $\{y_n\}$ that is constructed as follows

$$d(y_{n+1}, Ty_n) = d(A, B) \text{ for all } n \in \mathbb{N}. \tag{16}$$

with $x = y_0 \preceq y_1 \preceq y_2 \preceq \cdots y_n \preceq y_{n+1} \ldots.$ Arguing like above Theorem-(3.2), we obtain that $\{y_n\}$ is a non-decreasing sequence and $y_n \to y$ for certain $y \in A$. From (12), we have $y = \sup\{y_n\}$. Since, $x_n \preceq x = y_0 \preceq y_1 \preceq y_n \preceq y$ for all n, suppose that, $x \neq y$, then we have from (13) and (16), by T is proximal $C-$contraction of rational type,

$$\psi\Big(d(x_{n+1}, y_{n+1})\Big) \leq F\Big(\psi\Big(\max\Big\{\frac{d(x_n, x_{n+1})d(y_n, y_{n+1})}{d(x_n, y_n)}, d(x_n, y_n)\Big\}\Big), \phi\Big(\max\{\frac{d(x_n, x_{n+1})d(y_n, y_{n+1})}{d(x_n, y_n)}, d(x_n, y_n)\}\Big)\Big).$$

Taking limit as $n \to \infty$ in the above inequality, we have

$$\begin{aligned} \psi\Big(d(x, y)\Big) &\leq F\Big(\psi\Big(\max\{0, d(x, y)\}\Big), \phi\Big(\max\{0, d(x, y)\}\Big)\Big) \\ &= F\Big(\psi(d(x, y)), \phi(d(x, y))\Big) \end{aligned}$$

So $\psi\left(d(x, y)\right) = 0$ or $\phi(d(x, y)) = 0$. Hence, $d(x, y) = 0$ which is a contradiction. Hence, $x = y$. We have, $x = y_0 \preceq y_1 \preceq y_n = x$, therefore $y_n = x$, for all n. From (16), we obtain x is a best proximity point for T. The proof is complete. □

Corollary 3.5. *Assume the condition-(12) instead of continuity of T in the Corollary-(3.3), then the conclusion of Corollary-(3.3) holds.*

Now, we present an example where it can be appreciated that hypotheses in Theorem-(3.2) and Theorem-(3.4) do not guarantee uniqueness of the best proximity point.

Example 3.6. *Let $X = \{(0,2),(2,0),(-2,0),(0,-2)\} \subset \mathbb{R}^2$ and consider the usual order $(x,y) \preceq (z,t) \Leftrightarrow x \leq z$ and $y \leq t$. Thus, $(X,\preceq)$ is a partially ordered set. Besides, (X,d_2) is a complete metric space considering d_2 the euclidean metric. Let $A = \{(0,2),(2,0)\}$ and $B = \{(0,-2),(-2,0)\}$ be a closed subset of X. Then, $d(A,B) = 2\sqrt{2}, A = A_0$ and $B = B_0$. Let $T : A \to B$ be defined as $T(x,y) = (-y,-x)$. Then, it can be seen that T is continuous, proximally increasing mappings such that $T(A_0) \subseteq B_0$. The only comparable pairs of elements in A are $x \preceq x$ for $x \in A$ and there are no elements such that $x \prec y$ for $x,y \in A$. Hence, T is proximal $C-$contraction of rational type. It can be shown that the other hypotheses of the Theorem-(3.2) and 3.4 are also satisfied. However, T has two best proximity points $(0,2)$ and $(2,0)$.*

Theorem 3.7. *In addition to the hypotheses of Theorem 3.2 (respectively Theorem 3.4), suppose that*

$$\text{for every } x,y \in A_0, \text{ there exist } z \in A_0 \text{ that is comparable to } x \text{ and } y \tag{17}$$

then T has a unique best proximity point.

Proof. From Theorem 3.2 (resp. Theorem 3.4), the set of best proximity points of T is non-empty. Suppose that there exist elements x,y in A which are best proximity points. We distinguish two cases:

Case 1: If x and y are comparable. Since, $d(x,Tx) = d(A,B)$ and $d(y,Ty) = d(A,B)$. Since, T is a proximal $C-$contraction of rational type, we get

$$\begin{aligned}\psi\Big(d(x,y)\Big) &\leq F\bigg(\psi\Big(\max\Big\{\frac{d(x,x)d(y,y)}{d(x,y)}, d(x,y)\Big\}\Big), \phi\Big(\max\Big\{\frac{d(x,x)d(y,y)}{d(x,y)}, d(x,y)\Big\}\Big)\bigg)\\ &= F\Big(\psi(d(x,y)), \phi(d(x,y))\Big)\end{aligned}$$

So, $\psi\left(d(x,y)\right) = 0$ or $\phi(d(x,y)) = 0$. Hence, $d(x,y) = 0$ that is $x = y$.

Case 2: If x is not comparable to y. By the condition (17) there exist $z_0 \in A_0$ comparable to x and y. We define a sequence $\{z_n\}$ as $d(z_{n+1}, Tz_n) = d(A,B)$. Since, z_0 is comparable with x, we may assume that $z_0 \preceq x$. Since, T is a proximally increasing, $z_n \preceq x$ for all n. Suppose that there exist $n_0 > 1$ such that $x = z_{n_0}$, again by using T is proximally increasing, we get $x \preceq z_{n_0+1}$. But, $z_n \preceq x$ for all n. Therefore, $x = z_{n_0+1}$. Arguing like above, we obtain $x = z_n$ for all $n \geq n_0$. Hence, $z_n \to x$ as $n \to \infty$. On the other hand, if $z_{n-1} \neq x$ for all n. Now using T is a proximal $C-$contraction of rational type, we have

$$\begin{aligned}\psi\Big(d(z_n,x)\Big) &\leq F\bigg(\psi\Big(\max\Big\{\frac{d(z_{n-1},z_n)d(x,x)}{d(z_{n-1},x)}, d(z_{n-1},x)\Big\}\Big), \phi\Big(\max\Big\{\frac{d(z_{n-1},z_n)d(x,x)}{d(z_{n-1},x)}, d(z_{n-1},x)\Big\}\Big)\bigg)\\ &= F\Big(\psi(d(z_{n-1},x)), \phi(d(z_{n-1},x))) \leq \psi(d(z_{n-1},x)\Big).\end{aligned}$$

Since ψ is a nondecreasing, we get $d(z_n,x) \leq d(z_{n-1},x)$ Hence, the sequence $\{d(z_n,x)\}$ is monotone non-increasing and bounded. Thus, there exist $r \geq 0$ such that

$$\lim_{n\to\infty} d(z_n,x) = r \geq 0. \tag{18}$$

Suppose that $\lim_{n\to\infty} d(z_n,x) = r > 0$. Taking n$\to\infty$, $\psi(r) \leq F\Big(\psi(r),\phi(r)\Big)$. So $\psi\left(r\right) = 0$ or $\phi(r) = 0$. Hence $r = 0$ and therefore,

$$\lim_{n\to\infty}\left(d(z_n,x)\right) = 0. \tag{19}$$

Analogously, it can be proved that $\lim_{n\to\infty} d(z_n,y) = 0$. Finally, the uniqueness of the limit gives us $x = y$. ☐

Let us illustrate the above theorem with the following example.

Example 3.8. *Let $X = \mathbb{R}^2$ and consider the order $(x, y) \preceq (z, t) \Leftrightarrow x \leq z$ and $y \leq t$, where $\leq$ is usual order in $\mathbb{R}$. Thus, $(X, \preceq)$ is a partially ordered set. Besides, (X, d_1) is a complete metric space where the metric is defined as $d_1((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$. Let $A = \{(0, x) : x \in [0, 2]\}$ and $B = \{(2, x) : x \in [0, 2]\}$ be a closed subset of X. Then, $d(A, B) = 2, A = A_0$ and $B = B_0$. Let $T : A \to B$ be defined as $T(0, x) = (2, \frac{x}{4})$. Then, it can be seen that T is continuous and proximally increasing mappings such that $T(A_0) \subseteq B_0$. Now, we prove that T is a proximal $C-$contraction of rational type with $F(s, t) = \frac{s}{1+t}$, $\psi(t) = 4t$ and $\phi(t) = 3$. That is to prove*

$$\left.\begin{array}{r} x \leq y, x \neq y \\ d((0, \dfrac{x}{2}), T(0, x)) = 1 \\ d((0, \dfrac{y}{2}), T(0, y)) = 1 \end{array}\right\} \Rightarrow \psi\Big(d\big((0, \frac{x}{4}), (0, \frac{y}{4})\big)\Big) \leq F\Big(\psi\Big(m((0, x), (0, y))\Big), \phi\Big(m((0, x), (0, y))\Big)\Big)$$

where $m((0, x), (0, y)) = \max\{\frac{9xy}{16(y-x)}, y - x\}$.Note that $d\big((0, \frac{x}{4}), (0, \frac{y}{4})\big) = \frac{1}{4}|y - x|$ and

$$m((0, x), (0, y)) = \begin{cases} \frac{9xy}{16(y-x)} & \text{if } \frac{41}{16}xy \geq x^2 + y^2 \\ y - x & \text{if } \frac{41}{16}xy \leq x^2 + y^2. \end{cases}$$

so,

$$\begin{aligned} \psi\Big(d\big((0, \frac{x}{4}), (0, \frac{y}{4})\big)\Big) &\leq F\Big(\psi\Big(m((0, x), (0, y))\Big), \phi\Big(m((0, x), (0, y))\Big)\Big) \\ &= \frac{\psi(m((0, x), (0, y)))}{1 + \phi(m((0, x), (0, y)))} = \begin{cases} \frac{9xy}{16(y-x)} & \text{if } \frac{41}{16}xy \geq x^2 + y^2 \\ y - x & \text{if } \frac{41}{16}xy \leq x^2 + y^2. \end{cases} \end{aligned}$$

Thus, the mapping T is a proximal $C-$contraction of rational type. Hence all the hypotheses of the Theorem 3.7 are satisfied. where $(0, 0)$ is the unique best proximity point of the mapping T.

The following result, due to Nguyen Van Luong and Nguyen Xuan Thuan [10] is a corollary from the above theorem 3.7, by taking $A = B$.

Corollary 3.9. *In addition to the hypothesis of Corollary 3.3 (respectively Corollary 3.5), suppose that*

$$\text{for every } x, y \in A, \text{ there exist } z \in A \text{ that is comparable to } x \text{ and } y \tag{20}$$

then T has a unique fixed point.

References

[1] A.H. Ansari, *Note on φ -ψ-contractive type mappings and related fixed point*, The 2nd Regional Conference onMathematics And Applications, Payame Noor University, (2014), 377-380.

[2] A. Anthony Eldred and P. Veeramani, *Existence and convergence of best proximity points*, J. Math. Anal. Appl., 323(2006), 1001-1006.

[3] G.V.R. Babu and P.D. Sailaja, *A fixed point theorem of generalized weakly contractive maps in orbitally complete metric spaces*, Thai Journal of Mathematics, 9(2011).

[4] M. De la Sen and R.P. Agarwal, *Some fixed point-type results for a class of extended cyclic self-mappings with a more general contractive condition*, Fixed Point Theory Appl., 2011(2011), 59.

[5] J. Geno Kadwin and M. Marudai, *Fixed Point and Best Proximity Point Results for Generalised Cyclic Coupled Mappings*, Thai Journal of Mathematics, 14(2)(2015), 431-441.

[6] J.Geno Kadwin, Arslan Hojat Ansari, Choonkil Park and N. Annamalai, *Common fixed point results for weakly compatible mappings using C -class functions*, Journal of Computational Analysis and Applications, 25(1)(2018).

[7] D.S. Jaggi, *Some unique fixed point theorems*, Indian J. Pure Appl. Math., 8(1977), 223-230.

[8] M.S. Khan, M. Swaleh and S. Sessa,*Fixed point theorems by altering distances between the points*, Bulletin of the Australian mathematical society, 30(1)(1984).

[9] W.A. Kirk, S. Reich and P. Veeramani, *Proximinal retracts and best proximity pair theorems*, Numer. Funct. Anal. Optim., 24(2003), 851-862.

[10] N.V.Luong and N.X. Thuan, *Fixed point theorem for generalized weak contractions satisfying rational expressions in ordered metric spaces*, Fixed Point Theor. Appl., 46(2011).

[11] H.K.Nashine, P.Kumam and C. Vetro, *Best proximity point theorems for rational proximal contractions*, Fixed Point Theory Appl., 95(2013).

[12] V. Pragadeeswarar and M. Marudai, *Best proximity points for generalized proximal weak contractions in partially ordered metric spaces*, Optim. Lett., (2013).

[13] A.C.M. Ran and M.C.B. Reuring, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., 132(2004), 1435-1443.

[14] S. Sadiq Basha and P. Veeramani, *Best proximity pair theorems for multifunctions with open fibres*, J. Approx. Theory, 103(2000), 119-129.

[15] S. Sadiq Basha, *Discrete optimization in partially ordered sets*, J. Global Optim., 54(2012), 511-517.

[16] V. Sankar Raj, *A best proximity point theorem for weakly contractive non-self-mappings*, Nonlinear Anal., 74(2011), 4804-4808.