Int. J. Math. And Appl., 6(2–A)(2018), 53–62
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of *Mathematics And its Applications*

# Analytic Study on Fractional Implicit Differential Equations with Impulses via Katugampola Fractional Derivative

**M. Janaki[1,*], K. Kanagarajan[1] and D. Vivek[1]**

Department of Mathematics, Sri Ramakrishna Mission Vidyalaya College of Arts and Science, Coimbatore, Tamilnadu, India.

**Abstract:** In this paper, we establish certain conditions for the existence and uniqueness of solutions for such a class of fractional implicit differential equations using Katugampola fractional derivative in Caputo. The arguments are based on fixed point theorems. We provide an example, which shows the validity of our main results.

**MSC:** 26A33, 34A08, 34A09, 34A37.

**Keywords:** Generalized fractional derivative, Fractional implicit differential equations, Impulses, Fixed point.



*Accepted on: 26.03.2018*

## 1. Introduction

Fractional calculus is as old as the conventional calculus, and it is the generalization of integral order differentiation and integration to arbitrary noninteger order. For detailed study, see the books such as [10, 12–14]. The attraction towards this subject is due to the fact that fractional derivatives and integrals are not a local property. That is why fractional differential and integral models captured the reality of the nature better, as these models considered the history and nonlocal distributed effects. Fractional calculus has a large number of applications in different branches of science, engineering as well as in medical fields. The fractional differential models describe many real world phenomena in different fields, i.e., biology, dynamical systems, physics, control theory, chemistry and in many other fields, in a more efficient and realistic way. FDEs and control problems involving the Riemann-Liouville fractional derivative or the Caputo fractional derivative have been paid more and more attention. Very recently, a generalized Caputo-Katugampola derivative was proposed in [7, 8] by Katugampola, and further he proved the existence of solutions of Caputo-Katugampola FDEs in [9].

Impulsive differential equations, which provide a natural description of observed evolution processes, are regarded as important mathematical tools for the better understanding of several real world problems in applied sciences. The theory of impulsive differential equations of integer order has found extensive applications in realistic mathematical modeling of a wide variety of practical situations and has emerged as an important area of investigation in recent years. For the general theory and applications of impulsive differential equations, we refer the reader to the references [11, 15]. For some recent work on impulsive and implicit differential equations of fractional order, see [1–5, 16] and the references therein. Motivated by the

* *E-mail: janakimaths@gmail.com*

above-mentioned work, we treat fractional implicit differential equations with impulses involving Katugampola fractional derivative of the following form:

$$\begin{cases} {}^{\rho}D^{\omega}_{x_m}u(x) = h(x, u, {}^{\rho}D^{\omega}_{x_m}u(x)), & \text{for each } x \in (x_m, x_{m+1}],\ m = 0, 1, \ldots k,\ 0 < \omega \leq 1, \\ \Delta u|_{x=x_m} = I_m(u(x_m^-)), & m = 1, \ldots, k, \\ u(0) = u_0, \end{cases} \tag{1}$$

where ${}^{\rho}D^{\omega}_{x_m}$ is the Katugampola fractional derivative in Caputo sense, $h : \mathfrak{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $I_m : \mathbb{R} \to \mathbb{R}$, and $u_0 \in \mathbb{R}$, $0 = x_0 < x_1 < \cdots < x_k < x_{k+1} = T$, $\Delta u|_{x=x_m} = u(x_m^+) - u(x_m^-)$, $u(x_m^+) = \lim_{l\to 0^+} u(x_m + l)$ and $u(x_m^-) = \lim_{l\to 0^-} u(x_m + l)$ denotes the right and left limits of $u(x)$ at $x = x_m$.

In this paper, two results are presented for the problem (1). That is, the Banach contraction principle and the Schaefer's fixed point theorem.

## 2. Prerequisites

In this section, we introduce background definitions and lemmas that are needed for the proof of the main results.

Let $\mathfrak{C}(\mathfrak{J}, \mathbb{R})$ be the Banach space of all continuous functions from $\mathfrak{J}$ into $\mathbb{R}$ with the norm

$$\|u\|_{\infty} := \sup\{|u(x)| : x \in \mathfrak{J}\}.$$

**Definition 2.1** ([9])**.** *The generalized left-sided fractional integral* ${}^{\rho}I^{\omega}_{0+}h$ *of order* $\omega \in \mathbb{C}(Re(\omega) > 0)$ *is defined by*

$$({}^{\rho}I^{\omega}_{0+}h)(x) = \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_0^x (x^{\rho} - s^{\rho})^{\omega-1} s^{\rho-1} h(s) \mathrm{d}s, \tag{2}$$

*for* $x > 0$, *if the integral exists.*

**Definition 2.2** ([9])**.** *The generalized fractional derivative, corresponding to the generalized fractional integral* (2), *is defined by*

$$({}^{\rho}D^{\omega}_{0+}h)(x) = \frac{\rho^{\omega-n+1}}{\Gamma(n-\omega)} \left(x^{1-\rho}\frac{d}{dx}\right)^n \int_0^x (x^{\rho} - s^{\rho})^{n-\omega-1} s^{\rho-1} h(s) \mathrm{d}s, \tag{3}$$

*if the integral exists.*

**Lemma 2.3.** *Let* $\omega \geq 0$ *and* $n = [\omega] + 1$. *Then*

$${}^{\rho}I^{\omega}\left({}^{\rho}D^{\omega}_{0+}h(x)\right) = h(x) - \sum_{m=0}^{n-1} \frac{h^m(0)}{m!} x^m.$$

**Lemma 2.4.** *Let* $\omega > 0$, *then the differential equation* ${}^{\rho}D^{\omega}_{0+}u(x) = 0$ *has solutions*

$$u(x) = b_0 + b_1 x^{\rho} + b_2 x^{2\rho} + \cdots + b_{n-1} x^{(n-1)\rho},$$

$b_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\omega] + 1$.

**Lemma 2.5.** *Let $\omega > 0$, then*

$$^{\rho}I^{\omega}\left(^{\rho}D^{\omega}_{0+}u(x)\right) = u(x) + b_0 + b_1x^{\rho} + b_2x^{2\rho} + \cdots + b_{n-1}x^{(n-1)\rho},$$

*for some $b_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\omega] + 1$.*

**Theorem 2.6** (Banach's fixed point theorem [6])**.** *Let $\mathcal{C}$ be a non empty closed subset of a Banach space $\mathcal{X}$, then any contraction mapping $T$ of $\mathcal{C}$ into itself has a unique fixed point.*

**Theorem 2.7** (Schaefer's fixed point theorem [6])**.** *Let $\mathcal{X}$ be a Banach space, and $M : \mathcal{X} \to \mathcal{X}$ a completely continuous operator. If the set*

$$S = \{u \in \mathcal{X} : u = \mu M u, \text{for some}\, \mu \in (0,1)\}$$

*is bounded, then $M$ has fixed points.*

# 3. Existence of solutions

Let $\mathfrak{C}(\mathfrak{J}, \mathbb{R})$ be the Banach space of continuous functions $\mathfrak{J} \to \mathbb{R}$, with the supremum norm

$$\|u\|_{\infty} = \sup\{|u(x)|, x \in \mathfrak{J}\}.$$

Consider the set of functions

$$\mathfrak{PC}(\mathfrak{J}, \mathbb{R}) = \{u : \mathfrak{J} \to \mathbb{R} : u \in \mathfrak{C}\left((x_m, x_{m+1}], \mathbb{R}\right), m = 0, 1, \ldots, k, \text{ and there exist}$$
$$u(x_m^-) \text{ and } u(x_m^+),\ m = 1, 2, \ldots, k \text{ with } u(x_m^-) = u(x_m)\}.$$

$\mathfrak{PC}(\mathfrak{J}, \mathbb{R})$ is a Banach space with the norm

$$\|u\|_{\mathfrak{PC}} = \sup_{x \in \mathfrak{J}} |u(x)|.$$

Let $\mathfrak{J}_0 = [x_0, x_1]$ and $\mathfrak{J}_m = (x_m, x_{m+1}]$, where $m = 1, 2, \ldots, k$.

**Definition 3.1.** *A function $u \in \mathfrak{PC}(\mathfrak{J}, \mathbb{R})$ whose $\omega$-derivative exists on $\mathfrak{J}_m$, is said to be a solution of (1), if $u$ satisfies the equation*

$$^{\rho}D^{\omega}_{x_m}u(x) = h(x, u(x), {}^{\rho}D^{\omega}_{x_m}u(x))$$

*on $\mathfrak{J}_m$, and satisfies the conditions $\Delta u|_{x=x_m} = I_m(u(x_m^-))$, $m = 0, 1, \ldots, k$, $u(0) = u_0$.*

To prove the existence of solutions of (1), we need the following lemma.

**Lemma 3.2.** *Let $0 < \omega \leq 1$ and let $h : \mathfrak{J} \to \mathbb{R}$ be continuous. A function $u$ is a solution of the fractional integral equation*

$$u(x) = \begin{cases} u_0 + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_0^x (x^{\rho} - s^{\rho})^{\omega-1} s^{\rho-1}\ h(s)\mathrm{d}s, & \text{if } x \in [0, x_1], \\ u_0 + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \sum_{i=1}^{m} \int_{x_{i-1}}^{x_i} (x_i^{\rho} - s^{\rho})^{\omega-1} s^{\rho-1}\ h(s)\mathrm{d}s & \\ + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^{x} (x^{\rho} - s^{\rho})^{\omega-1} s^{\rho-1}\ h(s)\mathrm{d}s & \\ + \sum_{i=1}^{m} I_i(u(x_i^-)), & \text{if } x \in \mathfrak{J}_m := (x_m, x_{m+1}], \end{cases} \tag{4}$$

where $m = 1, 2, \ldots, k$, if and only if, $u$ is a solution of the fractional initial value problem

$$\begin{cases} {}^{\rho}D^{\omega}_{x_m} u(x) = h(x), & x \in \mathfrak{J}_m, \\ \Delta u|_{x=x_m} = I_m(u(x_m^-)), & m = 1, 2, \ldots, k, \\ u(0) = u_0. \end{cases} \tag{5}$$

Now, we prove the existence result for the problem (1) based on the Banach's fixed point theorem.

**Theorem 3.3.** *Assume that,*

*(A1) The function $h : \mathfrak{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.*

*(A2) There exist constants $c_1 > 0$ and $0 < c_2 < 1$ such that $|h(x, u_1, u_2) - h(x, \overline{u_1}, \overline{u_2})| \le c_1 |u_1 - \overline{u_1}| + c_2 |u_2 - \overline{u_2}|$, for any $u_1, u_2, \overline{u_1}, \overline{u_2} \in \mathbb{R}$ and $x \in \mathfrak{J}$.*

*(A3) There exists a constant $c_3 > 0$ such that $|I_m(u_1) - I_m(u_2)| \le c_3 |u_1 - u_2|$, for each $u_1, u_2 \in \mathbb{R}$ and $m = 1, 2, \ldots, k$.*

*If*

$$\frac{c_1(k+1)T^{\rho\omega}}{(1-c_2)\rho^{\omega}\Gamma(\omega+1)} + kc_3 < 1, \tag{6}$$

*then there exists a unique solution for the initial value problem (1) on $\mathfrak{J}$.*

*Proof.* Transform the problem (1) into a fixed point problem. Consider the operator $M : \mathfrak{PC}(\mathfrak{J}, \mathbb{R}) \to \mathfrak{PC}(\mathfrak{J}, \mathbb{R})$ defined by,

$$\begin{aligned} M(u)(x) = u_0 &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \sum_{0<x_m<x} \int_{x_{m-1}}^{x_m} (x_m^{\rho} - s^{\rho})^{\omega-1} s^{\rho-1} f(s) \mathrm{d}s \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^{x} (x^{\rho} - s^{\rho})^{\omega-1} s^{\rho-1} f(s) \mathrm{d}s + \sum_{0<x_m<x} I_m\left(u(x_m^-)\right), \end{aligned} \tag{7}$$

where $f \in \mathfrak{C}(\mathfrak{J}, \mathbb{R})$ is such that $f(x) = h(x, u(x), f(x))$. Clearly, the fixed points of operator $M$ are solutions of problem (1). Let $u_1, u_2 \in \mathfrak{PC}(\mathfrak{J}, \mathbb{R})$, then for $x \in \mathfrak{J}$, we have

$$\begin{aligned} |M(u_1)(x) - M(u_2)(x)| \le &\frac{\rho^{1-\omega}}{\Gamma(\omega)} \sum_{0<x_m<x} \int_{x_{m-1}}^{x_m} (x_m^{\rho} - s^{\rho})^{\omega-1} s^{\rho-1} |f_1(s) - f_2(s)| \mathrm{d}s \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^{x} (x^{\rho} - s^{\rho})^{\omega-1} s^{\rho-1} |f_1(s) - f_2(s)| \mathrm{d}s \\ &+ \sum_{0<x_m<x} \left| I_m(u_1(x_m^-)) - I_m(u_2(x_m^-)) \right|, \end{aligned}$$

where $f_1, f_2 \in \mathfrak{C}(\mathfrak{J}, \mathbb{R})$ are such that $f_1(x) = h(x, u_1(x), f_1(x))$, and $f_2(x) = h(x, u_2(x), f_2(x))$. By (A2), we have

$$|f_1(x) - f_2(x)| \le c_1 |u_1(x) - u_2(x)| + c_2 |f_1(x) - f_2(x)|.$$

Thus

$$|f_1(x) - f_2(x)| \le \left(\frac{c_1}{1-c_2}\right) |u_1(x) - u_2(x)|.$$

Then, for $x \in \mathfrak{J}$,

$$|M(u_1)(x) - M(u_2)(x)| \le \frac{\rho^{1-\omega}}{\Gamma(\omega)} \left(\frac{c_1}{1-c_2}\right) \sum_{m=1}^{k} \int_{x_{m-1}}^{x_m} (x_m^{\rho} - s^{\rho})^{\omega-1} s^{\rho-1} |u_1(x) - u_2(x)| \mathrm{d}s$$

$$+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\left(\frac{c_1}{1-c_2}\right)\int_{x_m}^{x}(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}\left|u_1(x)-u_2(x)\right|\mathrm{d}s+\sum_{m=1}^{k}c_3\left|u_1(x_m^-)-u_2(x_m^-)\right|$$
$$\leq\frac{c_1kT^{\rho\omega}}{(1-c_2)\rho^\omega\Gamma(\omega+1)}\|u_1-u_2\|_{\mathfrak{PC}}+\frac{c_1T^{\rho\omega}}{(1-c_2)\rho^\omega\Gamma(\omega+1)}\|u_1-u_2\|_{\mathfrak{PC}}+kc_3\|u_1-u_2\|_{\mathfrak{PC}}.$$

Thus,

$$\|M(u_1)-M(u_2)\|_{\mathfrak{PC}}\leq\left[\frac{c_1(k+1)T^{\rho\omega}}{(1-c_2)\rho^\omega\Gamma(\omega+1)}+kc_3\right]\|u_1-u_2\|_{\mathfrak{PC}}.$$

By (6), the operator $M$ is a contraction. Hence, by Banach's contraction principle, $M$ has a unique fixed point which is a unique solution of the problem (1). □

Now we prove the second result based on Schaefer's fixed point theorem.

**Theorem 3.4.** *Assume that (A1),(A2) and*

*(A4) There exists $p_1,p_2,p_3\in\mathfrak{C}(\mathfrak{J},\mathbb{R}_+)$ with $p_3^*=\sup_{x\in\mathfrak{J}}p_3(x)<1$ such that $|h(x,u_1,u_2)|\leq p_1(x)+p_2(x)\,|u_1|+p_3(x)\,|u_2|$, for $x\in\mathfrak{J}$ and $u_1,u_2\in\mathbb{R}$.*

*(A5) The functions $I_m:\mathbb{R}\to\mathbb{R}$ are continuous and there exist constants $M_1^*,\ M_2^*>0$ such that $|I_m(u)|\leq M_1^*\,|u|+M_2^*$, for each $u\in\mathbb{R}$, $m=1,2,\ldots,k$. If*

$$kM_1^*+\frac{(k+1)T^{\rho\omega}p_2^*}{(1-p_3^*)\rho^\omega\Gamma(\omega+1)}<1,$$

*then the initial value problem (1) has at least one solution on $\mathfrak{J}$.*

*Proof.* Consider the operator $M$ defined in (7). Now we use the Schaefer's fixed point theorem to prove that $M$ has a fixed point. The proof contains several steps.

**Step 1:** *M is continuous.* Let $\{u_n\}$ be a sequence such that $u_n\to u$ in $\mathfrak{PC}(\mathfrak{J},\mathbb{R})$. Then for each $x\in\mathfrak{J}$,

$$|M(u_n)(x)-M(u)(x)|\leq\frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{0<x_m<x}\int_{x_{m-1}}^{x_m}(x_m^\rho-s^\rho)^{\omega-1}s^{\rho-1}\left|f_n(s)-f(s)\right|\mathrm{d}s$$
$$+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}(x^\rho-s^\rho)^{\omega-1}s^{\rho-1}\left|f_n(s)-f(s)\right|\mathrm{d}s+\sum_{0<x_m<x}\left|I_m(u_n(x_m^-))-I_m(u(x_m^-))\right|,$$

where $f_n,f\in\mathfrak{C}(\mathfrak{J},\mathbb{R})$ are such that $f_n(x)=h(x,u_n(x),f_n(x))$ and $f(x)=h(x,u(x),f(x))$. By (A2), we have

$$|f_n(x)-f(x)|\leq c_1\left|u_n(x)-u(x)\right|+c_2\left|f_n(x)-f(x)\right|$$
$$\leq\left(\frac{c_1}{1-c_2}\right)\left|u_n(x)-u(x)\right|.$$

Since $u_n\to u$, then we get $f_n(x)\to f(x)$ as $n\to\infty$ for each $x\in\mathfrak{J}$. And let $\Omega>0$ be such that, for each $x\in\mathfrak{J}$, we have $|g_n(x)|\leq\Omega$ and $|f(x)|\leq\Omega$. Then, we have

$$(x^\rho-s^\rho)^{\omega-1}\left|f_n(s)-f(s)\right|\leq(x^\rho-s^\rho)^{\omega-1}\left[\left|f_n(s)\right|+\left|f(s)\right|\right]\leq2\Omega(x^\rho-s^\rho)^{\omega-1},$$

and

$$(x_m^\rho-s^\rho)^{\omega-1}\left|f_n(s)-f(s)\right|\leq(x_m^\rho-s^\rho)^{\omega-1}\left[\left|f_n(s)\right|+\left|f(s)\right|\right]\leq2\Omega(x_m^\rho-s^\rho)^{\omega-1}.$$

For each $x\in\mathfrak{J}$, the functions $s\to2\Omega(x^\rho-s^\rho)^{\omega-1}$ and $s\to2\Omega(x_m^\rho-s^\rho)^{\omega-1}$ are integrable on $[0,x]$; then by the Lebesgue dominated convergence theorem and (5) implies that $|M(u_n)(x)-M(u)(x)|\to0$ as $n\to\infty$, and hence

$\|M(u_n) - M(u)\|_{\mathfrak{PC}} \to 0$ as $n \to \infty$. Consequently, $M$ is continuous.

**Step 2:** *$\mathfrak{F}$ maps bounded sets into bounded sets in $\mathfrak{PC}(\mathfrak{J}, \mathbb{R})$.* Now, it is enough to show that for any $\Omega^* > 0$, there exists a positive constant $k_1$ such that for each

$$u \in B_{\Omega^*} = \left\{u \in \mathfrak{PC}(\mathfrak{J}, \mathbb{R}) : \|u\|_{\mathfrak{PC}} \leq \Omega^*\right\},$$

we have $\|M(u)\|_{\mathfrak{PC}} \leq k_1$. Then for each $x \in \mathfrak{J}$,

$$M(u)(x) = u_0 + \frac{\rho^{1-\omega}}{\Gamma(\omega)} \sum_{0<x_m<x} \int_{x_{m-1}}^{x_m} (x_m^\rho - s^\rho)^{\omega-1} s^{\rho-1} f(s) \mathrm{d}s \tag{8}$$
$$+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} f(s) \mathrm{d}s + \sum_{0<x_m<x} I_m\left(u(x_m^-)\right),$$

where $f \in \mathfrak{C}(\mathfrak{J}, \mathbb{R})$ is such that $f(x) = h(x, u(x), f(x))$. By $(A4)$, we have for each $x \in \mathfrak{J}$,

$$\begin{aligned}|f(x)| &= |h(x, u(x), f(x))| \\ &\leq p_1(x) + p_2(x)\,|u(x)| + p_3(x)\,|f(x)| \\ &\leq p_1(x) + p_2(x)\Omega^* + p_3(x)\,|f(x)| \\ &\leq p_1^* + p_2^*\Omega^* + p_3^*\,|f(x)|,\end{aligned}$$

where $p_1^* = \sup_{x \in \mathfrak{J}} p_1(x)$ and $p_2^* = \sup_{x \in \mathfrak{J}} p_2(x)$. Then

$$|f(x)| \leq \frac{p_1^* + p_2^*\Omega^*}{(1 - p_3^*)} := N$$

Thus (8) implies,

$$\begin{aligned}|M(u)(x)| &\leq |u_0| + \frac{kNT^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)} + \frac{NT^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)} + k(M_1^*\,|u| + M_2^*) \\ &\leq |u_0| + \frac{kNT^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)} + \frac{NT^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)} + k(M_1^*\Omega^* + M_2^*).\end{aligned}$$

Then,

$$\|M(u)\|_{\mathfrak{PC}} \leq |u_0| + \frac{(k+1)NT^{\rho\omega}}{\rho^\omega\Gamma(\omega+1)} + k(M_1^*\Omega^* + M_2^*) := k_1.$$

**Step 3:** *$\mathfrak{F}$ maps bounded sets into equicontinuous sets of $\mathfrak{PC}(\mathfrak{J}, \mathbb{R})$.* Let $t_1, t_2 \in \mathfrak{J}$, $t_1 < t_2$, $B_{\Omega^*}$ be a bounded set of $\mathfrak{PC}(\mathfrak{J}, \mathbb{R})$ as in Step 2, and let $u \in B_{\Omega^*}$. Then,

$$\begin{aligned}|M(u)(t_2) - M(u)(t_1)| &\leq \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_0^{t_1} \left|(t_2^\rho - s^\rho)^{\omega-1} - (t_1^\rho - s^\rho)^{\omega-1}\right| s^{\rho-1}\,|f(s)|\,\mathrm{d}s \\ &+ \frac{\rho^{1-\omega}}{\Gamma(\omega)} \int_{t_1}^{t_2} |t_2^\rho - s^\rho|^{\omega-1} s^{\rho-1}\,|f(s)|\,\mathrm{d}s + \sum_{0<x_m<(t_2-t_1)} \left|I_m(u(x_m^-))\right| \\ &\leq \frac{N\rho^{1-\omega}}{\rho^\omega\Gamma(\omega+1)} [2(t_2^\rho - t_1^\rho)^\omega + (t_2^{\rho\omega} - t_1^{\rho\omega})] + (t_2^\rho - t_1^\rho)(M_1^*\,|u| + M_2^*) \\ &\leq \frac{N\rho^{1-\omega}}{\rho^\omega\Gamma(\omega+1)} [2(t_2^\rho - t_1^\rho)^\omega + (t_2^{\rho\omega} - t_1^{\rho\omega})] + (t_2^\rho - t_1^\rho)(M_1^*\Omega^* + M_2^*).\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. Therefore, from Steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $M : \mathfrak{PC}(\mathfrak{J}, \mathbb{R}) \to \mathfrak{PC}(\mathfrak{J}, \mathbb{R})$ is completely continuous.

**Step 4:** *A priori bounds.* Now we need to show that the set

$$G = \{u \in \mathfrak{PC}(\mathfrak{J}, \mathbb{R}) : u = \mu M(u), \text{ for some } 0 < \mu < 1\},$$

is bounded. Let $u \in G$, then $u = \mu M(u)$ for some $0 < \mu < 1$. Thus, for each $x \in \mathfrak{J}$, we have

$$u(x) = \mu u_0 + \frac{\mu\rho^{1-\omega}}{\Gamma(\omega)} \sum_{0<x_m<x} \int_{x_{m-1}}^{x_m} (x_m^\rho - s^\rho)^{\omega-1} s^{\rho-1} f(s) \mathrm{d}s$$
$$+ \frac{\mu\rho^{1-\omega}}{\Gamma(\omega)} \int_{x_m}^{x} (x^\rho - s^\rho)^{\omega-1} s^{\rho-1} f(s) \mathrm{d}s + \mu \sum_{0<x_m<x} I_m(u(x_m^-)).$$

And, by $(A3)$, we have for each $x \in \mathfrak{J}$,

$$\begin{aligned} |f(x)| &= |h(x, u(x), f(x))| \\ &\leq p_1(x) + p_2(x)\,|u(x)| + p_3(x)\,|f(x)| \\ &\leq p_1^* + p_2^*\,|u(x)| + p_3^*\,|f(x)|\,. \end{aligned}$$

Thus,

$$|f(x)| \leq \frac{1}{(1-p_3^*)}(p_1^* + p_2^*\,|u(x)|).$$

This implies, by $(A4)$ and $(A5)$ (as in Step 2), that for each $x \in \mathfrak{J}$, we get

$$\begin{aligned} |u(x)| &\leq |u_0| + \frac{kT^{\rho\omega}(\frac{1}{1-p_3^*})\,(p_1^* + p_2^*\,|u(x)|)}{\rho^\omega\Gamma(\omega+1)} + \frac{T^{\rho\omega}(\frac{1}{1-p_3^*})\,(p_1^* + p_2^*\,|u(x)|)}{\rho^\omega\Gamma(\omega+1)} + k(M_1^*\,|u(x)| + M_2^*) \\ &\leq |u_0| + \frac{kT^{\rho\omega}(\frac{1}{1-p_3^*})(p_1^* + p_2^*\|u\|_{\mathfrak{PC}})}{\rho^\omega\Gamma(\omega+1)} + \frac{T^{\rho\omega}(\frac{1}{1-p_3^*})(p_1^* + p_2^*\|u\|_{\mathfrak{PC}})}{\rho^\omega\Gamma(\omega+1)} + k(M_1^*\|u\|_{\mathfrak{PC}} + M_2^*). \end{aligned}$$

Then, we get,

$$\|u\|_{\mathfrak{PC}} \leq |u_0| + \frac{kT^{\rho\omega}(\frac{1}{1-p_3^*})(p_1^* + p_2^*\|u\|_{\mathfrak{PC}})}{\rho^\omega\Gamma(\omega+1)} + \frac{T^{\rho\omega}(\frac{1}{1-p_3^*})(p_1^* + p_2^*\|u\|_{\mathfrak{PC}})}{\rho^\omega\Gamma(\omega+1)} + k(M_1^*\|u\|_{\mathfrak{PC}} + M_2^*).$$

Thus,

$$\|u\|_{\mathfrak{PC}} \leq \frac{|u_0| + kM_2^* + \frac{p_1^*(k+1)T^{\rho\omega}}{(1-p_3^*)\rho^\omega\Gamma(\omega+1)}}{1 - kM_1^* - \frac{p_2^*(k+1)T^{\rho\omega}}{(1-p_3^*)\rho^\omega\Gamma(\omega+1)}} := L$$

This shows that the set $G$ is bounded. By Schaefer's fixed point theorem, we conclude that $M$ has a fixed point which is a solution of the problem (1). □

# 4. Nonlocal Fractional Implicit Differential Equations with Impulses

In this section, we present the existence and uniqueness result for the following nonlocal fractional implicit differential equations with impulses involving Katugampola derivative,

$$\begin{cases} {}^\rho D_{x_m}^\omega u(x) = h(x, u, {}^\rho D_{x_m}^\omega u(x)), & \text{for each } x \in (x_m, x_{m+1}],\ m = 0, 1, \ldots k,\ 0 < \omega \leq 1, \\ \Delta u|_{x=x_m} = I_m(u(x_m^-)), & m = 1, 2, \ldots k, \\ u(0) + \psi(u) = u_0, \end{cases} \tag{9}$$

where $h, u_0, I_m$ are defined as in Section 3 and $\psi : \mathfrak{C}(\mathfrak{J}, \mathbb{R}) \to \mathbb{R}$ is a continuous function.

**Theorem 4.1.** *Assume* $(A1)$-$(A3)$ *and the following hypothesis holds:*

*(A6) There exists a constant* $\delta > 0$ *such that* $|\psi(u) - \psi(v)| \leq \delta\,|u - v|$, *for each* $u, v \in \mathfrak{PC}(\mathfrak{J}, \mathbb{R})$.

*If*

$$\left[\frac{c_1(k+1)T^{\rho\omega}}{(1-c_2)\rho^{\omega}\Gamma(\omega+1)}+mc_3+\delta\right]<1,$$

*then the problem* (9) *has a unique solution on* $\mathfrak{J}$.

*Proof.* We transform the nonlocal problem (8) into a fixed point problem. Consider the operator $\tilde{M}:\mathfrak{PC}(\mathfrak{J},\mathbb{R})\to\mathfrak{PC}(\mathfrak{J},\mathbb{R})$ defined by

$$\begin{aligned}\tilde{M}(u)(x)=u_0-\psi(u)+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\sum_{0<x_m<x}\int_{x_{m-1}}^{x_m}(x_m^{\rho}-s^{\rho})^{\omega-1}s^{\rho-1}f(s)\mathrm{d}s\\+\frac{\rho^{1-\omega}}{\Gamma(\omega)}\int_{x_m}^{x}(x^{\rho}-s^{\rho})^{\omega-1}s^{\rho-1}f(s)\mathrm{d}s+\sum_{0<x_m<x}I_m(u(x_m^-)),\end{aligned}$$

where $f\in\mathfrak{C}(\mathfrak{J},\mathbb{R})$ be such that $f(x)=h(x,u(x),f(x))$. Clearly, the fixed points of the operator $\tilde{M}$ are solution of the problem (8). From Section 3, we can easily prove that $\tilde{M}$ is a contraction. □

## 5. Examples

**Example 5.1.** *Consider the following fractional implicit differential equations with impulses Katugampola derivative,*

$$\begin{cases}{}^{\rho}D_{x_m}^{\frac{1}{2}}u(x)=\frac{1}{28e^{x+3}\left(1+|u(x)|+\left|{}^{\rho}D^{\frac{1}{2}}u(x)\right|\right)}, & \text{for each } x\in\mathfrak{J}_0\cup\mathfrak{J}_1,\\ \Delta u|_{x=\frac{1}{3}}=\frac{\left|u(\frac{1}{3}^-)\right|}{76+\left|u(\frac{1}{3}^-)\right|}, \\ u(0)=1,\end{cases}\tag{10}$$

*where* $\mathfrak{J}_0=[0,\frac{1}{3}]$, $\mathfrak{J}_1=(\frac{1}{3},1]$, $x_0=0$ *and* $x_1=\frac{1}{3}$. *Let us assume,*

$$h(x,u_1,u_2)=\frac{1}{28e^{x+3}\left(1+|u_1|+|u_2|\right)},\ x\in[0,1],\ u_1,u_2\in\mathbb{R}.$$

*Clearly, the function* $h$ *is jointly continuous. For each* $u_1,u_2,\overline{u}_1,\overline{u}_2\in\mathbb{R}$ *and* $x\in[0,1]$:

$$|h(x,u_1,u_2)-h(x,\overline{u}_1,\overline{u}_2)|\le\frac{1}{28e^3}\left(|u_1-\overline{u}_1|+|u_2-\overline{u}_2|\right).$$

*Hence the condition* $(A2)$ *is satisfied with* $c_1=c_2=\frac{1}{28e^3}$. *And let,*

$$I_1(u_1)=\frac{u_1}{76+u_1},\ u_1\in[0,\infty).$$

*Let* $u_1,u_2\in[0,\infty)$, *then we have,*

$$\begin{aligned}|I_1(u_1)-I_1(u_2)|&\le\frac{76\,|u_1-u_2|}{(76+u_1)(76+u_2)}\\&\le\frac{1}{76}\,|u_1-u_2|.\end{aligned}$$

*Thus the condition,*

$$\frac{c_1(k+1)T^{\rho\omega}}{(1-c_2)\rho^{\omega}\Gamma(\omega+1)}+kc_3=\frac{4}{(28e^3-1)(0.4)^{0.5}\sqrt{\pi}}+\frac{1}{76}<1$$

*is satisfied with* $T=1$, $k=1$, *and* $c_3=\frac{1}{76}$, $\omega=0.5$, $\rho=0.4$. *It folows from Theorem 3.3 that the problem* (10) *has a unique solution on* $\mathfrak{J}=[0,1]$.

**Example 5.2.** *Consider the following fractional implicit differential equations with impulses involving Katugampola derivative,*

$$\begin{cases} {}^{\rho}D^{\frac{1}{2}}_{x_m}u(x) = \frac{2+|u(x)|+\left|{}^{\rho}D^{\frac{1}{2}}u(x)\right|}{98e^{x+4}\left(1+|u(x)|+\left|{}^{\rho}D^{\frac{1}{2}}u(x)\right|\right)}, & \text{for each } x\in\mathfrak{J}_0\cup\mathfrak{J}_1, \\ \Delta u|_{x=\frac{1}{4}} = \frac{\left|u(\frac{1}{4}^-)\right|}{66+\left|u(\frac{1}{4}^-)\right|}, \\ u(0)=1, \end{cases} \tag{11}$$

*where* $\mathfrak{J}_0=[0,\frac{1}{4}]$, $\mathfrak{J}_1=(\frac{1}{4},1]$, $x_0=0$ *and* $x_1=\frac{1}{4}$. *Set*

$$h(x,u_1,u_2)=\frac{2+|u_1|+|u_2|}{98e^{x+4}\left(1+|u_1|+|u_2|\right)},\ x\in[0,1], u_1,u_2\in\mathbb{R}.$$

*Clearly, the function* $h$ *is jointly continuous. For any* $u_1,u_2,\overline{u}_1,\overline{u}_2\in\mathbb{R}$ *and* $x\in[0,1]$*:*

$$|h(x,u_1,u_2)-h(x,\overline{u}_1,\overline{u}_2)|\le\frac{1}{98e^4}\left(|u_1-\overline{u}_1|+|u_2-\overline{u}_2|\right).$$

*Hence the condition* $(A2)$ *is satisfied with* $c_1=c_2=\frac{1}{98e^4}$. *We get, for each* $x\in[0,1]$,

$$|h(x,u_1,u_2)|\le\frac{1}{98e^{x+4}}\left(2+|u_1|+|u_2|\right).$$

*Thus the condition* $(A4)$ *is satisfied with* $p_1(x)=\frac{1}{49e^{x+4}}$ *and* $p_2(x)=p_3(x)=\frac{1}{98e^{x+4}}$, *and let*

$$I_1(u_1)=\frac{u_1}{66+u_1},\ u_1\in[0,\infty),$$

*we have, for each* $u_1\in[0,\infty)$,

$$|I_1(u_1)|\le\frac{1}{66}u_1+1.$$

*Thus the condition* $(A5)$ *is satisfied with* $M_1^*=\frac{1}{66}$ *and* $M_2^*=1$. *Thus the condition*

$$kM_1^*+\frac{(k+1)T^{\rho\omega}p_2^*}{(1-p_3^*)\rho^{\omega}\Gamma(\omega+1)}=\frac{1}{66}+\frac{4}{(98e^4-1)(0.4)^{0.5}\sqrt{\pi}}<1$$

*is satisfied with* $T=1$, $k=1$ *and* $p_2^*(x)=p_3^*(x)=\frac{1}{98e^4}$, $\rho=0.4$, $\omega=0.5$. *It follows from Theorem 3.4 that the problem (11) has at least one solution on* $\mathfrak{J}=[0,1]$.

## References


[1] R.P.Agarwal, M.Benchohra and B.A.Slimani, *Existence results for differential equations with fractional order impulses*, Mem. Differ. Equ. Math. Phys., 44(1)(2008)1-21.

[2] M.Benchohra and J.E.Lazreg, *Existence results for nonlinear implicit fractional differential equations with impulse*, Comm. Appl. Anal., 19(2015), 413-426.

[3] M.Benchohra and J.E.Lazreg, *Nonlinear fractional implicit differential equations*, Comm. Appl. Anal., 17(2013), 471-482.

[4] M.Benchohra and D.Seba, *Impulsive fractional differential equations in Banach spaces*, Electron. J. Qual. Theory Diff. Eqn. Spec. Ed. I, 8(2009), 1-14.

[5] M.Benchohra and B.A.Slimani, *Existence and uniqueness of solutions to impulsive fractional differential equations*, Elec. J. Diff. Eqn., 10(2009), 1-11.

[6] A.Granas and J.Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, (2003).

[7] U.N.Katugampola, *A new approach to generalized fractional derivatives*, Bull. Math. Anal. App., 6(4)(2014), 1-15.

[8] U.N.Katugampola, *New approach to a generalized fractional integral*, Appl. Math. Comput., 218(3)(2011), 860-865.

[9] U.N.Katugampola, *Existence and uniqueness results for a class of generalized fractional differential equations*, Bull. Math. Anal. Appl., arXiv:1411.5229v2[math.CA] 9 Jun(2016).

[10] A.A.Kilbas, H.M.Srivastava and J.J.Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., Amsterdam, (2006).

[11] V.Lakshmikantham, D.D.Bainov and P.S.Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, (1989).

[12] K.S.Miller and B.Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, (1993).

[13] I.Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, (1999).

[14] S.G.Samko, A.A.Kilbas and O.I.Marichev, *Fractional Integrals and Derivatives*, Theory and Applications, Gordon and Breach, Yverdon, (1993).

[15] A.M.Samoilenko and N.A.Perestyuk, *Impulsive Differential Equations*, World Scientific, Singapore, (1995).

[16] S.Zhang, *Positive solutions for boundary value problems of nonlinear fractional differential equations*, Elec. J. Diff. Eqn., 36(2006), 1-12.