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On the Cartesian Product of Generalized Fibonacci Graphs

Handan Akyar[1,*]

1 Department of Mathematics, Science Faculty, Anadolu University, Eskisehir, Turkey.

Abstract: In this article generalized Fibonacci graphs and their Cartesian product graphs are considered. These graph's certain fundamental properties, such as number of edges, planarity, diameter, radius, center, distance center, Wiener index, etc. are studied.

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1. Introduction

In this study, generalized Fibonacci graphs, which were first defined by Golumbic and Perl and used for fast and efficient communication on networks, are considered (see [5]). Let the graph G contain two different vertices labelled s and t. Define $N(G)$ to be the number of different paths in G from s to t. Golumbic and Perl dealt with the following problem: Given integer m and n, find an acyclic digraph G with m edges and n vertices maximizing the number $N(G)$. As a solution to this problem, they have defined Fibonacci graphs (see [5]). Fibonacci graphs have been used for the purpose of performing efficient communications in networks (see [2] and references therein). More precisely, consider a network of n nodes, and assume that communications among the nodes proceed by a sequence of synchronous calls between neighboring vertices. A round is defined as the set of calls performed at the same time. Several studies address the problem of computing the minimum number of rounds necessary to perform an all-to-all broadcasting (that is, gossiping) between n nodes.

There are several different studies on generalized Fibonacci graphs. In [9] authors investigate the structure of mincuts in an n-vertex generalized Fibonacci graph of degree 3 and calculate exact value of mincuts in this graph. In [8] authors investigate the relationship between algebraic expressions and graphs. They consider Fibonacci graph which gives a generic example of non-series-parallel graphs and they simplify the expressions of Fibonacci graphs and find their shortest representations. Although generalized Fibonacci graphs are mostly used for communication on networks, but there are also a variety of applications in chemistry (see. [3, 4, 6]). A generalized Fibonacci graph of degree k has vertices $\{1, 2, 3, \ldots, n\}$ and edges

$$\big\{\{v, w\} | 1 \le v, w \le n \text{ and } |w - v| \le k\big\},$$

and it is usually denoted by $F_n(k)$ (see Figure 1). In particular, if $k = 2$ then it is called a Fibonacci graph.

* *E-mail: hakyar@anadolu.edu.tr*

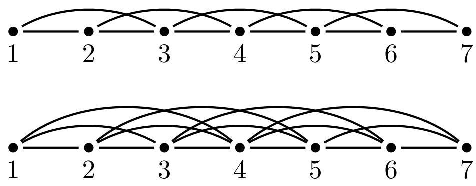

Figure 1. $F_7(2)$ **and** $F_7(3)$ **generalized Fibonacci graphs**

2. Cartesian Product of Generalized Fibonacci Graphs

In this section we discuss the certain properties of Cartesian product of two generalized Fibonacci graphs. The motivation behind this work comes from multi-computer interconnection networks, which are almost exclusively based on Cartesian product of graphs.

The Cartesian product of graphs G and H, written $G\Box H$, is the graph with vertex set $V(G) \times V(H)$ specified by putting (u_1, u_2) adjacent to (v_1, v_2) if and only if

(1). $u_1 = v_1$ and $\{u_2, v_2\} \in E(H)$, or

(2). $u_2 = v_2$ and $\{u_1, v_1\} \in E(G)$.

Thus, the Cartesian product $F_n(k)\Box F_m(l)$ of two generalized Fibonacci graphs $F_n(k)$ and $F_m(l)$ is a graph with nm vertices (see Figure 2). Since $F_n(k)$ and $F_m(l)$ are connected graphs, $F_n(k)\Box F_m(l)$ is also connected. But, the Cartesian product of generalized Fibonacci graphs $F_n(k)$ and $F_m(l)$ is not planar when $k > 1$ and $l > 1$. Because it is well-known that the Cartesian product of connected graphs G and H on at least three vertices is planar if and only if both G and H are paths or if one is a path and the other a cycle (see [7]).

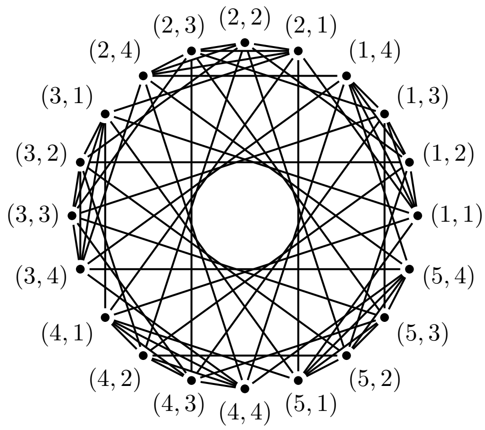

Figure 2. $F_5(3)\Box F_4(2)$ **Cartesian product graph of** $F_5(3)$ **and** $F_4(2)$ **generalized Fibonacci graphs.**

It is not difficult to prove that the degree sequence of generalized Fibonacci graph $F_n(k)$ is

$$(k, k, k+1, k+1, k+2, k+2, \ldots, 2k-1, 2k-1, 2k, 2k, \ldots, 2k),$$

when $k < n/2$. Here, the number of vertices having degree $2k$ is $(n-2k)$. Similarly, if $k \geq n/2$ then the degree sequence of generalized Fibonacci graph $F_n(k)$ is

$$(k, k, k+1, k+1, k+2, k+2, \ldots, n-1, n-1, \ldots, n-1).$$

Here, the number of vertices having degree $(n-1)$ is $2k-n+2$. Thus we have

$$\Delta\big(F_n(k)\big) = \begin{cases} 2k & , \text{ for } k < n/2 \\ n-1 & , \text{ for } k \geq n/2 \end{cases} \tag{1}$$

and

$$\delta\big(F_n(k)\big) = \begin{cases} k & , \text{ for } k < n \\ n-1 & , \text{ for } k \geq n. \end{cases} \tag{2}$$

Using formulas (1), (2) and following equalities

$$\Delta\big(F_n(k)\Box F_m(l)\big) = \Delta\big(F_n(k)\big) + \Delta\big(F_m(l)\big)$$

and

$$\delta\big(F_n(k)\Box F_m(l)\big) = \delta\big(F_n(k)\big) + \delta\big(F_m(l)\big)$$

one can calculate the maximum and minimum degrees of the Cartesian product of generalized Fibonacci graphs $F_n(k)$ and $F_m(l)$.

Proposition 2.1.

$$|E\big(F_n(k)\Box F_m(l)\big)| = \frac{m(2n-k-1)k}{2} + \frac{n(2m-l-1)l}{2}.$$

Proof. Let G and H be any graphs and $(u,v) \in V(G\Box H)$. Then

$$\deg_{G\Box H}\big((u,v)\big) = \deg_G(u) + \deg_H(v).$$

Using the hand-shaking lemma for graphs or alternatively, since for any graphs G and H,

$$|E(G\Box H)| = |V(G)|\,|E(H)| + |V(H)|\,|E(G)|$$

and

$$|E\left(F_n(k)\right)| = \frac{(2n-k-1)k}{2}$$

(see [1, 5, 7]), we obtain the desired result. □

Proposition 2.2. *For $k \geq 2$, $F_n(k)$ generalized Fibonacci graphs are Hamiltonian; that is, there exists a (Hamilton) cycle through the graph that visits each vertex exactly once.*

Proof. Indeed, if n is odd then

$$1 \to 2 \to 4 \to \ldots \to (n-1) \to n \to (n-2) \to \ldots \to 3 \to 1,$$

if n is even then

$$1 \to 2 \to 4 \to \ldots \to n \to (n-1) \to (n-3) \to \ldots \to 3 \to 1$$

is a Hamilton cycle in $F_n(k)$ for $k \geq 2$. Recall that if the graphs G and H are both hamiltonian, then $G\Box H$ is also hamiltonian. As a result, $F_n(k)\Box F_m(l)$ is hamiltonian for $k,l \geq 2$. □

For instance,

$$\begin{aligned}&(1,1)\to(1,2)\to(1,3)\to(1,4)\to(1,5)\to(1,6)\to(2,6)\to(2,4)\to(2,2)\to(2,1)\\&\to(2,3)\to(2,5)\to(3,5)\to(3,3)\to(3,1)\to(3,2)\to(3,4)\to(3,6)\to(4,6)\to(4,4)\\&\to(4,2)\to(4,1)\to(4,3)\to(4,5)\to(5,5)\to(5,3)\to(5,2)\to(5,4)\to(5,6)\to(6,6)\\&\to(6,4)\to(6,2)\to(6,1)\to(6,3)\to(6,5)\to(7,5)\to(7,6)\to(7,4)\to(7,2)\to(7,3)\\&\to(7,1)\to(5,1)\to(1,1)\end{aligned}$$

is a Hamilton cycle in $F_7(4)\Box F_6(3)$. We recall that the diameter $D(G)$ of a connected graph G is defined as

$$D(G)=\max_{u,v\in G} d_G(u,v),$$

the eccentricity $\epsilon(v)$ of a vertex v in G is defined as

$$\epsilon(v)=\max_{u\in U} d_G(v,u),$$

and the radius $R(G)$ is defined as

$$R(G)=\min_{v\in G}\epsilon(v).$$

Proposition 2.3.

$$D\big(F_n(k)\Box F_m(l)\big)=\left\lceil\frac{n-1}{k}\right\rceil+\left\lceil\frac{m-1}{l}\right\rceil,$$

and

$$R\big(F_n(k)\Box F_m(l)\big)=\left\lceil\frac{n}{2k}\right\rceil+\left\lceil\frac{m}{2l}\right\rceil.$$

Here $\lceil x\rceil$ *represents the smallest integer greater than or equal to* x.

Proof. For the connected graphs G and H, we have

$$D(G\Box H)=D(G)+D(H)\text{ and }R(G\Box H)=R(G)+R(H).$$

Therefore, since $D(F_n(k))=\lceil\frac{n-1}{k}\rceil$ (see [1]) we obtain

$$D\big(F_n(k)\Box F_m(l)\big)=\left\lceil\frac{n-1}{k}\right\rceil+\left\lceil\frac{m-1}{l}\right\rceil.$$

□

n \ k	1	2	3	4	5	6	7	8	9	10	11
2	1										
3	2	1									
4	3	2	1								
5	4	2	2	1							
6	5	3	2	2	1						
7	6	3	2	2	2	1					
8	7	4	3	2	2	2	1				
9	8	4	3	2	2	2	2	1			
10	9	5	3	3	2	2	2	2	1		
11	10	5	4	3	2	2	2	2	2	1	
12	11	6	4	3	3	2	2	2	2	2	1

Table 1. $D(F_n(k))$ **diameters of generalized Fibonacci graphs** $F_n(k)$ **for certain values of** n **and** k

Using Table 1 and Proposition 2.3 one can calculate the diameter of Cartesian product graph $F_n(k)\Box F_m(l)$.

Proposition 2.4.

$$R(F_n(k)) = \left\lceil \frac{n - \lceil \frac{n}{2} \rceil}{k} \right\rceil$$

Proof. Clearly, by the definition of eccentricity, the value of $\epsilon(\lceil n/2 \rceil)$ is the minimum value on the set of all eccentricities. Furthermore,

$$\epsilon(\lceil n/2 \rceil) = \left\lceil \frac{n - \lceil \frac{n}{2} \rceil}{k} \right\rceil .$$

Hence, we complete the proof. □

Using Proposition 2.4 one can obtain Table 2.

$n \backslash k$	1	2	3	4	5	6	7	8	9	10	11
2	1										
3	1	1									
4	2	1	1								
5	2	1	1	1							
6	3	2	1	1	1						
7	3	2	1	1	1	1					
8	4	2	2	1	1	1	1				
9	4	2	2	1	1	1	1	1			
10	5	3	2	2	1	1	1	1	1		
11	5	3	2	2	1	1	1	1	1	1	
12	6	3	2	2	2	1	1	1	1	1	1

Table 2. $R(F_n(k))$ **radii of generalized Fibonacci graphs** $F_n(k)$ **for certain values of** n **and** k

Therefore, in view of Proposition 2.4 we obtain,

$$R\big(F_n(k)\square F_m(l)\big) = \left\lceil \frac{n - \lceil \frac{n}{2} \rceil}{k} \right\rceil + \left\lceil \frac{m - \lceil \frac{m}{2} \rceil}{l} \right\rceil .$$

The center $C(G)$ of a graph G of radius r is the set of all vertices v that have the eccentricity $\epsilon(v) = r$. By the definition of generalized Fibonacci graphs and their centers it is easy to see that the center of the Cartesian product graph $F_n(k)\square F_m(l)$ is the Cartesian product of the centers of its factors (see Table 3).

$n \backslash k$	1	2	3	4	5	6	7	8
2	$\{1,2\}$							
3	$\{2\}$	$\{1,2,3\}$						
4	$\{2,3\}$	$\{2,3\}$	$\{1,2,3,4\}$					
5	$\{3\}$	$\{3\}$	$\{2,3,4\}$	$\{1,2,3,4,5\}$				
6	$\{3,4\}$	$\{2,3,4,5\}$	$\{3,4\}$	$\{2,3,4,5\}$	$\{1,2,3,4,5,6\}$			
7	$\{4\}$	$\{3,4,5\}$	$\{4\}$	$\{3,4,5\}$	$\{2,3,4,5,6\}$	$\{1,2,3,4,5,6,7\}$		
8	$\{4,5\}$	$\{4,5\}$	$\{2,3,4,5,6,7\}$	$\{4,5\}$	$\{3,4,5,6\}$	$\{2,3,4,5,6,7\}$	$\{1,2,3,4,5,6,7,8\}$	
9	$\{5\}$	$\{5\}$	$\{3,4,5,6,7\}$	$\{5\}$	$\{4,5,6\}$	$\{3,4,5,6,7\}$	$\{2,3,4,5,6,7,8\}$	$\{1,2,3,4,5,6,7,8,9\}$
10	$\{5,6\}$	$\{4,5,6,7\}$	$\{4,5,6,7\}$	$\{2,3,4,5,6,7,8,9\}$	$\{5,6\}$	$\{4,5,6,7\}$	$\{3,4,5,6,7,8\}$	$\{2,3,4,5,6,7,8,9\}$
11	$\{6\}$	$\{5,6,7\}$	$\{5,6,7\}$	$\{3,4,5,6,7,8,9\}$	$\{6\}$	$\{5,6,7\}$	$\{4,5,6,7,8\}$	$\{3,4,5,6,7,8,9\}$
12	$\{6,7\}$	$\{6,7\}$	$\{6,7\}$	$\{4,5,6,7,8,9\}$	$\{2,3,4,5,6,7,8,9,10,11\}$	$\{6,7\}$	$\{5,6,7,8\}$	$\{4,5,6,7,8,9\}$

Table 3. $C(F_n(k))$ **centers of generalized Fibonacci graphs** $F_n(k)$ **for certain values of** n **and** k

Example 2.5. *Let us consider the Cartesian product graph* $F_5(3)\square F_4(2)$ *(see Figure 2). The center of* $F_5(3)$ *and* $F_4(2)$ *are* $\{2,3,4\}$ *and* $\{2,3\}$ *respectively. Since* $R(F_5(3)\square F_4(2)) = 2$ *it is not difficult to see that*

$$C\big(F_5(3)\square F_4(2)\big) = \{(2,2),(2,3),(3,2),(3,3),(4,2),(4,3)\}.$$

The distance center $C_d(G)$ of a connected graph G is defined as

$$C_d(G) = \{v \in V(G) : \sum_{u \in V(G)} d_G(u,v) \text{ is minimum}\}.$$

It is different from the center of $F_n(k)$ (see Table 4). The distance center of a Cartesian product is the Cartesian product of the distance centers of the factors. Hence, we obtain

$$C_d\big(F_n(k)\Box F_m(l)\big) = C_d\big(F_n(k)\big) \times C_d\big(F_m(l)\big).$$

$n \backslash k$	1	2	3	4	5	6	7	8	9
2	{1, 2}								
3	{2}	{1, 2, 3}							
4	{2, 3}	{2, 3}	{1, 2, 3, 4}						
5	{3}	{3}	{2, 3, 4}	{1, 2, 3, 4, 5}					
6	{3, 4}	{3, 4}	{3, 4}	{2, 3, 4, 5}	{1, 2, 3, 4, 5, 6}				
7	{4}	{3, 4, 5}	{4}	{3, 4, 5}	{2, 3, 4, 5, 6}	{1, 2, 3, 4, 5, 6, 7}			
8	{4, 5}	{4, 5}	{4, 5}	{4, 5}	{3, 4, 5, 6}	{2, 3, 4, 5, 6, 7}	{1, 2, 3, 4, 5, 6, 7, 8}		
9	{5}	{5}	{4, 5, 6}	{5}	{4, 5, 6}	{3, 4, 5, 6, 7}	{2, 3, 4, 5, 6, 7, 8}	{1, 2, 3, 4, 5, 6, 7, 8, 9}	
10	{5, 6}	{5, 6}	{4, 5, 6, 7}	{5, 6}	{5, 6}	{4, 5, 6, 7}	{3, 4, 5, 6, 7, 8}	{2, 3, 4, 5, 6, 7, 8, 9}	{1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
11	{6}	{5, 6, 7}	{5, 6, 7}	{5, 6, 7}	{6}	{5, 6, 7}	{4, 5, 6, 7, 8}	{3, 4, 5, 6, 7, 8, 9}	{2, 3, 4, 5, 6, 7, 8, 9, 10}
12	{6, 7}	{6, 7}	{6, 7}	{5, 6, 7, 8}	{6, 7}	{6, 7}	{5, 6, 7, 8}	{4, 5, 6, 7, 8, 9}	{3, 4, 5, 6, 7, 8, 9, 10}

Table 4. $C_d(F_n(k))$ **distance centers of generalized Fibonacci graphs** $F_n(k)$ **for certain values of** n **and** k

Graphs arising in chemistry are a primary source of examples for graph theory such as benzenoid graphs, chemical trees, and fullerenes are just a few of the well-known examples. The main goal of chemical graph theory is to investigate the graph and to predict the molecule's properties. This is frequently obtained by computing cautious selected graph invariants. The Wiener index, introduced by Wiener (1947), is the oldest such invariant (see [10]). The Wiener index $W(G)$ of a graph G is defined as the sum of the distances between all pairs of vertices of G, that is,

$$W(G) = \frac{1}{2} \sum_{u\in V(G)} \sum_{v\in V(G)} d_G(u, v).$$

Thus the Wiener index of a generalized Fibonacci graph $F_n(k)$ is

$$W(F_n(k)) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} d_{F_n(k)}(i, j).$$

Since $d_{F_n(k)}(i, j) = \left\lceil \frac{|i-j|}{k} \right\rceil$, Table 5 can be easily constructed.

$n \backslash k$	1	2	3	4	5	6	7	8	9
2	1								
3	4	3							
4	10	7	6						
5	20	13	11	10					
6	35	22	18	16	15				
7	56	34	27	24	22	21			
8	84	50	39	34	31	29	28		
9	120	70	54	46	42	39	37	36	
10	165	95	72	61	55	51	48	46	45

Table 5. $W(F_n(k))$ **Wiener index values of generalized Fibonacci graphs** $F_n(k)$ **for certain values of** n **and** k

Since $F_n(1) \cong P_n$, $W(F_n(1)) = \frac{n^3-n}{6}$ and since for $k \geq n-1$, $F_n(k) \cong K_n$, $W(F_n(k)) = \frac{n(n-1)}{2}$ for $k \geq n-1$.

Proposition 2.6. *Wiener index of a generalized Fibonacci graph $F_n(k)$ satisfies following basic equality:*

$$W(F_n(k)) = \sum_{i=1}^{n} d_{F_n(k)}(n, i) + W(F_{n-1}(k))$$

Proof.

$$\begin{aligned} W(F_n(k)) &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d_{F_n(k)}(i,j) \\ &= \frac{1}{2}\Big(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\big[d_{F_n(k)}(i,j)\big] + d_{F_n(k)}(1,n) + d_{F_n(k)}(2,n) + \cdots + d_{F_n(k)}(n,n) \\ &\quad + d_{F_n(k)}(n,1) + d_{F_n(k)}(n,2) + \cdots + d_{F_n(k)}(n,n-1)\Big) \end{aligned}$$

Since $d_{F_n(k)}(i,j) = d_{F_n(k)}(j,i)$ for $1 \le i,j \le n$ and $d_{F_n(k)}(n,n) = 0$ we obtain

$$\begin{aligned} W(F_n(k)) &= \frac{1}{2}\left(\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\big[d_{F_n(k)}(i,j)\big] + 2d_{F_n(k)}(n,1) + 2d_{F_n(k)}(n,2) + \cdots + 2d_{F_n(k)}(n,n-1)\right) \\ &= \sum_{i=1}^{n} d_{F_n(k)}(n,i) + \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} d_{F_n(k)}(i,j) \\ &= \sum_{i=1}^{n} d_{F_n(k)}(n,i) + W(F_{n-1}(k)) \end{aligned}$$

which completes the proof. □

Theorem 2.7.

$$W(F_n(k)\Box F_m(l)) = n^2W(F_m(l)) + m^2W(F_n(k))$$

Proof.

$$\begin{aligned} W(F_n(k)\Box F_m(l)) &= \frac{1}{2}\sum_{1\le i,s\le n}\sum_{1\le j,t\le m} d_{F_n(k)\Box F_m(l)}\big((i,j),(s,t)\big) \\ &= \frac{1}{2}\sum_{1\le i,s\le n}\sum_{1\le j,t\le m}\big(d_{F_n(k)}(i,s) + d_{F_m(l)}(j,t)\big) \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{s=1}^{n}\sum_{j=1}^{m}\sum_{t=1}^{m}\big(d_{F_n(k)}(i,s) + d_{F_m(l)}(j,t)\big) \\ &= \sum_{i=1}^{n}\sum_{s=1}^{n}\left(\frac{1}{2}\sum_{j=1}^{m}\sum_{t=1}^{m} d_{F_m(l)}(j,t)\right) \\ &\quad + \sum_{j=1}^{m}\sum_{t=1}^{m}\left(\frac{1}{2}\sum_{i=1}^{n}\sum_{s=1}^{n} d_{F_n(k)}(i,s)\right) \\ &= n^2W(F_m(l)) + m^2W(F_n(k)). \end{aligned}$$

□

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