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Common Fixed Points of Generalized Quasi–Contractive Type Operators By a Three-Step Iterative Process in CAT(0) Spaces

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Abstract: The aim of this paper is to approximate common fixed points of a recent three step iterative process essentially due to Thakur et al [16] for two generalized quasi–contractive type operators and establish strong convergence result for the same operators in $CAT(0)$ spaces. The results obtained in this paper extend and improve the recent ones announced by Saluja [15] , Xu and Noor [17], Noor [13], Picard [14], Mann [12], Ishikawa [10] and many others.

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1. Introduction

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset R$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\Delta(x_1, x_2, x_2)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) = \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\bar{x}_i, \bar{x}_j) = (x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a $CAT(0)$ space if all geodesic triangles satisfy the following comparison axiom:
Let Δ be a geodesic triangle in X and let $\bar{\Delta}$ be a comparison triangle for Δ. Then Δ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$, $d(x, y) \leq d_{\mathbb{E}^2}(\bar{x}, \bar{y})$. If x, y_1, y_2 are points in a $CAT(0)$ space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the $CAT(0)$ inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2 \tag{CN}$$

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This is the (CN) inequality of Bruhat and Tits [7]. In fact, a geodesic space is a $CAT(0)$ space if and only if it satisfy (CN) inequality.

Lemma 1.1 ([9]). *Let X be a $CAT(0)$ space. Then $d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z)$ for all $x,y,z \in X$ and $t \in [0,1]$.*

In 1890, Picard [14] defined an iterative scheme $\{x_n\}_{n=0}^{\infty}$ as

$$x_{n+1} = Tx_n, \;\; n = 0,1,2,3,.....; \tag{1}$$

has been employed to approximate the fixed point of mappings satisfying the inequality

$$d(Tx,Ty) \leq ad(x,y) \tag{2}$$

for all $x,y \in X$ and $a \in [0,1)$. The above condition (2) is called Banach's contraction condition. In 1969, Kannan [11] defined a mapping T called Kannan mapping if there exists $b \in (0,1/2)$ such that

$$d(Tx,Ty) \leq b[d(x,Tx) + d(y,Ty)] \tag{3}$$

for all $x,y \in X$. In 1972, Chatterjea [8] defined a mapping T is called Chatterjea mapping as a generalization of Kannan mapping if there exists $c \in (0,1/2)$ such that

$$d(Tx,Ty) \leq c[d(x,Ty) + d(y,Tx)] \tag{4}$$

for all $x,y \in X$. In 1972, Zamfirescu [18] obtained the following fixed point theorem.

Theorem 1.2. *Let (X,d) be a complete metric space and $T : X \to X$ be a mapping for which there exist the real numbers a,b,c satisfying $a \in (0,1)$, $b,c \in (0,1/2)$ such that for any pair $x,y \in X$, at least one of the following conditions holds:*

(1). $d(Tx,Ty) \leq ad(x,y)$

(2). $d(Tx,Ty) \leq b[d(x,Tx) + d(y,Ty)]$

(3). $d(Tx,Ty) \leq c[d(x,Ty) + d(y,Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = Tx_n$, $n = 0,1,2,3,.....$; converges to p for any arbitrary but fixed $x_0 \in X$.

An operator T which satisfy at least one of the above conditions (1), (2), (3) is called a Zamfirescu operator or a Z-operator. In 2004, Berinde [3] gave the strong convergence result of Ishikawa iterative process [10] defined by: $x_0 \in C$,

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n Ty_n, \\ y_n &= (1-\beta_n)x_n + \beta_n Tx_n, \quad \text{for all } n \geq 0; \end{aligned}$$

to approximate fixed points of Zamfirescu operator in an arbitrary Banach space E. While proving the result, Berinde used the condition,

$$\|Tx - Ty\| \leq \delta \|x - y\| + 2\delta \|x - Tx\| \tag{5}$$

which holds for any $x, y \in E$ where $0 \leq \delta < 1$. In 2014, Saluja [15] gave a more generalized condition than (5) in the framework of normed linear spaces as: Let C be a nonempty closed convex subset of a normed linear space E and $T: C \to C$ be a self map of C. There exists a constant $L \geq 0$ such that for all $x, y \in C$, we have

$$\|Tx - Ty\| \leq e^{L\|x-Tx\|}(\delta \|x - y\| + 2\delta \|x - Tx\|)$$

where $0 \leq \delta < 1$ and e^x denotes the exponential function of $x \in C$. We modify this condition in the framework of $CAT(0)$ spaces as follows: Let C be a nonempty closed convex subset of a $CAT(0)$ space and $T: C \to C$ be a self map of C. There exists a constant $L \geq 0$ such that for all $x, y \in C$, we have

$$d(Tx, Ty) \leq e^{Ld(x,Tx)}(\delta d(x, y) + 2\delta d(x, Tx)) \tag{6}$$

where $0 \leq \delta < 1$ and e^x denotes the exponential function of $x \in C$. We call this condition (6) a generalized quasi-contractive type operator.

Remark 1.3. *If $L = 0$ in the above condition, then $d(Tx, Ty) \leq \delta d(x, y) + 2\delta d(x, Tx)$, which is the Zamfirescu condition used by Berinde [3] where $\delta = max\{a, b/1 - b, c/1 - c\}, 0 \leq \delta < 1$, while constants a, b, c are defined in the same manner as in Theorem 1.*

In this paper, our purpose is to approximate common fixed points of two generalized quasi-contractive type operators by using Thakur et. al. [16] iterative process in the framework of $CAT(0)$ spaces. Therefore, we need to modify the generalized quasi-contractive type operator (6) to the case of two mappings. One simple way is that we force both of our mappings to satisfy above kind of condition separately. That is, S and T satisfy

$$\begin{aligned} d(Sx, Sy) &\leq e^{Ld(x,Sx)}(\delta d(x, y) + 2\delta d(x, Sx)) \\ \textit{and} \quad d(Tx, Ty) &\leq e^{Ld(x,Tx)}(\delta d(x, y) + 2\delta d(x, Tx)) \end{aligned}$$

respectively. However, we can modify this to a more general extension as:

$$\max\{d(Sx, Sy), d(Tx, Ty)\} \leq e^{Lmax\{d(x,Sx),d(x,Tx)\}}(\delta d(x, y) + 2\delta max\{d(x, Sx), d(x, Tx)\}) \tag{7}$$

This condition (7) reduces to (6) as follows when either $S = T$ or one of the mappings is identity.

- The case $S = T$ is obvious.
- When one of the mappings, say S, is identity, then (1.7) reduces to

$$\max\{d(x, y), d(Tx, Ty)\} \leq e^{Ld(x,Tx)}(\delta d(x, y) + 2\delta d(x, Tx)). \tag{8}$$

- If $max\{d(x, y), d(Tx, Ty)\} = d(Tx, Ty)$, then clearly (8) reduces to (6).
- If $max\{d(x, y), d(Tx, Ty)\} = d(x, y)$, then (8) reduces to

$$d(Tx, Ty) \leq d(x, y) \leq e^{Ld(x,Tx)}(\delta d(x, y) + 2\delta d(x, Tx)).$$

Thus, we conclude that (7) reduces to (6) when either $S = T$ or one of the mappings is identity.

In 2016, Thakur et al. [16] established a new three step iterative process in Banach spaces and they showed by an example that this iteration is much faster than the iteration due to Picard [14], Mann [12], Ishikawa [10], Noor [13], Agarwal et al. [2], Abbas et al. [1]. Let C be a nonempty closed convex subset of a uniformly convex Banach space E and $T : C \to C$ be a nonexpansive mapping. For $x_1 \in C$,

$$x_{n+1} = (1-\alpha_n)Tz_n + \alpha_n Ty_n \tag{9}$$

$$y_n = (1-\beta_n)z_n + \beta_n Tz_n \tag{10}$$

$$z_n = (1-\gamma_n)x_n + \gamma_n Tx_n \tag{11}$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences of positive numbers in $(0,1)$. We modify the iterative process (11) into $CAT(0)$ spaces for two mappings as follows: Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $T : C \to C$ be a mapping. Then the sequence $\{x_n\}$ in C is defined as For $x_1 \in C$,

$$x_{n+1} = (1-\alpha_n)Sz_n \oplus \alpha_n Ty_n \tag{12}$$

$$y_n = (1-\beta_n)z_n \oplus \beta_n Tz_n \tag{13}$$

$$z_n = (1-\gamma_n)x_n \oplus \gamma_n Sx_n \tag{14}$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences of positive numbers in $[0,1]$. If we take $S = T$ in the above iterative process (14), then this process reduces to (11) in $CAT(0)$ space settings. Our result extend and improve many results in the existing literature due to Abbas et al. [1], Agarwal et al. [2], Ishikawa [10], Mann [12], Noor [13], Picard [14], Saluja [15], Xu and Noor [17] and many others.

2. Main Results

Let X be a complete $CAT(0)$ space and $T : X \to X$ be a self mapping of X. Suppose $F(T) = \{x \in C : Tx = x\}$ is the set of fixed points of T.Therefore, $F(S) \cap F(T) = \{x \in C : Tx = x = Sx\}$. We need the following useful lemma to prove our main results in this paper.

Lemma 2.1 ([4]). *Suppose that $\{p_n\}_{n=0}^{\infty}$, $\{q_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=0}^{\infty}$ are three sequences of nonnegative real numbers satisfying the following condition: $p_{n+1} \le (1-s_n)p_n + q_n + r_n$, $n = 0,1,2,3,...,$where $\{s_n\}_{n=0}^{\infty} \subset [0,1]$. If $\sum\limits_{n=0}^{\infty} s_n = \infty$, $\lim\limits_{n\to\infty} q_n = O\{s_n\}$ and $\sum\limits_{n=0}^{\infty} r_n < \infty$, then $\lim\limits_{n\to\infty} p_n = 0$.*

Theorem 2.2. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $S,T : C \to C$ be generalized quasi-contractive type operators given by (7) with $F(S) \cap F(T) \ne \phi$. For $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by (14). If $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $[0,1]$ such that $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$, then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Proof. Assume that $F(S) \cap F(T) \ne \phi$. Let $p \in F(S) \cap F(T)$. Since S and T satisfies the generalized quasi–contractive type condition given by (7). Taking $x = p$ and $y = y_n$, we see from (7) that

$$\max\{d(Sy_n,p), d(Ty_n,p)\} \le e^{L(0)}\{\delta d(p,y_n) + 2\delta(0)\},$$

which gives

$$d(Ty_n, p) \leq \delta d(y_n, p) \tag{15}$$

Similarly, by taking $x = p$ and $y = x_n, z_n$ in (7), we have

$$d(Sz_n, p) \leq \delta d(z_n, p) \tag{16}$$

$$d(Tz_n, p) \leq \delta d(z_n, p) \tag{17}$$

$$d(Sx_n, p) \leq \delta d(x_n, p) \tag{18}$$

Now, using (14), we have

$$d(x_{n+1}, p) = d((1 - \alpha_n)Sz_n \oplus \alpha_n Ty_n, p) \leq (1 - \alpha_n)d(Sz_n, p) + \alpha_n d(Ty_n, p)$$

From (15) and (16), we have

$$d(x_{n+1}, p) \leq \delta(1 - \alpha_n)d(z_n, p) + \alpha_n \delta d(y_n, p) \tag{19}$$

But

$$\begin{aligned} d(y_n, p) &= d((1 - \beta_n)z_n \oplus \beta_n Tz_n, p) \\ &\leq (1 - \beta_n)d(z_n, p) + \beta_n d(Tz_n, p) \end{aligned}$$

Using (17), we have

$$\begin{aligned} d(y_n, p) &\leq (1 - \beta_n)d(z_n, p) + \beta_n \delta d(z_n, p) \\ &= (1 - \beta_n(1 - \delta))d(z_n, p) \end{aligned} \tag{20}$$

But

$$\begin{aligned} d(z_n, p) &= d((1 - \gamma_n)x_n \oplus \gamma_n Sx_n, p) \\ &\leq (1 - \gamma_n)d(x_n, p) + \gamma_n d(Sx_n, p) \end{aligned}$$

Using (18), we have

$$\begin{aligned} d(z_n, p) &\leq (1 - \gamma_n)d(x_n, p) + \gamma_n \delta d(x_n, p) \\ &= (1 - \gamma_n(1 - \delta))d(x_n, p) \end{aligned} \tag{21}$$

Using (21) in (20), we have

$$d(y_n, p) \leq (1 - \beta_n(1 - \delta))(1 - \gamma_n(1 - \delta))d(x_n, p) \tag{22}$$

Therefore, using (21) and (22) in (19), we have

$$\begin{aligned} d(x_{n+1}, p) &\leq \delta(1 - \alpha_n)(1 - \gamma_n(1 - \delta))d(x_n, p) + \alpha_n \delta(1 - \beta_n(1 - \delta))(1 - \gamma_n(1 - \delta))d(x_n, p) \\ &= \delta(1 - \gamma_n(1 - \delta))[1 - \alpha_n + \alpha_n(1 - \beta_n(1 - \delta))]d(x_n, p) \end{aligned}$$

$$
\begin{aligned}
&= \delta(1-\gamma_n(1-\delta))[1-\alpha_n(1-(1-\beta_n(1-\delta)))]d(x_n,p)\\
&= \delta(1-\gamma_n(1-\delta))[1-\alpha_n\beta_n(1-\delta)]d(x_n,p)\\
&\leq (1-\{1-\delta\}^3\alpha_n)d(x_n,p)\\
&\leq (1-B_n)d(x_n,p);
\end{aligned}
$$

where $B_n = \{1-\delta\}^3\alpha_n$, since $0 \leq \delta < 1$ and by assumption of the theorem $\sum_{n=1}^{\infty}\alpha_n = \infty$, it follows that $\sum_{n=1}^{\infty}B_n = \infty$, therefore from Lemma 2, we get that $\lim_{n\to\infty} d(x_n,p) = 0$. Thus $\{x_n\}$ converges strongly to a common fixed point of S and T. To show the uniqueness of the fixed point p, assume that $p_1, p_2 \in F(S)\cap F(T)$ and $p_1 \neq p_2$. Applying the generalized quasi–contractive type condition given by (7) and using the fact that $0 \leq \delta < 1$, we obtain

$$
\begin{aligned}
d(p_1,p_2) &= d(Sp_1,Sp_2)\\
&\leq e^{Ld(p_1,Sp_1)}\{\delta d(p_1,p_2)+2\delta d(p_1,Sp_1)\}\\
&\leq e^{Ld(p_1,p_1)}\{\delta d(p_1,p_2)+2\delta d(p_1,p_1)\}\\
&= e^{L(0)}\{\delta d(p_1,p_2)+2\delta(0)\}\\
&= \delta d(p_1,p_2)\\
&< d(p_1,p_2);
\end{aligned}
$$

which is a contradiction. Therefore, $p_1 = p_2$. Thus, $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique common fixed point of S and T. □

3. conclusion

The generalized quasi–contractive type condition is more general than Zamfirescu operators which, in turn, is more general than Kannan mappings, Chatterjea mappings; and Thakur's new iterative scheme is more general than iterative schemes comparing with Picard [14], Mann [12], Ishikawa [10], Noor [13], Agrawal et al. [2] and Abbas et al. [1] iterative schemes. Thus the result presented in this paper is an extension and generalization of corresponding result proved in [1, 2, 6, 10, 12, 13, 14, 15, 16, 17] and some others given in the contemporary literature.

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